NPTEL ONLINE CERTIFICATIONS COURSE
Mathematical Modeling
Analysis and Applications
Lecture-13
Continuous Time Models in
Populations Dynamics-1
With
Dr.Ameeya kumar Nayak
Department of Mathematics
Indian Institute of Technology Theories

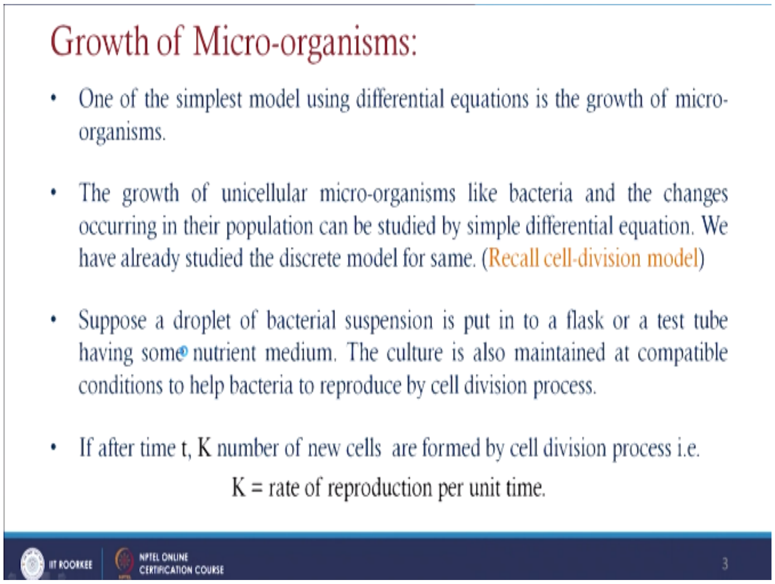
Welcome to the lecture series on mathematical modeling analysis and application, in the last lecture we have discussed the basic definition of differential equations and how it is just appearing in the natural system or like industrial applications or engineering problems and we have discussed also some of this solutions techniques is for first order linear difference equation and in this lecture, we will start about to this continuous time models, on population dynamics. Especially this population dynamics models that has developed based on this differential equation, and we will construct some of this population model based on this ordinary differential equations first then, we will just go for the higher order differential equations.

(Refer Slide Time: 01:15)

So, first in this Lecture, we will just go for like growth of micro- organisms, and how this model is developed based on this differential equations that will just discuss, then we will just go for dependency of Rate of Reproduction on Time and Resources, and in the third stage, we will just taste the steady states, how we can just find a like a physical state or critical points for the development of this population level.

Then, if the population level is increasing then, there should be some resources, and how the resources are acting? In development of this population level or increase of this population level that, we will discuss in last page of this lecture.

(Refer Slide Time: 01:59)



So, one of this simplest model using differential equations is the growth of micro- organisms, since whenever we will just consider any change of variable, with respect to any other parameter definitely, this will just provide you for any differential equations , since this growth of micro-organisms, is nothing but, it change of a size of the population or the change of size of, like different cells with this respective time or with this respective resources or with respective like different aspects of a physical phenomenal, that can be constructed, as in the form of differential equations.

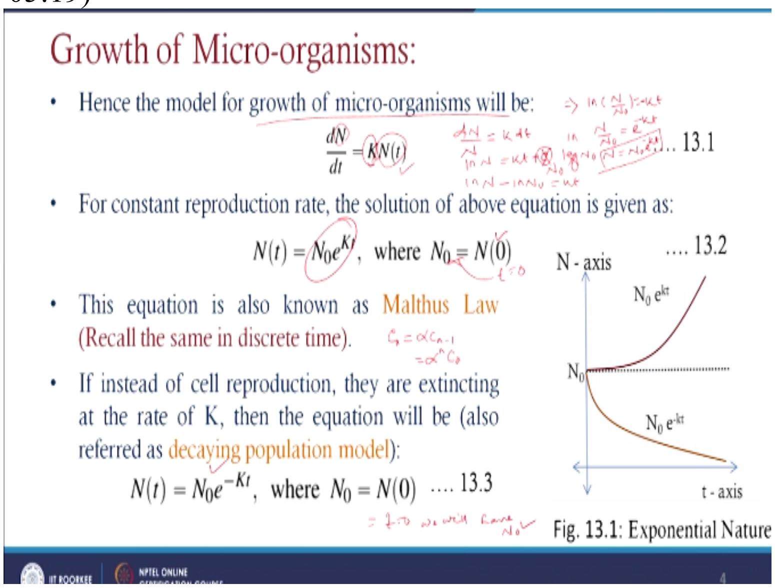
So the growth of use unicellular micro-organisms likes bacteria, unicellular especially since, there like individually, they can grow up, or they can be break to form like new cell, and the changes occurring in their population can be steadied by simple differential equation, since either it can be like divided in to the cells, to growth they, like level of population there, so that is why? We can just construct a model, where each of the individual or uniformly it can be divided to get the new cell, there over and, we can just formulate this model using these differential equations.

We have already studied the descript model for the same, since there itself , we have just written this model as in the form of CNS α CN-1, which is called your cell division model, and especially for the growth of this bacteria , so you have to consider like a droplet of bacterial suspension is put in to a flask or a test tube having some nutrient medium , this means that we have to provide a like compatible medium , where this cell or this bacterial growth can possible so then just we can formulated this model.

So this culture is also maintained computable conditions, since if like favorable conditions will not be supplied there, then this growth cannot be possible also, so many restrictions are there but first we will just go for this development of this model where there is a uniform growth of this populations level due to supportive or the compatible conditions to growth process of this bacteria should be provided.

So that is why we have just considered here the culture is also maintained at compatible conditions to help bacteria to reproduce by cell division process, at each time or at each level the cell will be divided to give new cells that, this cell can be like growth or it can be increase, so after suppose certain time we can just find that this cell division process already if it is started then after certain time you will have like this number of population levels for the cells and for a particular time level suppose T we are just considering here, K is the number of new cells formed by cell division process suppose that is we can just write rate of reproductions for unit time is define as K here rate of reproductions means that with respective times how the cell division is occurring and how this populations growth rate, it is just activated that depends on this parameters K.

(Refer Slide Time: 05:19)



So obsessively if we will just consider this exchange of population level with respective to time that depends on K and the number of population present at the beginning, hence the model for growth of micro organism will be, in the form of like change of population of N there which depends on the rate of change of this population level into the total number of populations, so if you will just considered that suppose this reproductions rate is constant K as constant here then the, solutions of this above equations.

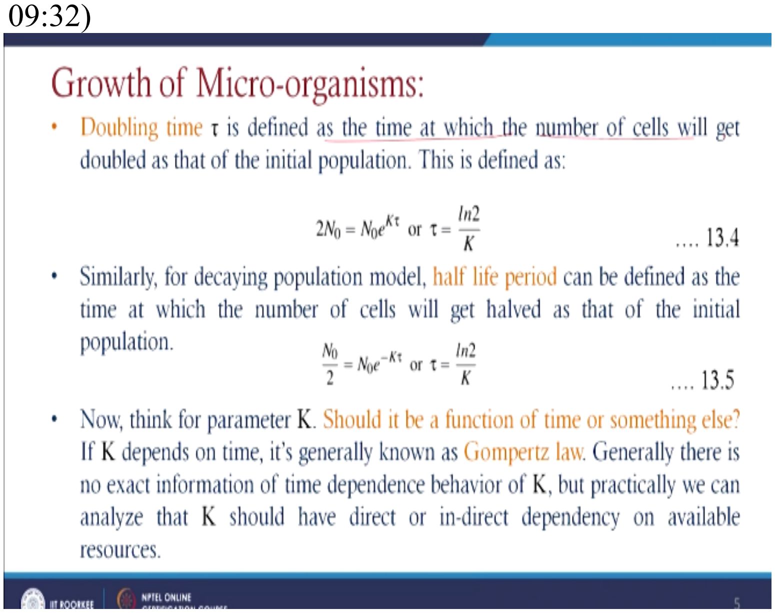
Especially we can just write as DN by N as KDT here and then we can just write LN of N as KT+C here any obituary constant so this obituary constant, we can just assume as suppose like N0 here that is the total number of populations present at the beginning, then we can just write this one as N=R we can just define this one, as in the form of log of N0 suppose, so then we can just write this one as log of N, LN of N- this we can also write also LN of N0 so LN of N0 which can be written here KT here then this implies that wean just write as LN of N by N0 is KT here so N by N0 can be written as E to the power KT so N is equal to N0 E to the power Kt here this Is the final solutions.

If the initial level or at 0^{th} or T is equal to 0 the total number of populations sizes is N0 at that point, so this equation is known as Malthus Law or the Malthus Law of population's dynamics so if we just recall this one as in a diffract time so obviously we can just write since in that sense, we are just writing that model CN is equal to α CN-1, obviously we can just write this one as a α to the power N C0 there, which is also increasing in a extensional order.

If instead of like cell reproductions if there is a decaying or like extension at the rate of K suppose this means that the population level will be decline there, then the equation will be referred as decline population level and especially in that senses we can just consider that this K is negative there so that is why this direct population level we can just put it as – KT there then we will have like NY, N0 so this E to the power-K TN final level we will have like N-,N=N0 into E to the power – KTL where N0 is equal to N of a 0 that is nothing but yet T is equal to 0 we will have this populations level as N0.

So if you just show this one in a graphical sensor then we can just find that there is a growth decaying rate so this would come in the form show there is a expensively growth of a rate that is N0 E to the power KT and if there is a decaying of a growth rate so that should be like this is the population decaying rate there, it is just approaching towards 0 and finally we can just find that there will be no survival of the populations if we just a tens D tens 2 infinite in that rates and that especially in a like physical senses this does not happens since the resource are like many physical restrictions it will just provide this growth of this populations level, so certainly we can just find that this population level will not increase in a infinite manner or it can be decade to 0 that population will vanish, so that cannot be happen especially in a physical scenario so many restrictions it will be there and we will be just consider this restrictions are toward.

(Refer Slide Time: 09:32)

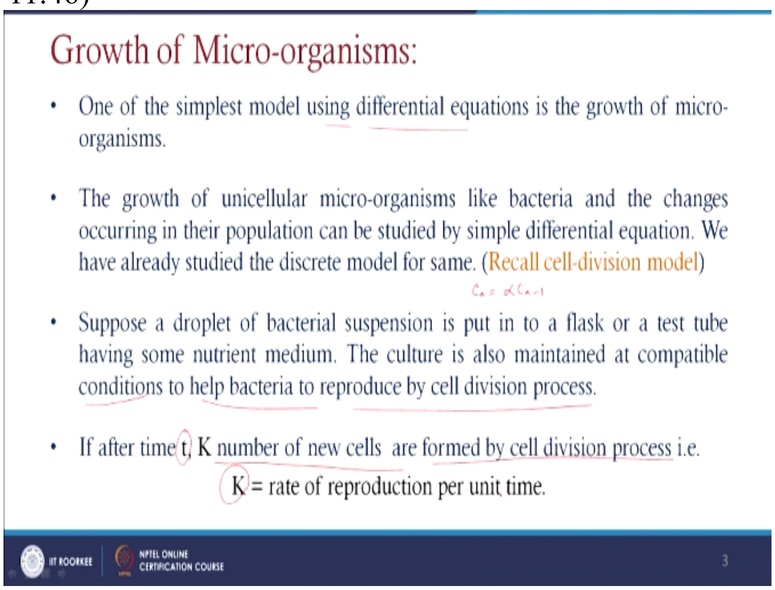


And if we will just go for like if the population size is double at level then this is called your doubling time period and how must time is required if you want to calculate so doubling time T is define as the rate at which the number of a cells will get doubled as that of the initial populations initial populations means we can just considered our solutions that is in the form of N of T is written as N0 E to the power KT so if the population size will be double form the initial population size senses we are just considering at initial T is equal to 0 N is consider as N0 there so for that if you will just consider like N equal to 2 N0 at a level then we can directly replace here N of T as a 2 N0 this is nothing but N0 E to power KT and finally we can just write N0 will cancel it out so E to the power KT is equal to 2 and KT is equal to L and 2 an T we want to calculate so it is a L and 2 YK there this is the final solution.

Similarly for decaying population level so if we required like the population size it will be like half of the initial population size then this is the special called half life period that the initial survival population should be have to there. So it can be define at the time at which the number of cells will get out this means that this N of T can be replaced as like N0Y 2 there which can be written as N0 E to the power KT so if you just cancel it out then we can just find TS like L and 2 by K, since if you just see here that is the half it is there and this is a like –KT it should be present there so that is why we just finding this LN2Y K there if it is high.

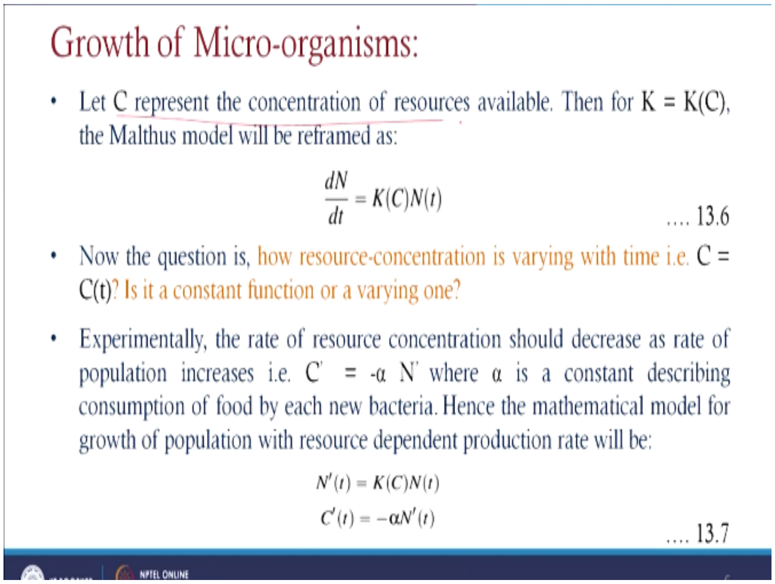
Now think of the parameter K here since a K is the rate at which like we have just define here K is nothing.

(Refer Slide Time: 11:46)



But the rate of reproduction for unit time sometime it may be like decaying, sometimes, it is just infusing the increment of the population level so that is why, we can just think of this parameter K how it is just depending or how it just acting inside the population level for the increment and the decrement of this populations level should it be function of time or something else, so with respective time or whether K will get change or not that we have to also to consider if K defines on time suppose it is generally known as Gompertz Law and generally there is no exact information of time dependences that means the time dependences behavior of K, but practically we can analysis that K should have direct or indirect dependences on available resources since if the resources are self sufficient then this populations level only grows off and if suppose there is a limitations in the resource level then this population level after certain time we can just find that this cells will be struggle for the food and they will just decline.

(Refer Slide Time: 12:54)

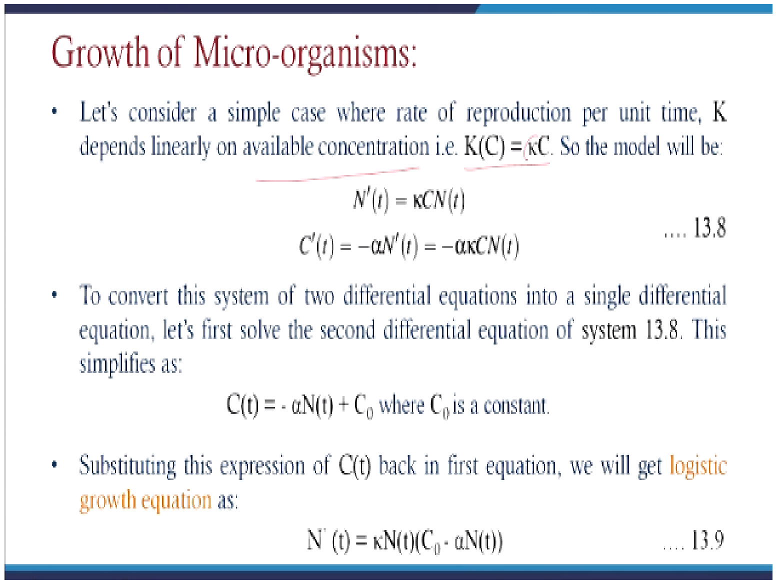


Let C represent the concentrations of resource available, then for K equal to suppose K is a function of C here that is the resources available there then the Malthus model can be reframed as like since K is a function of C here are the concentrations level or the food or the resources whatever it is available there then this change of population level that dependences directly on K of C there and it can be multiplied or used by the total populations size, now the question how the resource concentrations is varying with time.

So may be sometimes we can just find that whether C is getting changed or this resources level it will get like less after like five days or ten days or certain period of time we can just find that C also got changed, so whether it is constant function or varying one that we have to determine experimentally is it tested that the rate of resources concentrations should decrease as rate of population increases definitely this population size they will use this resources hence the resources will get reduced there.

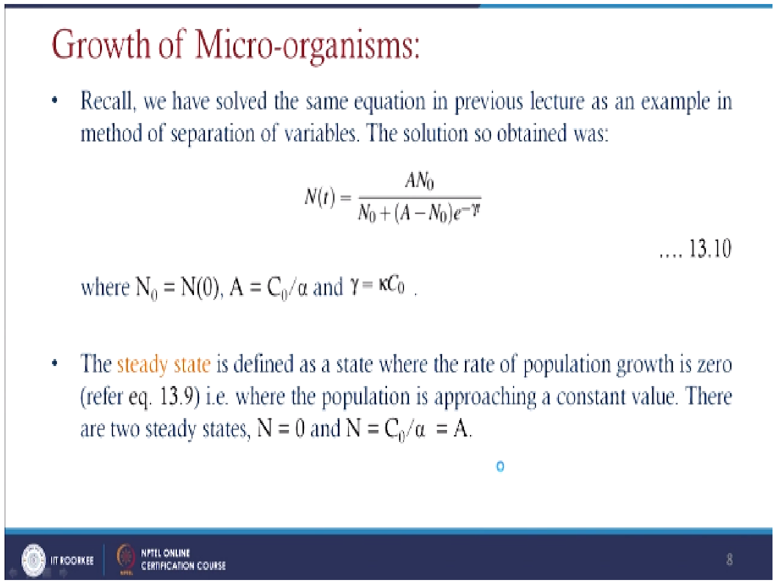
So that is we can just write C prime as –α N prime there that is the change of this resources concentrations will depend directly in a decline form without changes of this populations size where α is constant describing consumption of food by each new bacteria hence the mathematical model for growth of population with resource dependent reproduction rate will be like NST we can just right direct model here K of C into N of T and this consumption of this concentrations by this bacteria which can be written as C does T as-α NST.

(Refer Slide Time: 14:47)



If you just consider a simple case where rate of reproduction for unit time K dependence linearly on available concentrations since there is no other restrictions directly there are just eating the food and there are just getting grown up that is a K of C we can just write K into C there where K is a like a constant there so the model we can just a write it as N does T which was earlier written as K of C into N of T so that is why we can just directly replace K of C as N of KC into N of T and C does T it is especially written as –α into N does T and N does T directly it can be replaced by this –α K C N of T there to convert this system of two differential equations into single differential equations since we can directly we can just find the solutions from the second equations and we can just put in the first in the equations and this simplified case if you just integrate this equation here that is C does T as –α and does T here directly we can integrate then and then we can just find C of T –α is a constant here so –α N of T + proportionality constant C0 here, where C0 is the proportionality constant and if we just substitute this expression in a C of T that in the first equation here, then we can just write this equations as N does T and finally KC N of T it is there so K N of T it is present so C is just replaced by C0-α N of T.

(Refer Slide Time: 16:32)



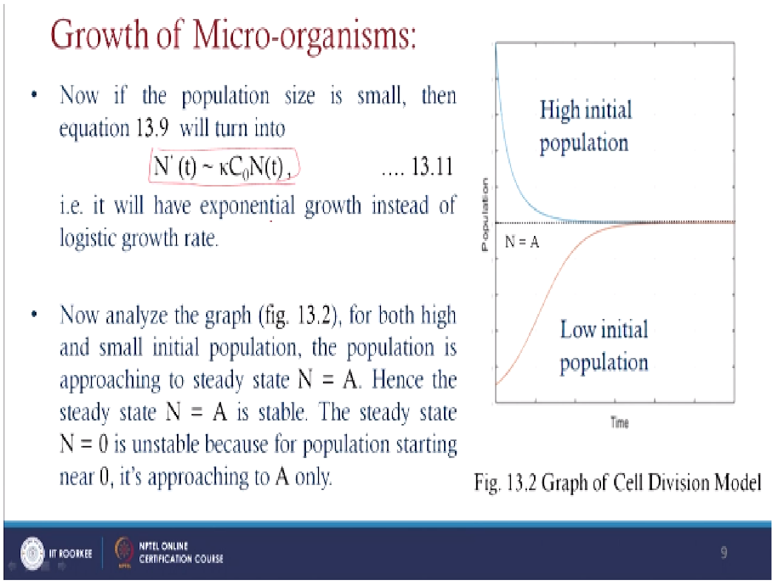
And if we will proceed for then we can just find that the final solutions of this equation is in the form of N of T equal to AN0 by NO + A- N0 E to the power T ,so last lecture also we have obtain this same equations and the solutions we have just get it over there in a differential equations form where we have done this operations of variable and we have just like to have do some manipulations to get this solution there where we have assumed N0 as n of 0 here that is the population size at T is equal to 0 that is N0 and the constant like A is define as in the form of C0 Y α and γ it has been kept as a C0 if you just follow the previous lecture we have obtain this solutions for this differential equations.

So that is why we need not have to like do it again if we want to go for this steady state or the critical points we want to find for this differential equations here then the steady study is define as a state where the rate of population growth is 0 especially if you just would here D and by DT is equal to 0 then we will have study state there this means that we can find there is a, a chance of populations with respective to time it is 0 there then we will have like steady state with respective time there is no variation we have to find and if you just consider this one if we will just to find like our equation 13.9 here that is N does T it is just to written or N prime of T it is equal to 0 here so then we can just to find that KN of T either it is equal to 0 or C0 –α N of T is equal to 0.

So if K N of t is equal to 0 this means that K is not equal to 0 since it is proportionality constant so obviously we can just get N is equal to 0 there and for the other side if you just say that we can just to get as C0 –α N of T this equals to 0 obviously N of T can be written as C0 by α there so this is the parameter we have just to define.

So the steady state is assumed either when we can consider N is equal to 0 or we can just consider N is equal to 0 by α as K there

(Refer Slide Time: 19:03)

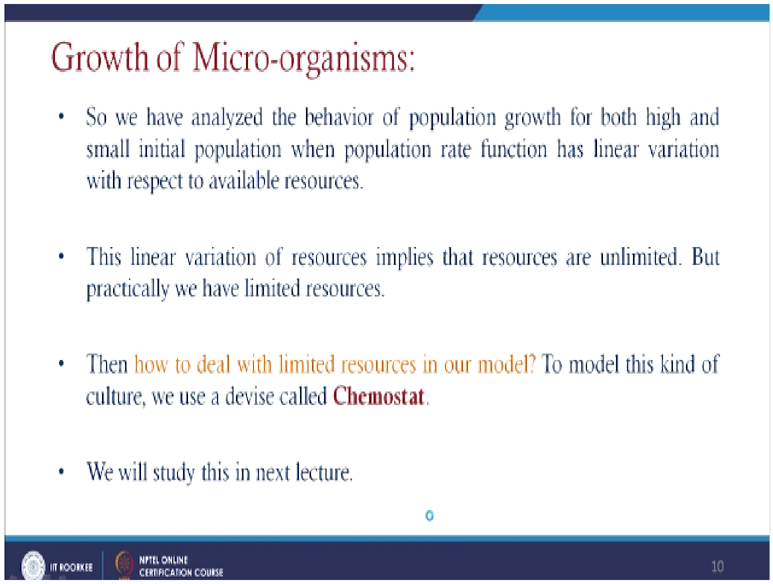


So if you just go for like, for a growth size is suppose small here very small suppose we can just consider then this square in a higher power thorns can be neglected and further if you just follow like our population level here that is define as N prime of T here, so which is written as like KN of T into 0 –α N of T here, so first team it will just keep you like a KN of TC0 and second team it will just give you like K α into N square T since N is a small here then N square can be negated so we will have this faster that is in the form K of N of TC0.

So that is why we have written here N does T is approximated as like K C0 N of T here that is it will have a extensional growth in state of logistic growth rate directly it will just find the solution it will just provide your extensional solution there N of T is equal to some constant into E to the power like C0, KC0 we can just get it over there KC0 into T we will just get and now if you just analysis this like populations level at a critical point or like steady state if you just to try to find it out, so from the graphical if you just see here for both high and small initial populations we are just find the populations is approaching to a steady state that is N is equal to A here and the steady state N is equal to 0 is unstable since we will start with certain population there and we will, we can find at that this population level will be 0 and we can start the experiment or just we can do , we can expect some like some population level if we do not have any initial any population sizes so that can be accepted so that is why we can just consider this steady state N is equal to 0 is unstable because we need certain populations at the beginning to get the growth of the population size.

So it is just a, if we will have certain populations level then we can just find the distances population is approaching to A whenever this time proceeds at a critical states.

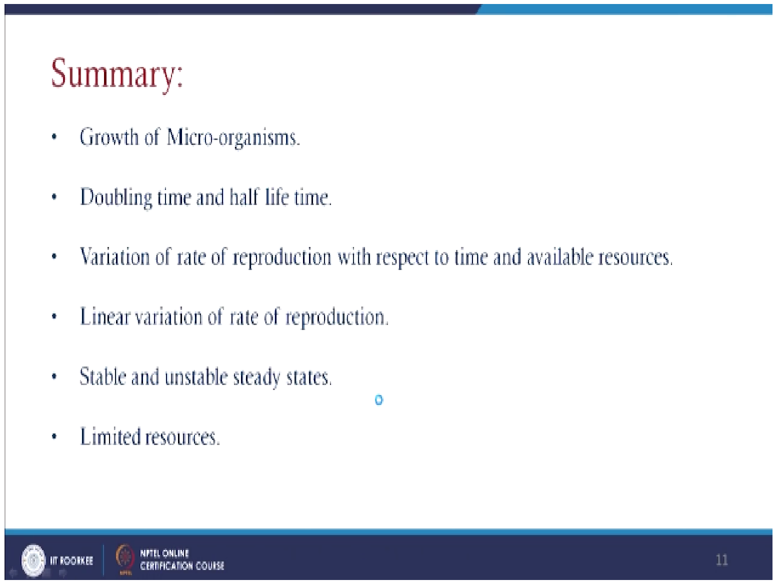
(Refer Slide Time: 21:26)



So if we just consider this growth of micro organism show we have analysis the behavior of population growth, for both high and small initial population when population rate function has linear variations with respective available resources this means that if we just consider this behavior of this population growth at a like high population size or like a small population size so this population size variations it is also showing a linear variations with respective to this available recourses.

Since whenever this recourses or like limited then either if the high population it is there they can eat and after certain time whatever this resources it is will be like feedback or like feuded the error it can be available so certain population will be dilute out and it will just maintain a proper ration there and if it is a small population level suddenly if they have huge amount of foods they can just grown it off and after certain time they will struggle for the food and they will maintain a proper linear variations and this linear variation of resources implies that resources are unlimited. But practically we have limited resources, so the resources either we can just provide or whatever it is available so it is always limited, so that is why the population level always they will just try to maintain their balance for the decline process then if the limited resources is available, how our model will like treat there, so model this kind of culture we use to device or we use a device called Chemostat since that if the culture medium it is there so we are just supplying some food how this proper utilization of food is occurring inside this system and how the output is just coming, the total process it is called Chemostat.

(Refer Slide time: 23:31)

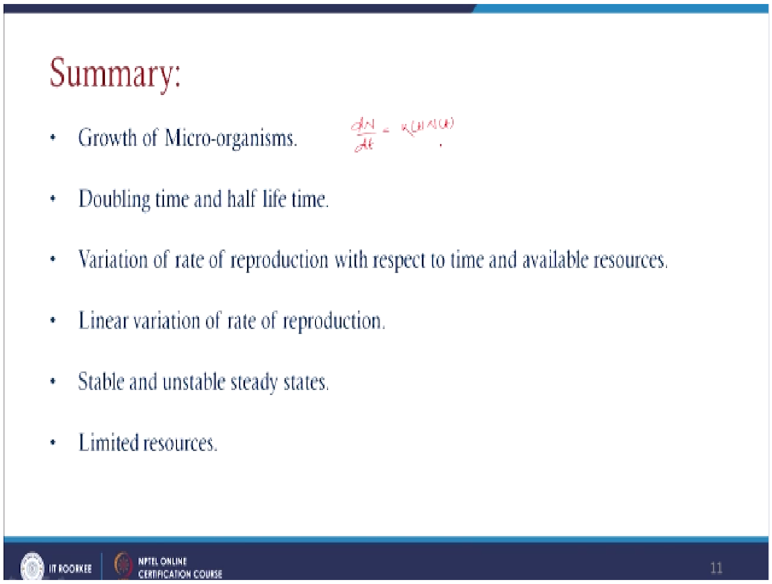


So this things we will just discuss in the next lecture and in the present lecture, we have discussed like first the growth of a micro organisms that is, we have just consider a simple differential equations that is the population size it is just to getting change with respective to time which is define in the form of DN/ DT and which is directly proposal to the a, like rate of change of this population size and with respective like the total population available at the beginning and based on that we have just find some solutions, so that solutions just reflect as like sometimes this population level is growing are like increasing in expansion manner and if is decline or if K is just taking a negative value there then this decreed it is just a occurring in a extensional decreed manner and if you want to calculate also the doubling time this means that the population size it will doubled after certain time. so that time level we can calculate also that we have computed in our calculations and sometimes if the decaying is also occurring and after like certain time state we can just fined this half of this population level that time level we have just completed also .

Then in the third phase we have a discussed the variations of rate of production reproduction with respective to time and available resources so C as a function of like K we have just consider or K is a function of C then C can be like dependent on this population size increment and decrement level that we have discussed.

Then in the fourth stage we have just consider this linear variations since the available resources they are limited and based on that how this population level is increasing or decreasing that we have tested and in the last to second one, we have just consider this stable and unstable steady state so it is very important since we have to consider like critical points to find this solution growth rate or like decaying rate or how it is just approaching to a stable solution process since stable means like whatever this resources are like barriers we will just provide them but after certain restrictions also they will just maintain this same level so that is why it is very important to discuss this stability and the on stability is used for this like growth of populations.

(Refer Slide Time: 26:13)



So last is we have just discussed that how this population level will like varied with respective to the resources available, so if the resources are like limited so even if this restriction it is there but the population it will just goes it of there and if the resources can be like plenty available sometimes we can just find that the population level will also maintained the same level when we just to go for like stability condition for the steady state, thank you for the listen this lecture.

For Further Details Contact
Coordinator Educational Technology Cell
Indian Institute of Technology Roorkee
Roorkee – 247667
E Mail:e0cel@i0cermet.in.etcelLitrke@gmail.com
Website: www.nptel.itm.ac.in
Acknowledgement
Prof. Ajit Kumar Chatuwedi
Director IIT Roorkee
NPTEL Coordinator
Prof. B. K. Gandhi
Subject Expert
Dr. Ameeya Kumar Nayak
Department of Mathematics
IIT Roorkee
Produced By
Sarath K.V
Graphics
Binoy V. P
Production Team
Mohanraj. S
Pankaj Saini
Web Team
Dr. Nibedita Bisoyi
Neetesh Kumar

Jitender Kumar
Vivek kumar
Djaramaveer Singh
Gaurav Kumar
An Educational Technology Cell
IIT Roorkee Production
IIT Roorkee Production