Matrix Analysis with Applications
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Lecture – 08
Basis and Dimension

Hello, friends. Welcome to lecture series on matrix analysis with applications. So, our today talk is basis and dimension. So, we have already discussed about vector spaces subspace of a space V. Now, what do you mean by basis of vector space and how can we calculate its dimension? So, let us discuss these things in this lecture.

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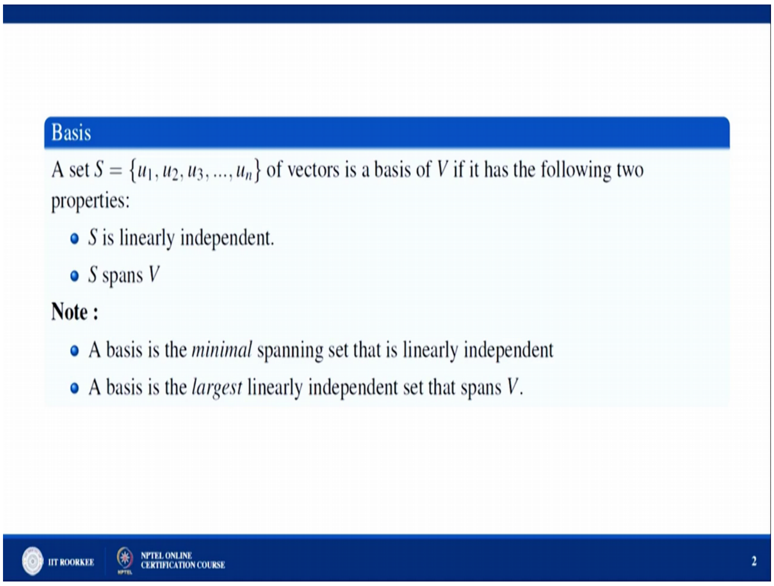

So, first of all what is a basis? Now, a set S which is given by u 1, u 2, u 3 up to u n of vectors is a basis of V if it has the following two properties; number – 1, S is linearly independent. So, the first property is you consider a as a subset of V, ok. Now, this subset of V is will be a basis of this V if number 1, this is linearly independent and number – 2, the span of S generates V a span of S generates V means the span of S is equal to V that means, by the linear combination of the elements of S we are getting this vector space V.

So, we will discuss few examples based on this that things will be clear. Now, a basis is a minimal spanning set that is linearly independent number 1, ok. It is a minimum minimal spanning tree a spanning set that is linearly independent and number 2, a basis is the largest linearly independent set that spans V, because if you are having a more one more element in a this basis then it will become linearly independent. So, it is a largest linearly independent set that spans V that generates V, ok.

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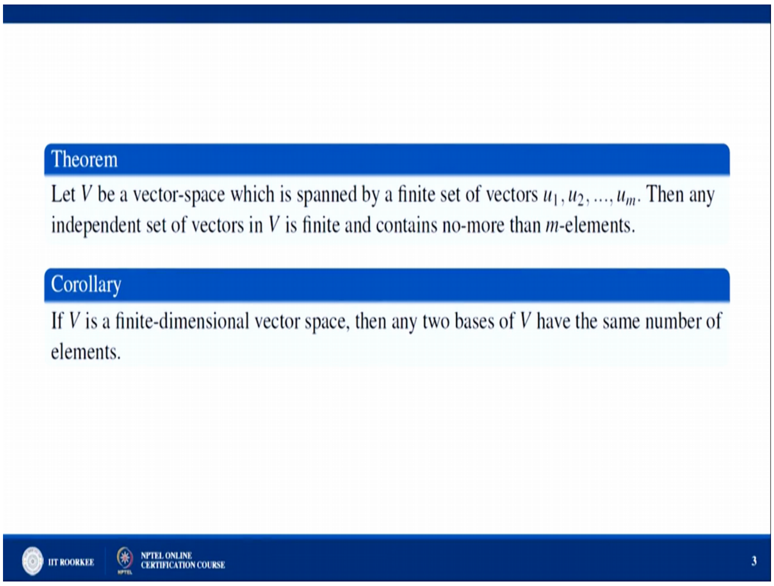

Now, now the first theorem is let V be a vector space which is spanned by a finite set of vectors u 1, u 2, up to u n, ok. Then any independent set of vectors in V is finite and contains no more than m-elements.

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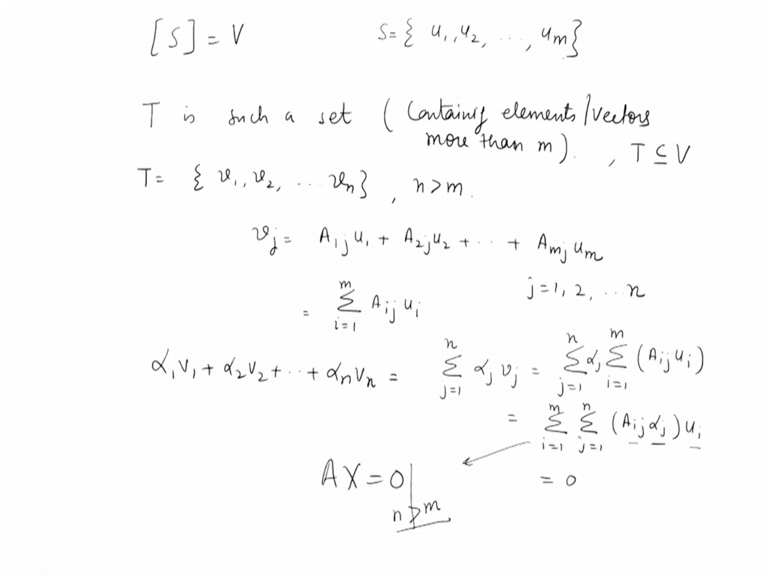

Now, what I want to say that suppose S is equal to suppose you have a set of elements u 1, u 2 up to u m and this spans V, then any independent set of vectors in V is finite and contains no more than m-elements that means, if you would take any independent set of vectors in V, the number of element in that set it will not exceed m.

So, that means, if you take any set S which contain more than m elements is always linearly independent is always linearly dependent. Indirectly if you want to say that if a independent set is finite and contains no more than m-element that means, if we take any set any subset S of V contains more than m more than m-elements that will be always linearly dependent ok. So, let us suppose S is that set or say T is that set ok, T is such a set such a set meaning containing elements or vectors more than m, ok.

So, let us suppose T is equals to say you can take v 1, v 2 up to v n, where n is greater than m. Now, now what to show we have to show that this T is linearly dependent, ok. So, first of all since span of this set since span of this set generates V and T is a subset of V of course, T is a subset of V so, that means every element of T will be will be in some linear combination of elements of S ok. So, we can write this v j as A 1 j u 1 plus A 2 j u 2 and so on up to A m j u m, where j is varying from 1 to n; that means, which is equals to summation i varying from 1 to it is it is it can be written as i varying from 1 to m A i j u j A i j u y, and j is varying from 1 to n, ok.

Now, in order to show that this set is linearly independent take some linear combination of this, put it equal to 0 and try to show that not all scalars are 0, this means this set is LT ok. So, so let us take set us let us take some linear combination of these vectors,. Let us say take some linear combination of these elements. Now, it is it is you see that it is a summation alpha it is the summation alpha j v j ,where j is varying from 1 to n. So, this is summation j varying from 1 to n and this v j's are nothing, but given by this expression here. So, this can be written as summation i from 1 to m this is alpha j is as it is and this is A i j u i.

Now, this can be re arranged and can be written as summation i varying from 1 to m, summation j varying from 1 to n this is A i j alpha j into u i. Now, if it if this is equal to 0, means this is equal to 0. Now, if this is equal to 0 means this is equal to 0 means this is a system of linear homogeneous equations, you have to find alpha alpha j's. Alpha j's are unknowns u j u i's are known, A i j's are known, alpha j alpha j's are unknown. So, it is it is this is this is something like AX equal to 0 type, where X is alpha j's, ok, it is to find out all other things are known.

Now, for this system we are having how many alpha j j's are varying from 1 to n. So, number of unknowns are n and number of equations are m and n is more than m. So, this system will be having more than one solution, I mean infinitely many solutions and if it is having infinitely many solution this means there will be some non-zero solution also. So, we have shown that this set of vectors v 1, v 2, v 3 up to v n of T are linearly dependent. Hence we have shown that if we are if we are having a set S which is spans V which spans V then the linear then the linearly independent set of V is always finite and will not contain more than more than m elements, ah.

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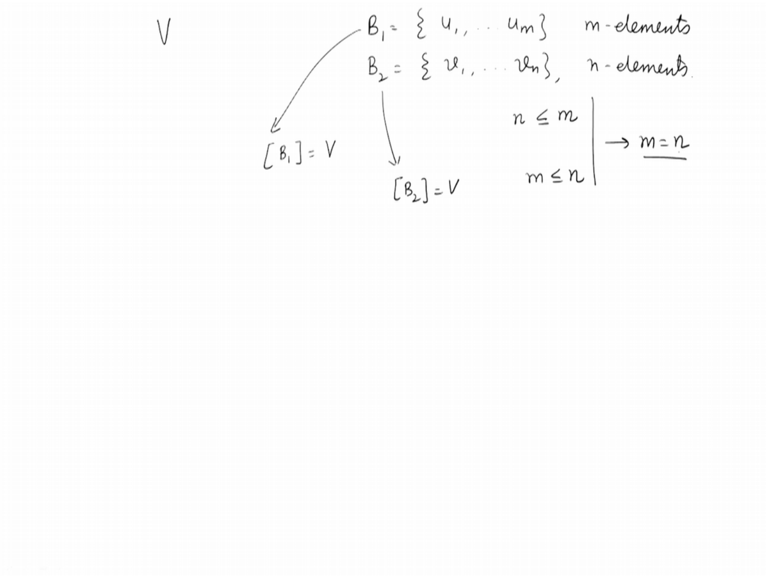

Now, based on this we have another result if V is a finite dimensional vector space now, what do you mean by dimension? Dimension is simply number of elements in the vector space. I mean number of elements in a basis of vector space ok, then any two basis of V have same number of elements you see suppose we have vector space V, and say B 1 is one basis which consists of say m number of elements; So, and another basis B 2 which con contains say n number of elements.

So, we have to show that m is equal to n, ok. Now, you see here since span since this is a basis this means span of B 1 will be V because we know that basis's are set which are linearly independent and spans is equal to V, span of that set is equal to V, ok. Now, span of this set is equal to V and this is this B 2 is linearly independent. So, by the previous result we can say that n will be less than equal to m. Now, similarly B 2 is also the basis

B 2 is a basis. Now, that means, span of B 2 will be equal to V, now the span of B 2 equal to V and B 1 is linearly independent; that means, m will be the less than equal to n by the previous result. So, from these two we can easily say that m equal to n.

So, what I want to say that if V has a finite dimension, then it may be having infinitely many basis, but number of element in each basis are same, ok, number of elements in the basis are same and that is called its dimension.

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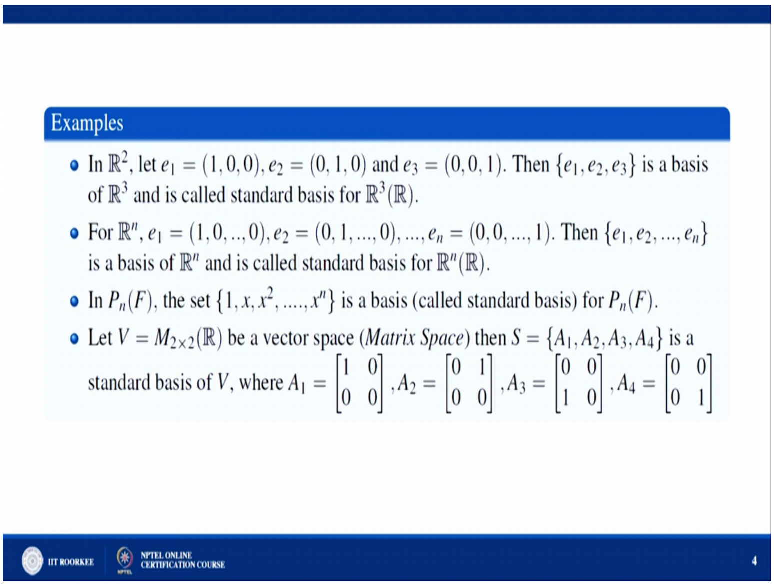

Now, say you consider R 3 in R 3 if you take e 1 as 1, 0, 0 e 2 as 0, 1, 0 and e 3 as a 0, 0, 1 then this is a basis of R 3 and is also called standard basis of R 3.

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$V = \mathbb{R}^3(\mathbb{R})$, $S = \{(1,0,0), (0,1,0), (0,0,1)\}$

$\alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1) = (0,0,0)$

$[S] = V$

$\alpha_1 = \alpha_2 = \alpha_3 = 0 \Rightarrow LI$

$[S] \subseteq V$

$V \subseteq [S]$

let $(x,y,z) \in V$

$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$\in [S]$

$\Rightarrow V \subseteq [S] \Rightarrow [S] = V$.

Now, you can easily see you see if you take if you take vector space as R 3 over the real field of course, and your taking a set which is 1, 0, 0, 0, 1, 0 and 0, 0, 1. Now, first of all this set is linearly independent, it is very easy to show. You take linear combination of this set linear combination elements of this set put it equal to 0 and this implies alpha 1 equal to alpha 2 equal to alpha 3 equal to 0 that means, set is LI.

Now, secondly, we have to show that a span of S equal to V ok. Now, the span of S is equal to V; that means, any element of V can be expressed as linear combination of elements of S you see. We have to show that span of S is equal to V. It is very clear that span of S is a subset of V, it is obvious because V is a vector space a span means linear combination of elements of S ok, because it is a vector space. So, it must be close to with respect to scalar multiplication. So, that means, this span will be automatically be a sub set of V. Now, we have to show that V is a sub set of a span of S to show that span of S is equal to V.

So, take an elements say x, y, z in V and we have to show that this x, y, z can be expressed as some linear combination of elements of S, then only we can say that it is in a span of S. So, this x, y, z can clearly expressed as x times 1, 0, 0 plus y times 0, 1, 0 plus z times 0, 0, 1 these are alpha, beta, gamma the scalars and that means, belongs to the span of S and that means, V is a sub set of span of S and hence implies a span of S is equals to V. So, we can say that this S is a is a basis of this vector space R 3 and this is also called standard basis of R 3, ok.

The second example is similarly, we can go for R n, if you take R n then you can take e 1 as 1, 0, 1, 0, 0 up to I mean n-th term. Similarly, e 2 and e n and e 1, e 2 up to e n it is a basis of R n similarly and is also called standard basis for R n.

Similarly, if you take P n which is the consisting of all the polynomials of degree less than equal to n over the field F and you take a set 1, x, x square, up to x skip x is to power n is a basis for P n and is also called standard basis for P n, ok. This is also very simple to show.

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$$S = \{1, x, x^2, \ldots, x^n\}. \quad V = P_n(F)$$

$$\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n = 0 + 0x + \cdots + 0x^n$$

$$\Rightarrow \alpha_i = 0 \ \forall i \Rightarrow LI$$

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n \in V$$

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n =$$

$$= a_0 \cdot$$

If you take you see if you take this set which is 1, x, x square and so on up to x raised to power n, you are taking P n over F, but space. Now, first in order to show that it is linearly independent you take a linear combination of linear combination of this set put it equal to 0. Now, this 0 can be written as 0 plus 0 x and so on up to 0 x raised to power n. Now, when it will be hold for every x this equal to this will hold for every x only when all alpha i is 0. So, this implies LI.

Now, in order to show that a span of this is equal to V, you take an arbitrary element in V ok, you take any polynomial say an say you take a naught x raised to power n plus a 1 x x raised to power n minus 1 and so on a naught in V and this element can be written as this is a n, plus a n this element can be written as a naught times first plus I mean this element can be written as you can see a naught a naught time this vector a n times the previous vector and so on a n time the first vector. So, we have shown that this element can be written as linear combination of elements of S and hence span of S generates V, that means, it is a basis of V.

Now, similarly if you consider a matrix of order all matrix of order 2 cross 2 over a real field and you take a sub set A 1, A 2, A 3, A 4, where A 1 is this, A 2 is this, A 3 is this and A 4 is this that is a standard basis of this vector space and this is this will this will be a basis because first of all it is linearly independent, second you take any element any matrix of order 2 cross 2 that can be expressed as linear combination of A 1, A 2, A 3 and A 4 ah.

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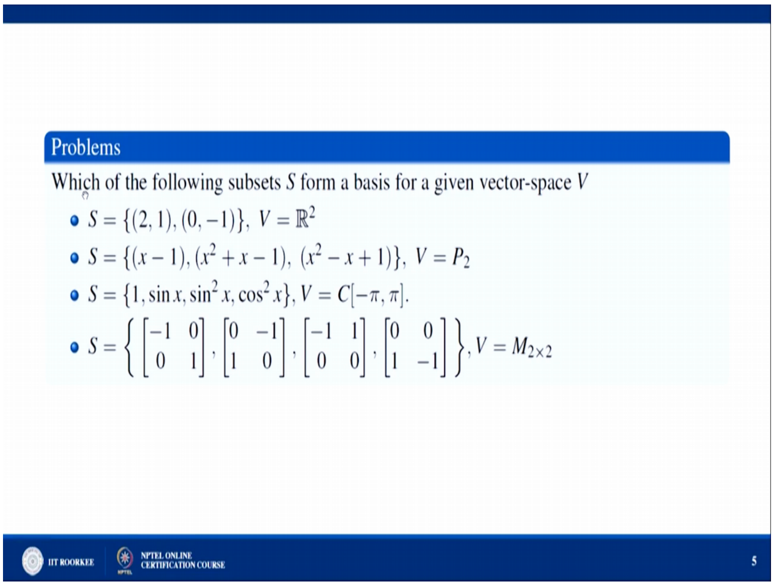

Now, in this problems which of the following subset S form basis for a given vector space V? Say you take first example, ok.

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$$S = \{(2,1), (0,-1)\}, \quad V = \mathbb{R}^2(\mathbb{R}).$$

(i) $\alpha_1(2,1) + \alpha_2(0,-1) = (0,0) \Rightarrow (2\alpha_1, \alpha_1 - \alpha_2) = (0,0)$

$\Rightarrow \alpha_1 = \alpha_2 = 0 \Rightarrow LI$

(ii) let $(x,y) \in V$

$(x, y) = \alpha(2,1) + \beta(0,-1)$

$\Rightarrow x = 2\alpha, \quad y = \alpha - \beta$

$\Rightarrow \alpha = \frac{x}{2}, \quad y = \frac{x}{2} - \beta \Rightarrow \beta = \frac{x}{2} - y$

$\Rightarrow (x,y) = \left(\frac{x}{2}\right)(2,1) + \left(\frac{x}{2} - y\right)(0,-1)$

$\Rightarrow [S] = V$

The first example is you are taking S as 2 comma 1, 0 comma minus 1 and here V is R 2 over the real field. So, we have to see whether it will constitute a whether it will constitute a basis for this vector space or not. So, first of all we have to see whether these vectors are linearly independent or not, number one. So, take linear combination of these vectors, put it equal to 0, 0. So, this implies to alpha 1 and alpha 1 minus alpha 2 equal to 0, 0. So, this implies alpha 1 equal to alpha 2 equal to 0. So, this means LI. So, first property hold that means, this set is LI.

The second property is the second property is the span of this must be equal to V. So, that means, you take any x, y in V. So, they are not exist some alpha beta in field such that this x, y can be expressed as linear combination of linear combination of these vectors. So, let us try to find those. So, let x, y is equals to alpha times 2, 1 plus beta times 0, minus 1. So, this implies x is equals to 2 alpha and y is equals to alpha minus beta. So, this implies x alpha is x by 2 and y is x by 2 minus beta or beta is x by 2 minus y. So, here we can write as this x, y as x by 2 times 2, 1 plus x by 2 minus y times 0 comma minus 1.

So, for every x, y in V we have find alpha and beta such that every x, y can be expressed as a linear combination of these two vectors. You change x and y, you change x, y, x and y you will get you will get this multipliers or these are scalars correspondingly for which this x, y can be expressed as a combination of these two vectors. So, what we have shown? We have shown that we have shown that span of S is equal to V, I mean we have shown that any x y can be expressed as a combination of elements of S and hence we can say that this is a basis of V.

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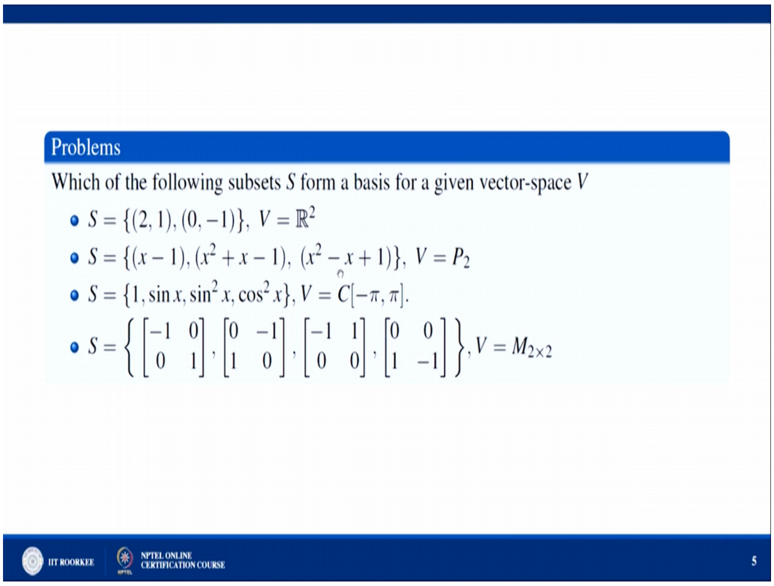

The second example is we have taken these vectors. These vectors I mean these subset of V and we have to see whether it will constitute a sub subspace I mean if I constitute a basis for this P 2 or not. So, how will show again?

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$S = \{(x-1), (x^2+x-1), (x^2-x+1)\}$ $\quad V = P_2(\mathbb{R})$

(i) $\alpha(x-1) + \beta(x^2+x-1) + \gamma(x^2-x+1) = 0$

$\Rightarrow \beta + \gamma = 0, \quad \alpha + \beta - \gamma = 0, \quad -\alpha - \beta + \gamma = 0$

$\Rightarrow \beta = -\gamma \quad \alpha = 2\gamma \quad -2\gamma + \gamma + \gamma = 0$

$\Rightarrow$ satisfied.

$\gamma = 1, \ \alpha = 2, \ \beta = -1$

$2(x-1) + (-1)(x^2+x-1) + 1(x^2-x+1)$

$= 0$

$\Rightarrow$ LD

So, let us try to prove this, if it is. So, you take as first factor x, minus 1, second is x square plus x minus 1 the third is x square minus x plus 1, ok. Here this is P 2 over R. Now, first of all if it is a basis of this P 2, then must be linearly independent. So, first we will see whether it is linearly independent or not.

So, take linear combination of these vectors, x plus 1 equal to 0. So, now, you collect the like terms. You see the power of x square here is no term of x square, here it is beta here it is gamma. So, beta plus gamma must be 0 all the all the coefficients must be 0 because it is equal to 0 and the coefficient x is alpha plus beta minus gamma should be 0 and the constant term is minus alpha minus beta plus gamma is equal to 0. Now, from here you can you get beta equal to minus gamma. When you substitute beta equal to minus gamma here it is alpha is equal to 2 gamma; Now, when you substitute alpha as minus alpha as 2 gamma here and beta as gamma so, this is automatically satisfied, ok.

So, this means this means it has many solutions. So, there is a non zero solution also. Say a few if you put gamma equal to 1 so, alpha equal to 2 and beta equal to minus 1 and if you take if you take 2 x minus 1 plus minus 1 times x square plus x minus 1 plus 1 times x square minus x plus 1. So, what it is it is minus x square plus x square cancel out, minus x plus x minus x minus x minus 2 x plus 2 x cancel out and this plus 1 plus 1, 2, minus 2 cancel, 0. So, this means what? this means LD because there are non zero scalars which is whose linear combination equal to 0 and that means, it is not a basis of V because for basis it must be linearly independent, ok.

Similarly, we can show for the other two also that whether it will be subspaces I mean basis or not. For example, if you see the third example, now this one can be written as sine square x plus cos square x that is this one is the linear combination of these two elements. So, this set is LD. So, this will not be a basis of this vector space, ok.

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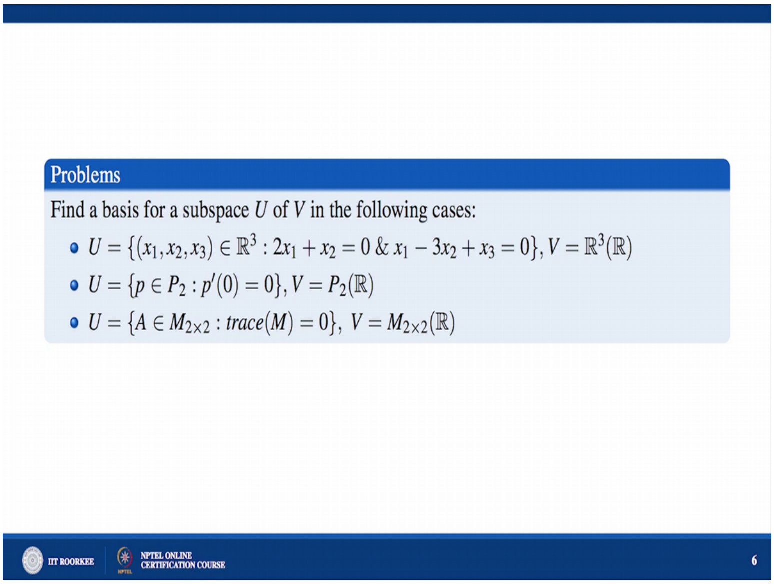

Now, for a basis for a subspace find the basis for a subspace U of V in the following cases, ok. Now, in these problems we have to find the basis. Now, the first example first is you take x 1, x 2, x 3 in R 3 such that 2 x 1 plus x 2 equal to 0 and x 1 minus 3 x 2 plus x 3 equal to 0 in R 3. So, how we can find this?

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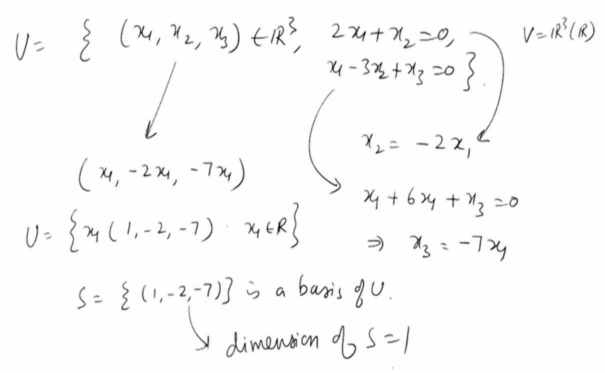

So, here we are taking all x 1, x 2, x 3 in R 3 such that 2 x 1 plus x 2 equal to 0 and x 1 minus 3 x 2 plus x 3 equal to 0. So, this will this is a subspace of this vector space V this is very clear. We can show it because it satisfy closure property with respect to a addition and scalar multiplication. Now, we have to find out if it is a subspace so, it must be having some basis. So, how can you find out the basis of this subspace of this V? So, how can you find it let us see. Now, here x 2 is equals to minus 2 x 1 from this equation.

Now, from this equation if you take x 1 and substitute x 2 as minus 2 x 1 here it is plus 6 x 1 plus x 3 equal to 0. So, this implies x 3 is equal to minus 7 x 1. So, this set is x 1 is x 1, x 2 is minus 2 x 1 and x 3 is minus 7 x 1. So, this is basically x 1 times 1 minus 2 minus 7 we are, where x 1 belongs to R this is basically U, ok.

So, pick out a linearly independent set which generates this set, that will be the basis. So, we can say that basis is simply 1, minus 2, minus 7 is a basis of U. In fact, in fact, any alpha times any x 1 times this vector is a basis you see this is this is LI and if you take alpha time this so, this generate the entire U. So, hence this is a basis of this vector this subspace U. So, what the dimension of this sub space dimension of this sub space is 1, and this an also this can also be computed like this you see here it is in R 3 and how many equations how many independent equations we are having? Two so, 3 minus 2 so, dimension will be 1, ok.

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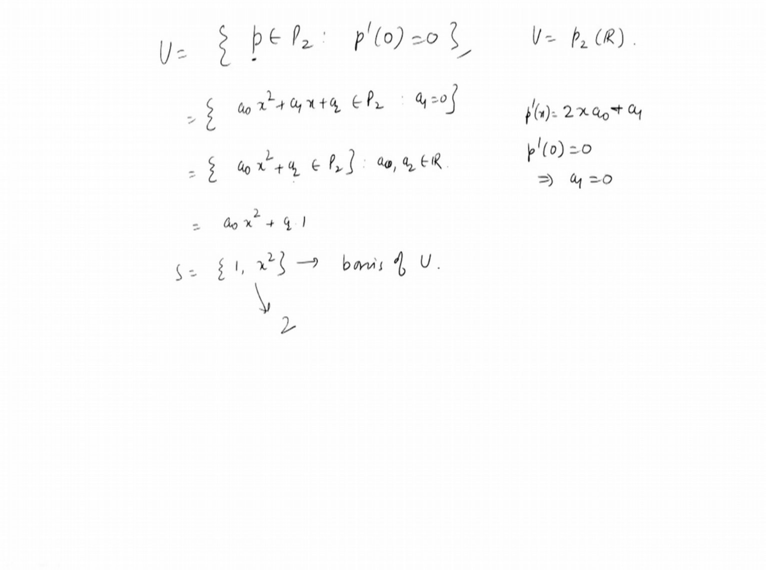

The next example is the next example is you see you are taking all the polynomials p in P 2 such that p dash 0 equal to 0. It is clearly a subspace and now what to find? Its basis p 2 over real field ok; So, let us write this p as a naught x square plus a 1 x plus a 2 belongs to p 2 such that derivative is this derivate as 0 equal to 0; that means, what is the derivative of this? It is 2 x a naught plus a 1, this is a derivative of this and p dash at 0 is 0. So, this implies a 1 equal to 0.

So, a 1 equal to 0 means what? a naught x square plus a 2 where a naught a 2 a naught a 2 are real numbers. So, basically U is this set, all a naught x square plus a 2 in P 2 such that a naught a 2 are in real numbers. So, how to find out the basis of this, is very easy. You see you can write this as a naught times x square plus a 2 times 1, that is a 1 comma x square will be the basis of this.

Because you see 1 and x square are linearly independent. 1 cannot be expressed as alpha times a x square or x square cannot be expressed as alpha times 1, number 1 and number 2, if you take a linear combination of these two vectors 1 and x square it generates a naught x square plus a 2 all such vectors. So, it is a basis of this subspace and the dimension of this is 2, ok. This is not the only basis you can you can find out infinitely many basis, but the set must be ally and it should generate entire sub space, ok.

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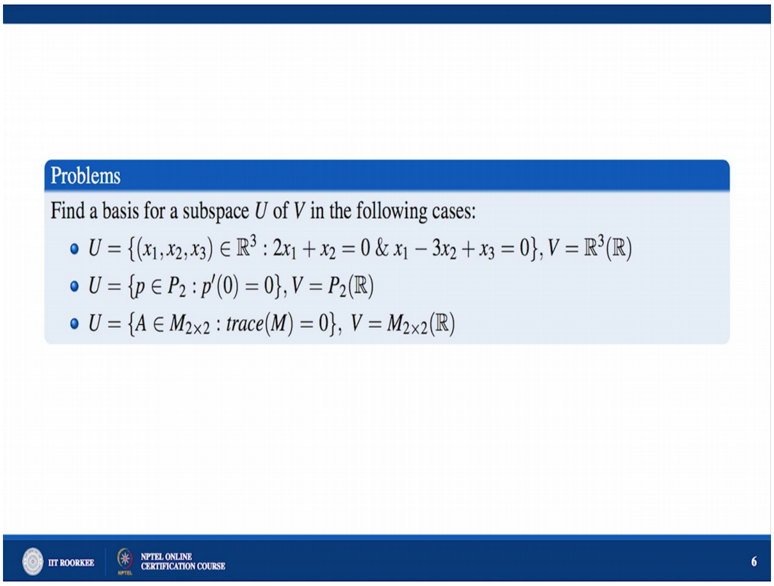

The last example here is, consider all matrices of order 2 cross 2 such that trace of M is 0.

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$$V = M_{2\times 2}(\mathbb{R}), \quad U = \{ A \in M_{2\times 2} : \text{trace}(A) = 0 \}$$

$$= \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M_{2\times 2} : x + w = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \in M_{2\times 2} \right\}, \quad x, y, z \in \mathbb{R}$$

$$= x \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

↳ basis of this $\underline{U}$.

You see here vector space we are taking as m of order 2 cross 2 over R and sub space we are talking as all matrix of order 2 cross 2 such that trace of M is trace of A is 0. So, basically we are taking all x, y z, w in M of order 2 cross 2 such that x plus w equal to 0, trace is what some more elements. So, this is this is basically all x, y, z you can write it minus x here because x plus w is 0 in M of order 2 cross 2, where x, y, z in R. So, this can be written as x times 1, 0, 0, minus 1, y times 0, 1, 0, 0 and z times 0 0, 0, 1, 0.

So, you can write this set as set as 1, 0, 0, minus 1, 0, 1, 0, 0 and 0, 0, 1, 0 this will be the basis of this U, because first is this is linearly independent and this the linear combination of this generates the entire U. So, this will be the basis of this subspace U. So, hence we have seen that how to find out a subspace of a how to find out the basis of a sub space and how we can see that that a given set is a basis of vector space or not.

So, thank you.