Matrix Analysis with Applications
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Lecture – 07
Subspaces

Hello friends, welcome to lecture series on Matrix Analysis with Applications. So, in this lecture, we will talk about Subspaces. In the last lecture, we have seen that what vector spaces are, and how can we see that a given set is a vector-space or not, over a over a given real field ok, real or complex field.

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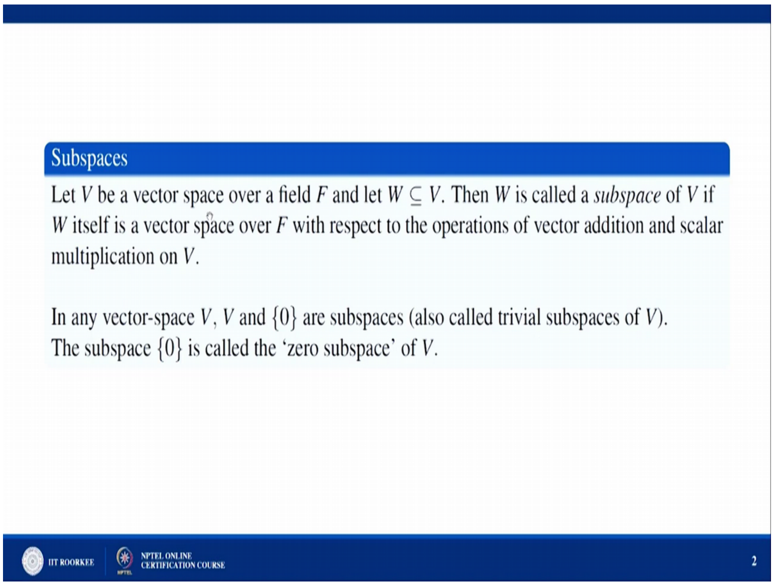

So, what subspaces are. Let us see. Let V be a vector-space over the field F, and let W be a subset of V ok; this V is a vector-space over the field F, and this W is a subset of this V. Then this V this W is called subspace of V, if W itself is a vector-space over the field F with respect to the same operations of vector addition and scalar multiplication on V. Whatever scalar, whatever vector addition and scalar multiplication is in V over F, the same vector the same vector addition scalar multiplication we apply on W, which is a subset of V. If it is itself vector-space of the same field F, then we say that is a subspace of V.

So, in any vector-space V, two subspaces are trivial, the first one is V itself ok, because V is a subset of V, and it is a vector-space. And this 0 subspace, which is a trivial subspace of trivial subspaces of vector spaces V. The subspace this 0, this 0 is simply additive identity you see, and is also called ‘zero subspace’ of V. So, if a set is given to you ok, if a subset of a vector-space V is given to you, and if this W itself is a vector-space over the same field F, with respect to the same set of vector additions and scalar multiplication that we say that W is a subspace over vector-space V.

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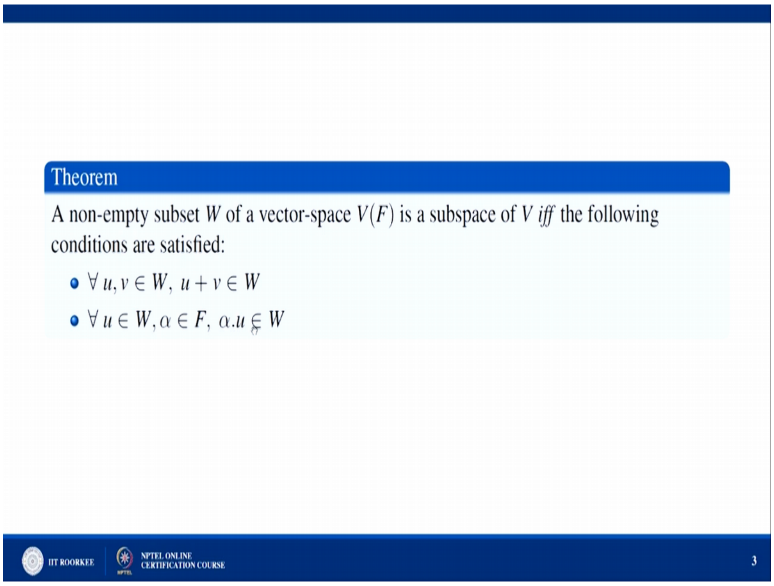

Now, how can we show that a given a subset W of V is a subspace. So, instead of showing all the properties of a vector-space V, if we show these two properties that is closure respect to addition vector addition, and closure with respect to a scalar multiplication, then it is sufficient to show that that W is a subspace of this vector-space V.

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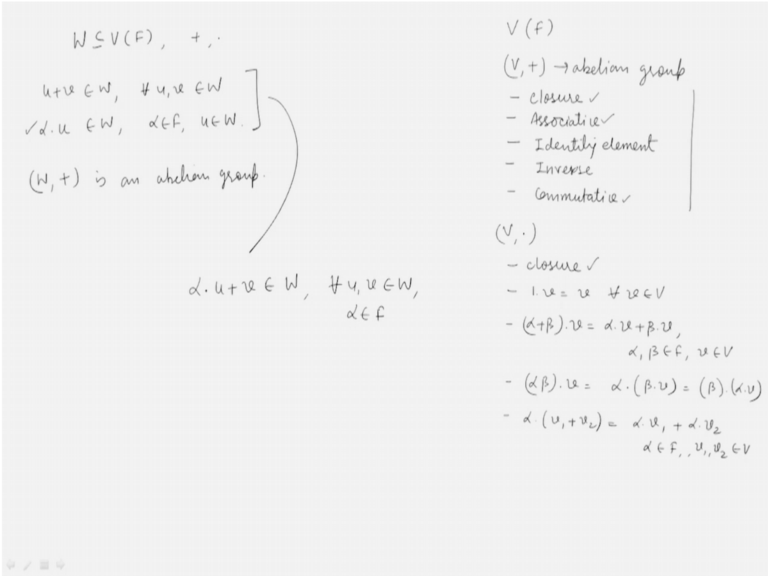

Now, how can we say this, you see. You see that we say that V is a vector-space over the field F if V with respect to plus is an Abelian group. Abelian group means, first is closure, next is associative ok, then identity element, then inverse element ok, and then commutative. If these property holds this with respect to addition vector addition, then we say that V with respect to addition is the Abelian group.

And with respect to dot, the first is closure, the second property is 1 dot V is equal to V for all V in V ok, third property is alpha plus beta dot with V is equals to alpha dot V plus beta dot V, where alpha beta are in field, and V is any vector in V. Next is alpha beta dot V is equals to alpha alpha dot beta dot V ok, and same as beta; beta dot alpha dot V. And next is alpha dot V 1 plus V 2 is must be equals to alpha dot V 1 plus alpha dot V 2 for all alpha in field, and for all V 1 V 2 in vector pace V.

So, if this property hold with respect to addition and scalar multiplication, then we say that V is a vector-space over the field F. Now, now we are taking W as a sub set of V, over the same field F with respect to same vector addition in scalar multiplication. Now, now you see you see if you say these closure, closure is a automatically, because of this property, the first property itself says that it is close with respect to vector addition ok.

Now, the second property is associative. Since, associative holds for every element u, v, w in V, so it will hold for a subset of V also. Now, we have to show the existence of identity element, and inverse with respect to addition. Now, we know that u plus v belongs to w, for all u v belongs to w. And alpha dot u belongs to w, for all alpha belongs to field, and u belongs to w.

Now, if you take alpha equal to 0, say for example, because it hold for every alpha, so it will hold for alpha equal to 0 also. And 0 dot u is nothing but I did not I did not expect to addition, so that means, 0 belongs to w, so, we have shown that I existence of identity element with respect to addition. Now, for inverse for inverse, you simply you simply replace you simply replace alpha by minus 1. So, minus 1 dot u is minus u, we have already shown, so that means, inverse element with respect to addition also exist.

Now, commutativity property holds, because this property holds for every u v in v, so it will hold for a subset also, because subset elements of w are nothing but the elements of v. And this v holds commutative property for every u v in v, so it will hold for subset also. So, so we can say that w with respect to plus is an Abelian group.

Now, we have to show with respect to dot. Now, with respect to dot closure is automatically satisfied, because of the second property. Now, 1 dot v is v, because it is it is now this property is holding for every alpha ok. You see, you see this property is holding for every v in v, so it will hold for a subset also. So, we can say that this property hold. And similarly, all the remaining three properties also hold for w, because w is a subset of v.

So, hence, we can say that if we show the these two properties, then we can say that w is a subspace of v, or we can club these two properties as we can club these properties to property alpha dot u plus v should belong to w for all u v in w, and alpha in F. So, either we have two ways to show that W is a subspace of vector-space V, either we show these two properties separately, or we can show the single property, and we can show that, then we can say that W is a subspace of a vector-space V.

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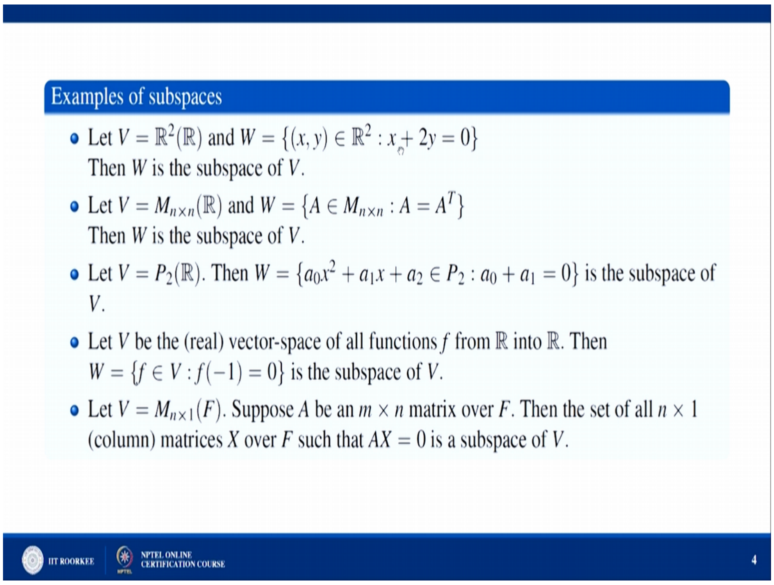

So, now let us discuss few examples based on this. Now, first example is V, we are taking as R 2, over the real field R. Now, W, which is given as all x, y naught to such that x plus 2, y is equal to 0, and we have to show that W is a subspace of this vector-space V ok.

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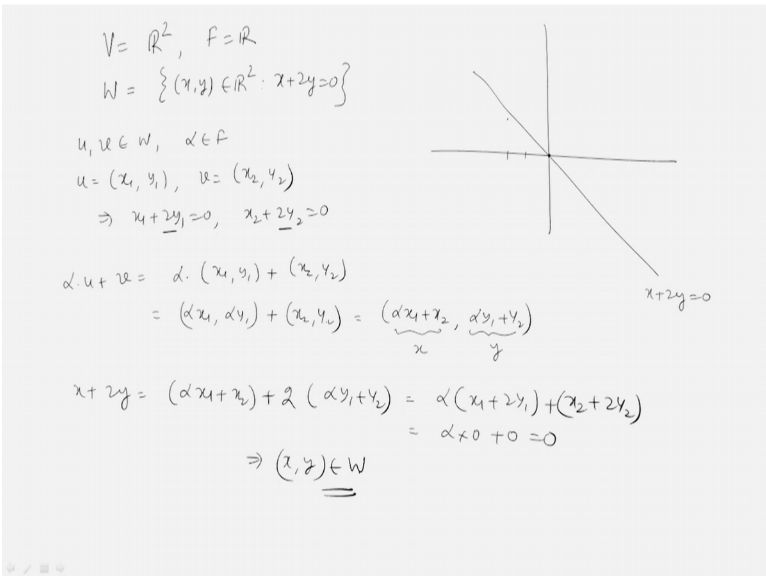

Now, what is V here? V is entire R 2 and field is R. So, this is something like this, this entire R 2 is simply a vector-space V over a real field over the real filed. Now, W is simply all x y and R 2, such that x plus 2, y is equal to 0. So, it is passing through origin, then x is 0 y is 0. And when x is minus 1, when x is minus 2 for example, then y, then y is 1 then y is 1, so it is something this line ok, this is x plus 2, y is equal to 0. So, it is of course, it is of course, a subset of this R 2. Now, we have to see that whether, it is a now it is show that it is a subspace of this V. So, you take to arbitrary element in w, and alpha in field.

So, u is something you can says, u is as x 1, y 1, and say V is x 2, y 2. So, this implies x 1 plus 2 y 1 is equal to 0, and x 2 plus 2 y 2 is equal to 0, because u v are from w, so it will satisfy these properties. Now, if it is a subspace, then we have to show that alpha u plus v must be in w. So, take alpha dot u plus v, so it is alpha dot u is x 1, y 1 plus x 2, y 2, so it is alpha x 1 alpha y 1. So, here I am taking the usual scalar multiplication ok, because I am taking R 2.

So, I have we have taken, the standard addition, vector addition, scalar multiplication ok. It is x 2, y 2, and when you add them, so it is simply alpha x 1 plus x 2 alpha y 1 plus y 2. Again by the standard vector addition now, we have to show that this is in this is in w, so that means, we have to show that x this is x and this is y. So, we have to show that x plus 2 y must be 0, then only we can say that this element is in w.

So, you can take x plus 2 y, now x plus 2 y is alpha x 1 plus x 2 plus 2 times alpha y 1 plus y 2, it is equal to alpha times x 1 plus y 1 plus x 2 plus 2 y 2. And it is 0 x 1 plus 2 y 1 is 0 from this equation, and x 2 plus 2 y 2 is 0 from this equation, so it is alpha into 0 plus 0, which is 0. And this implies x comma y belongs to w. So, we can say that this x, and this y is in w, that means, w is a subspace of v.

Now, now similarly if we want to show that this matrix, this vector-space we have taken as all M cross n matrices all the real field R, and W is a subset of this V, we have taken as all collection of all symmetric matrices, where a equal to a transpose.

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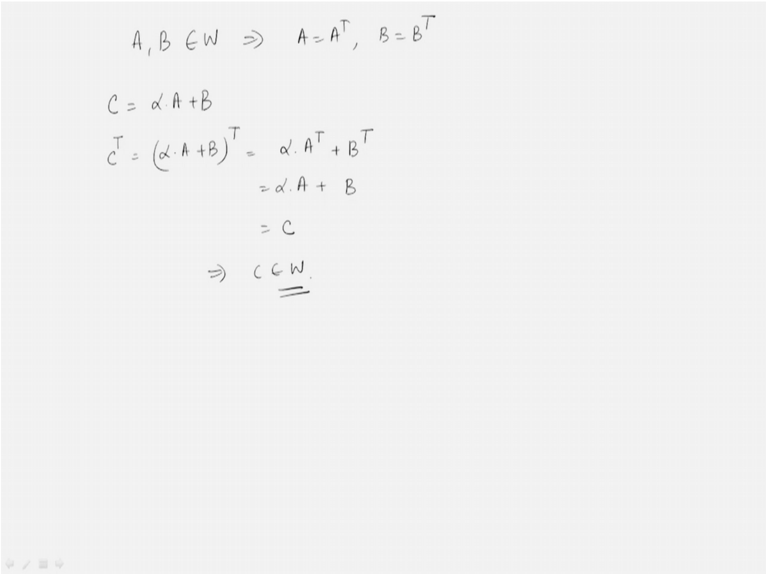

And again if you want to show that a subspace of v, so again it is very easy to show you simply, you simply take two matrices, A comma B belongs to w. So, this implies A is equal to A transpose, and B is equal to B transpose.

And we have to show that a matrix C, which is alpha dot A plus B is also in W, so that means, you have to show that C is equal to C transpose. So, take C transpose, which is alpha dot A plus B whole transpose, which is alpha dot A transpose plus B transpose. And it is alpha dot A plus B, because A transposes A, and B transposes B, so it is equal to C, so this implies C is in W.

So, we have shown that W is a subspace of this vector-space V, because all the matrices, which is equal to its transpose R in W. And we have shown that C is equal to C transpose, and C is nothing but alpha A plus B, that means, that means W is a subspace of this vector-space V.

Similarly, similarly we can show for the third problem also, you take two arbitrary element, two arbitrary vectors of p 2. And try to show that the, it is closed under addition, and the scalar multiplication. All you can show that you can take u and V in P 2, and you can try to show that alpha dot u plus V is also in p 2.

Now similarly, if you take the next example, we are we have taken V as a collection of all functions from R in to R. And we have consider a subset W of V such that f of minus

1 is 0 ok. All those functions, we which are which vanishes at minus 1, these we are taken as W.

So, again in order to show that it is a subspace of this vector-space V, you take any two arbitrary element of this subset W, and try to show that it is closed with respect to addition and scalar multiplication. Now, then the last example, last example of this slide that we are we are taking V as all M cross n cross 1 matrices over the field F. Suppose, A be an M cross n matrix over F is a fix matrix. There are set of all n cross 1 column matrices X over F, such that AX equal to 0 is a subspace of V.

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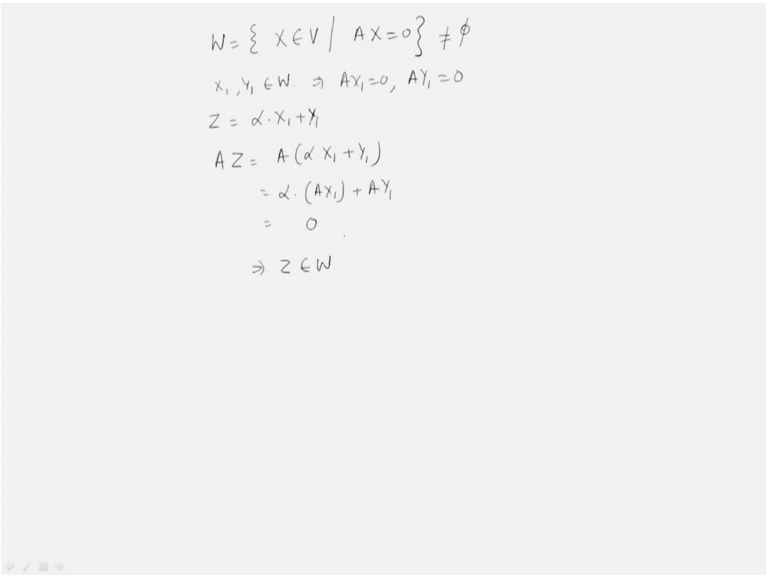

You see that here. Here we have taken a W as all X all X in V such that AX is equal to 0. Here A is a fixed matrix of order M cross n, and this X is a vector of order M cross n cross 1 ok. And the collection of all those X, we are claiming that this W is nothing but a subspace of the vector-space V. Now, now it is easy to show you see that if you again, again you take again you take two elements say x 1 and y 1 in W. Now, of course, of course, this set is never empty, this set is never empty, because this is never empty, because at least X equal to 0 is in the set.

And if it contains a single term 0, so that is a trivial subspace of trivial subspace of V. And if it is if it is not a trivial sub space, so it will contain infinitely many X, such that X equal to 0, because this is a linear system of homogeneous equations, which is either a unique solution infinitely many solutions. If it is as a unique solution, which is X equal to 0 that is a trivial subspace of V, that is automatically satisfied. But, if and if it has infinitely many solution, then we can prove it like in this way. We will assume that X 1 and Y 1 are 2 elements in W, that means, A X 1 A X 1 equal to 0, and A Y 1 equal to 0.

And we have to show that Z, which is a alpha dot X 1 plus Y 1 is also in is also in W. So, you take A into Z, A into Z is A alpha dot X 1 plus Y 1, which is alpha dot A X 1, because alpha is a scalar, and it is A Y 1, and it is 0, and it is 0, so it is 0, so that means, Z belongs to W. And hence, we can say that W is nothing but a subspace of this vector-space V.

So, basically to show that this a subspace, we have to take two cases, first is a unique solution of this, if it is a unique solution that means, trivial subspace, if not unique that is infinitely many solutions, and the and the proof follows on this line ok. So, in this way, we can see that whether, a given subset of a vector-space V consider with the subspace or not. In this way, we can show that it is a subspace of vector given vector-space V.

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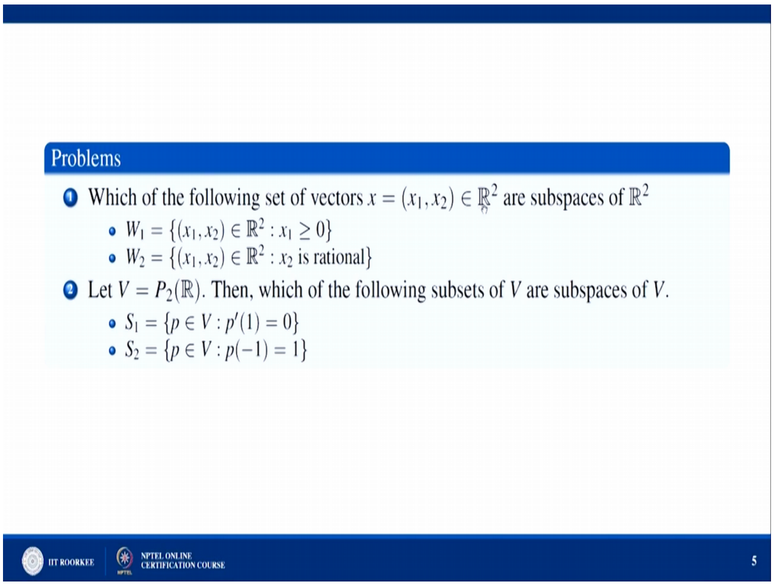

Now, let us consider few more problems based on this. Now, here in the first problem, we have taken V as R 2 over the real over the field R. Now, we have consider W 1, which is all x 1, x 2, such that x 1 is greater than equal to 0 ok.

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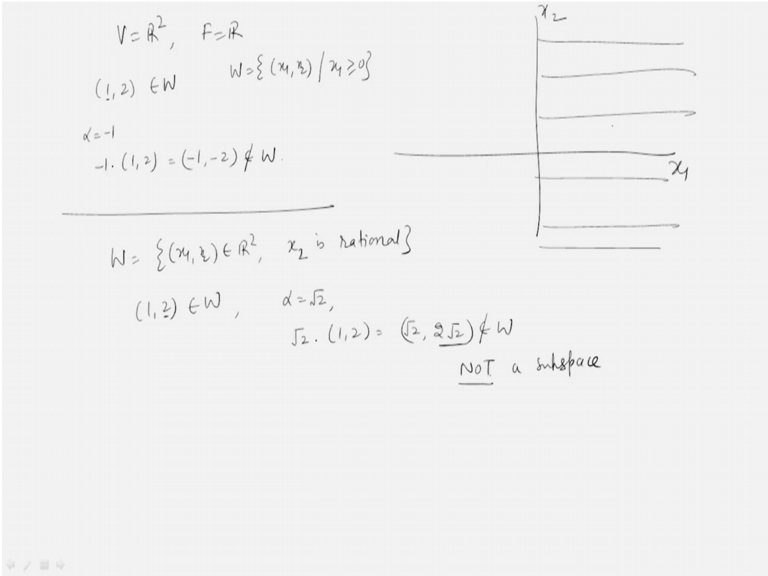

So, what W here is? So, the first example, we have taken V as R 2 field as R this is this is the thing. And x 1 greater than equal to 0, x this is x 1, and this is x 2, x 1 greater than equal to 0 means this thing. So, whether, whether this half of this R 2, we constitute a subspace of this vector-space or not. So, so you see that if it is a subspace, if it is a subspace of this vector-space, then it must be closed with respect to addition as scalar multiplication ok.

Now, if you take element say, say 1 comma 2, 1 comma 2 is in W, yes, because W are all those x 1, x 2 such that x 1 is greater equal to 0, here, x 1 is greater equal to 0, so it is in W. And you take and alpha is alpha is any real number it is come from the field.

So, if you take alpha as minus 1 says, and minus 1 dot 1 comma 2 will be minus 1 comma minus 2, which is not in W, so that means, it is not closed with respect to the scalar multiplication, and that means, it will not constitute a subspace of this vector-space V because, if it is a subspace of the vector-space V, it must be closed with respect to vector addition and scalar multiplication ok.

So, if you have to show that is a subspace of vector-space, then you must show that it is closed respect to addition and scalar multiplication. And if it is not a subspace at all, then simply give a counter example to show that it is not closed with respect to addition or a scalar multiplication.

Now, take second example. Now, here we have considered all x 1, x 2, in R 2. So, x 2 is irrational x 2 is rational. So, again when you take W as all x 1, x 2 all x 1, x 2 in R 2, such that x 1 is rational ok, x 2 is rational ok. Now, again if you take say 1 comma 2, it is in W, because here x 2 is rational.

Now, if you take alpha is under root 2, then under root 2 dot 1 comma 2 is simply under root 2 comma under root 2 under root 2, which is not in W, because now x 2 is not rational. So, it is not closed with respect to vector I mean scalar multiplication, so that means, it is not a subspace of this vector-space not a subspace, is it clear. Now, discuss second problem, the second problem is we have taken p as V as p 2, p 2 are all polynomials degree less than equal to 2 over the real field. Then, which are the following subsets of V are the subspaces of V. The first one is all p in V such that p dash 1 equal to 0.

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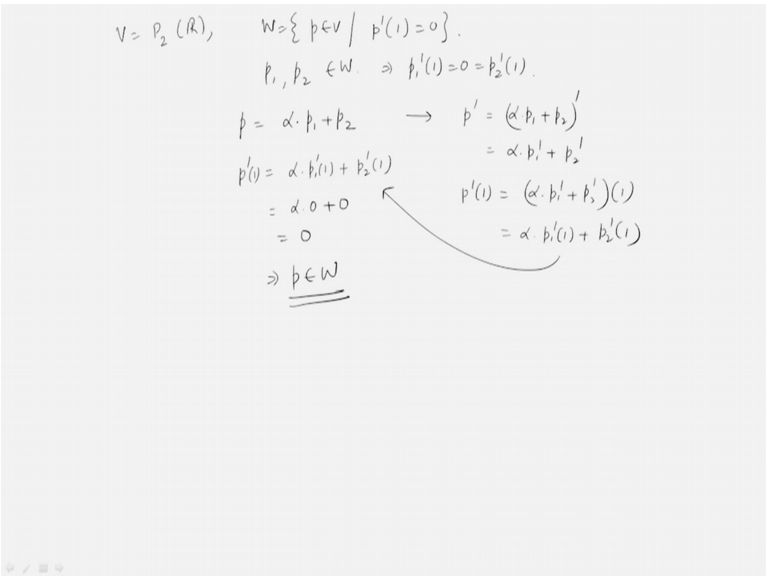

So, here V are all p 2 over real field R, and W we have taken as all p in V, such that p dash 1 is equal to 0. Now, you take two polynomials, p 1 and p 2 in W that means, p 1 dash 1 equal to 0 and p 2 dash 1 is also 0. And if it is a subspace, then it must be closed with respect to vector addition and scalar multiplication, so that means, you take an arbitrary you take this p, and we have if it is a subspace, then we have to show that it is in. Then we have to show that p dash of this p p dash p dash of this p at 1 must be 0. So, what is p dash of this, this is alpha dot p dash p dash at 1. Suppose, because this implies that p dash is simply alpha dot p 1 plus p 2 dash. Dash means derivative, which is alpha dot p 1 dash plus p 2 dash.

Now, p 1 dash at 1 is nothing but alpha dot p 1 dash plus p 2 dash at 1, and which is alpha dot p 1 dash at 1 plus p 2 dash at 1, which is which is this statement ok. And this is alpha dot 0 plus 0, which is 0, so that means, this p also is in W, and that means, W is a that means, this W is a subspace of the vector-space V.

Now, if you take the second example, second problem here, we have taken p in V, such that p minus 1 equal to 1 ok. Now, one important thing is that the additive identity of a vector-space, and all subspaces of that vector-space are same.

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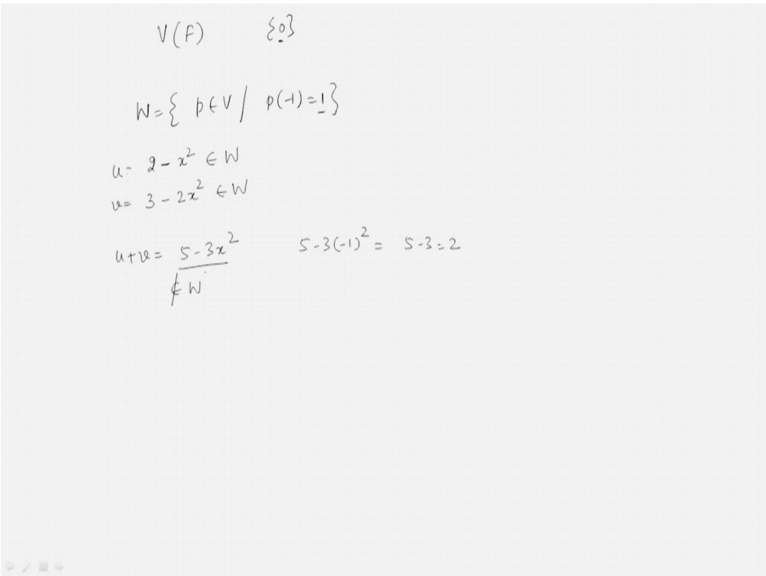

You take you take a vector-space V, suppose 0 suppose 0 is the you take a vector-space V over the real over the field F, suppose this 0 denotes the additive identity of this vector-space. And you take any subspace of this vector-space V over the field F it will also have the same additive identity 0.

Now, here, here you can easily see that if you take collection of all those polynomials, where p at minus 1 is 1. Then that that polynomial then 0 polynomial is I mean what I want to say basically that here we are taken all p in V, such that p at minus 1 is 1. Now, now what is the identity of this p 2 is 0; this p 2 has a identity 0. And that is that that polynomial is not there in this S S 2, because it contains all those polynomials, which are at minus 1 is 1.

So, the 0 polynomial is not there, because 0 polynomial at minus 1 will will attain will equal to 0 only will not be equal to 1, so that means, 0 polynomial is not there, that means, additive identity is not in S 2. And hence, we can say that this is not a subspace of this vector-space V, or the other way out is or the other way out is you take a two polynomial say you take 1, say you take 1 plus x square 1 plus x square is 2. So, you take you take any two polynomials, which are at minus 1 is 1 ok. So, at minus 1 is 1 means, you take 1 plus 1 is it is 2 minus 1.

If you take ok, if you take this polynomial, it is in W, because at minus 1 this gives 1. Now, you take another polynomial say 3 minus 2 x square, again it gives at minus 1 it gives 1. Now, this is u and this is V, if you add them u plus V, this is 5 minus 3 x square. And this polynomial this polynomial at minus 1 is nothing but is equal to 2, which is not 1, that means, this element does not belongs to W. So, we have shown that it is not closed with respect to vector addition. And hence, we can say that it is not a subspace of this vector-space V.

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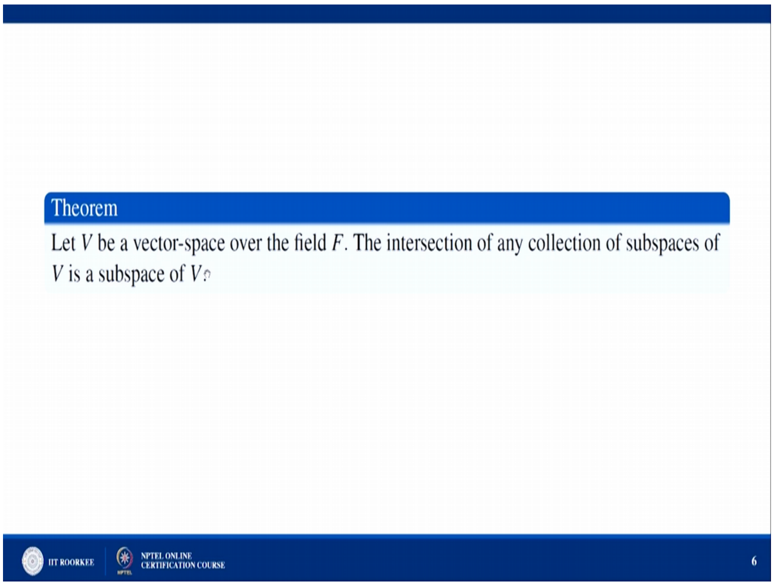

Now, let vector as V is a vector-space of field F. Then the intersection of any collection of subspaces of V is a subspace of V.

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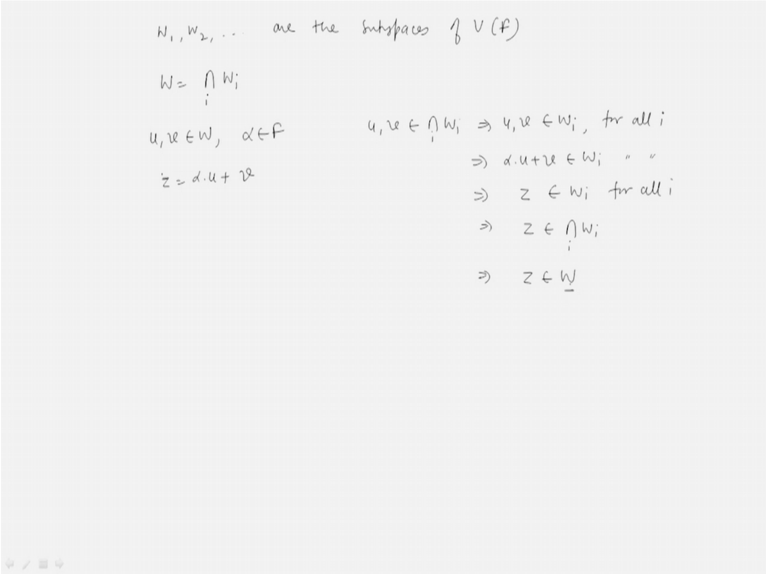

If you take, if you say if you take W, suppose W 1, W 2 and so on are the subspaces of V over F. Then if you take intersection of all thesesubspaces, then it will also of a subspace of this vector-space V. Things are very easy to show, you take u and V in F w, and alpha in field. Then since, and you have to show that this W this Z, which is alpha dot u plus V is also in W. So, say since u and V is in intersection of w i, this implies u and V are belongs to w i for all i. And this implies alpha dot u plus V belongs to w i for all i, because all w i are subspaces of V. So, it must be closes with respect to addition and scalar multiplication

So, this implies this Z, we also belong w i for all i. And this implies Z belongs to intersection of w i also, because if it is in w i for all i, that means, it is in intersection also, and that means, Z belongs to w, and that means, this W is a subspace of this vector-space V.

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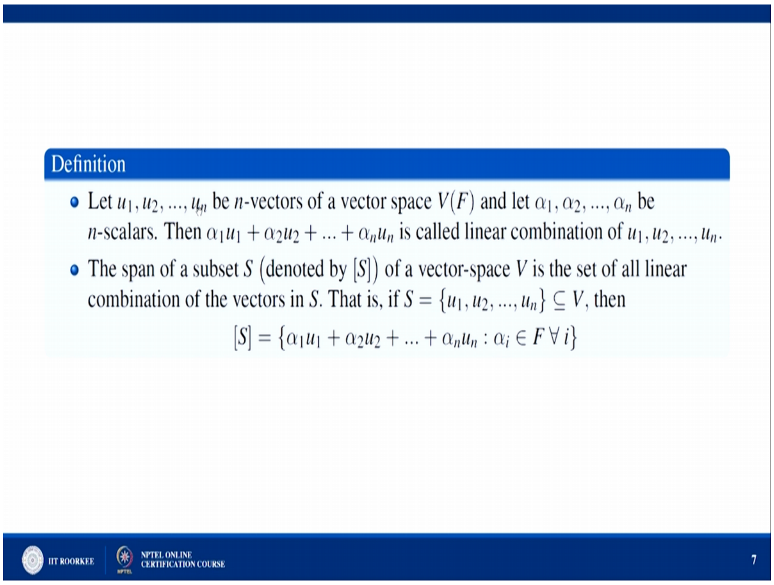

Now, let u 1, u 2, up to u n be n vectors of a vector space V, and let alpha is our scalars and n-scalars. Then if you take alpha 1 u 1 plus alpha 2 u 2 and so on up to alpha n u n, then this is called linear combination of u 1, u 2, up to u n. So, this and this will be a new vector ok, this will be a vector. Though this and it is definitely belongs to V, because V is a V is vector space. So, by the closing closure property of a scalar multiplication, vector addition, this element will be in V itself, and this element is called linear combination of u 1, u 2, up to u n.

Now, we defined we define subset of subset of V, which we called as span of S. Span of a subset S of vector-space V is simply, the collection of all linear combination of vectors in S, now suppose, S is u 1, u 2, up to u n ok. Then a span of S denoted by this bracket span of S is simply, the collection of all linear combinations of u 1, u 2, up to u n. Use you see you vary alpha, these u i S are fixed, you vary alpha, alpha 1, alpha 2, up to alpha n, you varies these scalars. You will get so many vectors, and the collection of all those vectors, we are call as a span of S.

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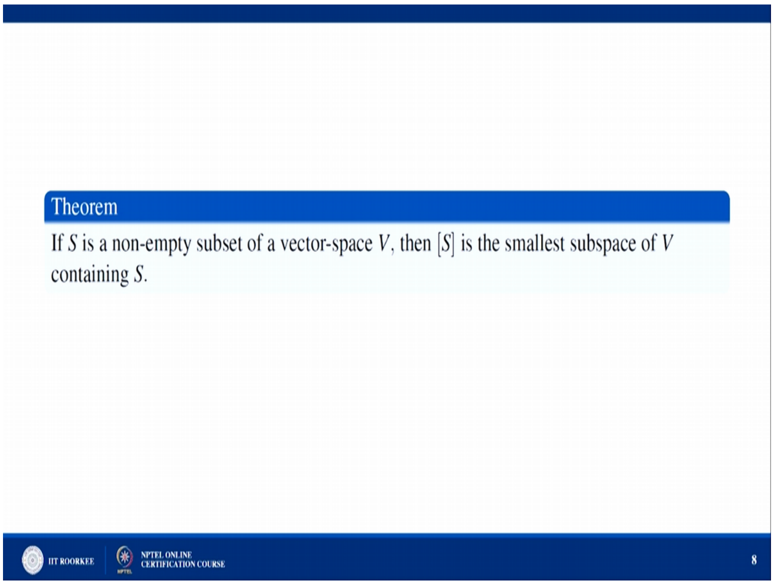

Now, a span of S is nothing but the small S subspace of V containing S.

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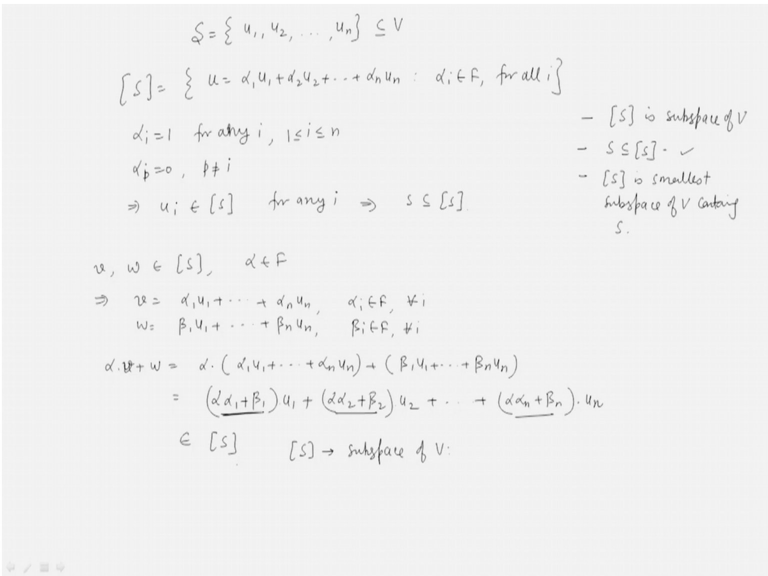

So, let us try to prove this. So, so what is S, S is simply u 1, u 2, up to u n it is subset of V all u i is are from V. What is span of S, a span is a as a all those use such that this u 1 alpha 1 u 1 plus alpha 2 u 2 and so on up to alpha n u n, such that alpha i belongs to field for all i.

Now, in this theorem we have to show three things. First we have to show that it is a subspace of V, it contains S we have to show that is a span of S is a subspace of V.

Second S is containing span of S,. And third is a span of has a smallest subspace, a smallest subspace of V containing S.

Now, the now the second property is very easy to show, you see you see this span is containing all the linear combinations of u 1, u 2, u to u n ok. If you take alpha 1 equal to 1, because alpha is are vary, alpha is varying from the field. If you take alpha 1 equal to 1, and all other alpha equal to 0, then u equal to u 1, so u 1 is definitely span of S.

Similarly, if you take alpha 2 equal to 0, and all remaining alpha i is are equal to 0. I mean alpha 2 equal to 1, and all remaining alpha equal to 0, then u 2 will be in span of S ok. So, so we can say that if alpha, alpha i you take as 1 for any i for any i i from n to 1. And alpha i is equal to 0, alpha p equal to 0, and p is not equal to i, then this implies u i belongs to span of S for any i, and this implies as is containing span of S. So, the first part is very easy to show, I mean this part second part is very easy to show. If it contain all the linear combinations of vectors u 1, u 2, up t u n, so it will definitely contains u 1 u 2 up to u n also.

Now, we have to show that it is a subspace of V. So, again you take a two two arbitrary elements of this sets say V and W belongs to the span of S, and alpha belongs to field. And then we have to show that alpha dot V plus W also in span of S. So, since V belongs to the span of S, and this implies V is some linear combination of vectors u 1, u 2, up to u n for alpha i belongs to field for all i. And similarly, W will be some other linear combination of u 1, u 2 up to u n, so scalars will be different. So, this will be some scalar combinations.

Now, you take alpha dot u plus w alpha dot v plus w. So, it is alpha dot alpha 1 u 1 and so on up to alpha n u n plus w is beta 1 u 1 and so on up to beta n u n. So, when you applied a definition of standard scalar multiplication, vector addition, you simply get alpha alpha 1 plus beta 1 times u 1 plus alpha alpha 2 plus beta 2 times u 2, and dot is already there. So, it is alpha alpha n plus beta n dot with u n.

Now, it is it is some it is some scalar, it is some scalar, it is some scalar, so we can say that it is also some linear combination of u 1, u 2, up to u n. So, we can simply say that it belongs to it belongs to span of S. So, so we can say that it belongs to span of S. Hence and hence, we can say that span of S is nothing but a subspace of field.

Now, we have to show that it is a smallest subspace of V containing S, so how we can say, how we can prove it. In order to show that is the smallest subspace containing as you take an arbitrary subspace say T of V containing S, and try to show that the span of S is also containing T.

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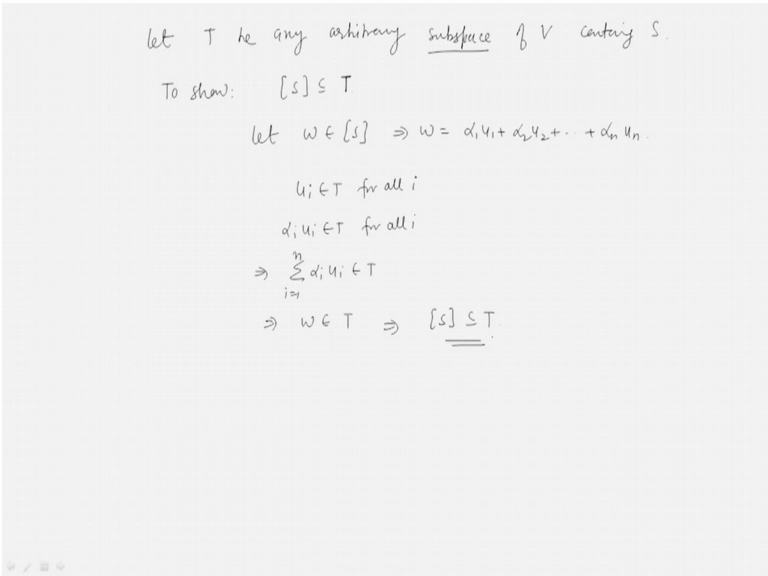

So, so let T be any arbitrary any arbitrary subspace of V containing S. And in order to show that span is the smallest such subspace, so we have to show that a span of S is a subset of T. Then we can then only, this will be the smallest one containing S.

So, in order to show that this is a subset of this take an arbitrary element in span of S, and try to show that that element is also in T. So, let say w belongs to span of S, so this implies w is some linear combination of elements u 1, u 2, up to u n and now, what to show, we have to show that this w is also in T, then only we can say that this span is a smallest such subspace.

Now, since u i belongs to T for all i, and T is a subspace. So, we can say that alpha i u i also in T for all i by the see by the closure property of a scalar multiplication, because T is a subspace ok. And this implies summation of i from 1 to n alpha u i also belongs to T. Again by the closure property of vector addition, because alpha 1 u 1 is in T, alpha 2 u 2 is in T, alpha n u n is in T. So, by the closure property of vector addition this sum is also in T ok, so that this implies w is in T ok, and this implies span of S is a subset of T. So, hence we have shown that this span is a smallest subspace of V containing S.

So, so in this lecture, we have seen that a subset of a vector-space V. If you want to show that it is S subspace of given vector-space V, we have to simply show that it is closure with respect to vector addition scalar multiplication. If it is not a subspace of that vector space, simply give a counter example, which content either closure properties with respect to addition or scalar multiplication. We have also seen that span of S, which is the collection of all linear combination of vectors in S, a simply as smallest subspace of V containing S.

Thank you.