Matrix Analysis with Applications
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Lecture – 06
Introduction to Vector Spaces

Hello friends. Welcome to lecture series on Matrix Analysis with Applications. So, today we will discuss about vector spaces that what vector spaces are. So, let us start.

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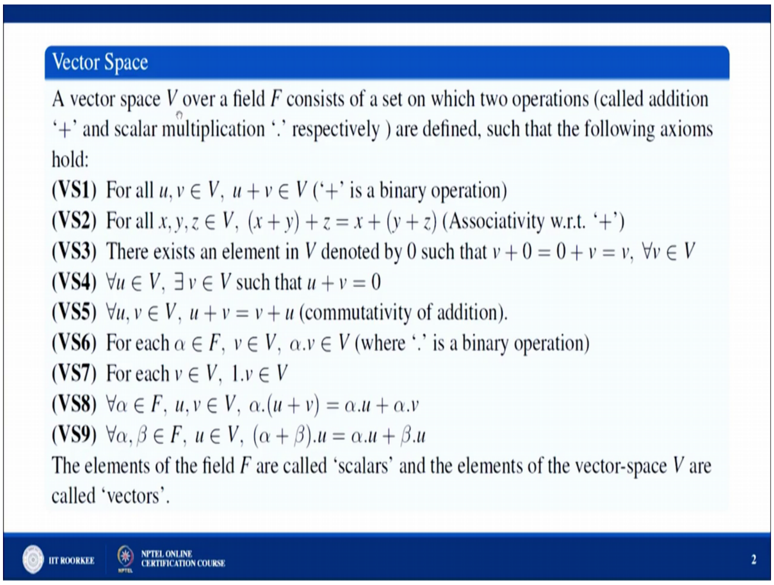

Now, vector spaces V over a field F consist of a set on which two operations called addition which is denoted by a plus sign and a scalar multiplication which is denoted by a dot, respectively are defined such that the following axioms are satisfied. So, what are axioms? The first axiom is you take any u v in V then u plus v also belongs to V that means, this addition is a binary operation that means, this satisfy a closure property. You take any arbitrary u and v in V then the with respect to addition the sum is also in V, ok.

Now, for all x y z in V x plus y plus z, and then is equal to x plus y plus z if you put the brackets in the first two or in the last two both are same. So, that is associative property with respect to addition. Next property is there exists an element V, then there exists an element in V denoted by a 0. This not this 0 is not usual 0 this is simply a notation and this denote identity element with respect to addition such that v plus 0 equal to 0 plus v equal to V for every v in V, ok.

Then for every u in V, there exists v in V such that u plus v is 0 and this V is called inverse of u with respect to addition. And the next is for every u v in V u plus v is equals to V plus u that is commutativity of addition. So, these 5 properties are with respect to addition. And we can also say that vector space V with respect to plus must be an abilean group, ok.

Now, with respect to dot for each alpha belongs to field v belongs to V alpha dot v also belongs to V, ok. So that means, dot is a binary operation that means, but this binary operation is defined one vector is from, one is from field other is from vector space. Now, for each v in V, 1 dot v also belongs to V. The next is for every alpha belongs to field u v belongs to V alpha dot u plus v is equal to alpha dot u plus alpha dot v, and for every alpha beta belongs to field u belongs to V alpha plus beta dot u is equal to this. So, if a vector space V over the field F with respect to addition and scalar multiplication dot satisfy these 9 properties then we say that that is a vector space over the field F.

Now, the elements of the vector the elements of the field F are called scalars and the elements of the vector space V are called vectors. Now, these vectors are not the usual vectors which we study in physics. We simply called the elements of vector spaces are is vectors that is all, ok. So, what is the vector space? We satisfy this property, that means, if you are talking about of set V over the field F we sometimes at like this, ok.

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$V(F)$ $\quad (V,+) \rightarrow$ abelian group

$(V,\cdot) \rightarrow \forall \alpha \in F,\ v \in V,\ \alpha \cdot v \in V$

$1 \cdot v \in V,\ \forall v \in V$

$(\alpha+\beta)\cdot v = \alpha \cdot v + \beta \cdot v,\ \forall \alpha, \beta \in F,\ v \in V$

$\alpha \cdot (u+v) = \alpha \cdot u + \alpha \cdot v,\ \forall \alpha \in F,\ u, v \in V$

Now, V plus 2 plus it must be an abelian group and with respect to dot this V must satisfy the following properties, which are mean which are for a every alpha belongs to field v belongs to V alpha dot v must belongs to V. 1 dot V must belongs to V, for every v belongs to V, then alpha plus beta dot v must be equal to alpha dot v plus beta dot v for all alpha beta from field and v belongs to V and alpha dot u plus v must be equals to alpha dot u plus alpha dot v for all alpha belongs to field and u v belongs to V

So, with respect to addition this V must be an abelian group and with respect to dot V must satisfy these properties that we said at V over F is a vector space. So, now, let us discuss few examples based on this. Now, you take V as R 2, a simple example V as R 2 and say field is R.

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$V = \mathbb{R}^2$, $F = \mathbb{R}$ $\qquad \mathbb{R}^2 = \{(x, y) \,/\, x, y \in \mathbb{R}\}$

$+ : \underbrace{(a_1, a_2)} + \underbrace{(b_1, b_2)} = (a_1+b_1, a_2+b_2)$

$\cdot : \alpha \cdot (a_1, a_2) = (\alpha a_1, \alpha a_2)$

$(V, +)$ - let $u, v \in V$, then $u = (a_1, a_2)$, $v = (b_1, b_2)$

$u + v = (a_1, a_2) + (b_1, b_2) = (a_1+b_1, a_2+b_2) \in \mathbb{R}^2$

- $[(a_1, a_2) + (b_1, b_2)] + (c_1, c_2) = (a_1, a_2) + [(b_1, b_2) + (c_1, c_2)]$

- $\underset{u}{(a_1, a_2)} \underset{+}{+} \underset{v}{(b_1, b_2)} = (a_1+b_1, a_2+b_2) = (b_1+a_1, b_2+a_2)$
$= (b_1, b_2) + (a_1, a_2) = v + u$, $\forall u, v \in V$

- $(a_1, a_2) + \underline{(0, 0)} = (a_1, a_2)$, $\forall (a_1, a_2) \in \mathbb{R}^2$

- $(a_1, a_2) \longrightarrow (-a_1, -a_2) \in \mathbb{R}^2$

What is a R 2? R 2 is simply a R 2 means all x comma y such that x and y both are in R. Now, the next is how addition and scalar multiplication is defined, then only it will constitute a vector space over that additional scalar multiplication.

So, addition is defined as you take any 2 elements in R 2 and this is simply a 1, a 1 plus b 1 sorry a 1 plus b 1, you guys simply write here b 1 a 1 plus b 1 and this is a 2 plus b 2 this is now, this is about addition. Now, the scalar multiplication is alpha dot a 1 a 2, this alpha a 1 and alpha a 2 if we defined.

If we defines these operation this plus, this plus which is defined like this, now this is this is 1 vector in R 2 this second vector in R 2. This is the operation addition which is defined on the 2 elements of vector spaces, this additional usual addition the addition of 2 real numbers, ok. And this is a dot which is operated on a element of field and a element of a vector space.

And how it is defined? This is simply alpha a 1 alpha a 2. Now, first you have to see that V over plus must be an abelian group if it is a vector space. So, the first element is this plus must be a binary operation or we can say that with respect to plus it must be closed closure property must be satisfied. So, this is very clear that if you take this element is in R 2, you take let u and v belongs to R 2 or V then u will be equal to sum a 1, a 2 and V will be equal to sum b 1, b 2. Now, if you take u plus v which is a 1 a 2 plus b 1 b 2 it is equal to a 1 plus b 1 by this definition and a 2 plus b 2 which also belongs to R 2. That means what? That means, it is a satisfied closure property, ok.

The second property is associative. If you take the associative property it is very obvious that if you take 3 elements a 1 a 2 and b 1 b 2 c 1 c 2 in R 2 you take the bracket in the first 2 you add them So, you can you can by simple calculation you can easily show that this is equal to a 1 a 2 plus b 1 b 2 plus c 1 c 2 that means, weather take the bracket in the first two elements or it take the bracket in the next 2 element last 2 elements both are same. It is very easy to show. You guys simply proceed using this addition operation.

The third property is commutative the commutative is very simple if you take a 1 a 2 and plus b 1 b 2, any two elements of R 2 if you add them by this addition by this addition operation you can see that is equal to a 1 plus b 1, a 2 plus b 2 which can be written as b 1 plus a 1. Because real numbers are always commutative 2 plus 3 same as 3 plus 2 and it is equals to and end it is b 2 plus a 2. So, again by this definition we can write it b 1 b 2 plus a 1 a 2. So, we can say that we can say that u plus v is same as v plus u for all u v belongs to V so that means, commutative property is satisfied.

Now, if you talk about the existence of identity element then you can simply see that if you add 0 0 to any element we will get itself. So, if you take any a 1, a 2 add 0 0 you will get a 1 a 2, and it is true for all a 1 a 2 in R 2 that means, you are shown the existence of identity element. And the inverse of any a 1, a 2 is minus a 1 minus a 2 which is again in R 2. Inverse means if you add these 2. So, it is equal to 0 0. So, we have shown all the 5 property with respect to addition that means, that means it satisfy that will be V respect to plus is a abilean group.

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- let $\alpha \in F$, $u = (a_1, a_2) \in \mathbb{R}^2$

$$\alpha \cdot (a_1, a_2) = (\alpha a_1, \alpha a_2) \in \mathbb{R}^2$$

- $1 \cdot v = 1 \cdot (a_1, a_2) = (a_1, a_2) = v, \quad \forall v \in V$

- $$\begin{aligned}(\alpha+\beta)\cdot v = (\alpha+\beta)\cdot(a_1, a_2) &= ((\alpha+\beta)a_1, (\alpha+\beta)a_2) \\ &= (\alpha a_1 + \beta a_1, \alpha a_2 + \beta a_2) \\ &= (\alpha a_1, \alpha a_2) + (\beta a_1, \beta a_2) \\ &= \alpha \cdot (a_1, a_2) + \beta \cdot (a_1, a_2) \\ &= \alpha \cdot v + \beta \cdot v, \quad \forall \alpha, \beta \in F, \ v \in V\end{aligned}$$

- $$\begin{aligned}\alpha \cdot (u+v) &= \alpha \cdot [(a_1, a_2) + (b_1, b_2)] \\ &= \alpha \cdot [a_1+b_1, a_2+b_2] = (\alpha(a_1+b_1), \alpha(a_2+b_2)) \\ &= (\alpha a_1 + \alpha b_1, \alpha a_2 + \alpha b_2) \\ &= (\alpha a_1, \alpha a_2) + (\alpha b_1, \alpha b_2) = \alpha \cdot u + \alpha \cdot v\end{aligned}$$

$\forall \alpha \in F$, $u, v \in V$

Now, with respect to dot, with respect to dot if you see the first property you take alpha belongs to field where field is real here and you take any u which is given by a 1, a 2 in R 2. If you take alpha dot a 1 a 2 then by the definition of scalar multiplication we can easily write alpha a 1, alpha a 2 and it again belongs to R 2 that means, closer with respect to scalar multiplication is satisfied.

Now, the next property is you take 1 dot v that means, 1 dot a 1, a 2 which is by definition is again a 1 a 2 that is v though 1, so 1 dot v is equal to V, for every v in V. That means that means, the second property holds. Now, if you take alpha plus beta dot with v. So, it is alpha plus beta dot v, v here is a 1 a 2. So, it is alpha plus beta a 1 by the definition, alpha plus beta a 2.

So, it is equals to alpha a 1 plus beta a 1 it is alpha a 2 plus beta a 2 which is equals to alpha 1 alpha a 2 plus beta a 1 beta a 2 by the definition by the definition of addition. And this is alpha dot a 1 a 2 plus beta dot a 1 a 2 which is equals to alpha dot v plus beta dot v. And it holds for every alpha beta in field and v belongs to V that means, this property hold for every alpha beta in field and v belongs to V.

And the last property is alpha dot u plus v, alpha dot, this is alpha dot u plus v must be equals to alpha dot u plus alpha dot v. So, to hold this property to put this property alpha dot say u is a 1 a 2 and say v is b 1 b 2. So, this will be alpha dot this is a 1 plus b 1 comma a 2 plus b 2, by the definition of addition. And this is equals to by the scalar multiplication this is alpha times a 1 plus b 1 and this is alpha times a 2 plus b 2.

This is alpha a 1 plus alpha b 1 this is alpha a 2 plus alpha b 2 and that can be written as alpha a 1 alpha a 2 plus alpha b 1 alpha b 2. And this is equals to alpha times u alpha dot u plus alpha dot v where u is a 1 a 2 and v is b 1 b 2 and this property holds for every alpha belongs to field and for every u v belong to V.

So, we have shown that all the 9 properties are satisfied that means, this that means, that means, this R 2 over the field R under these operations of additional scalar multiplication is a vector space. So, these are few samples of this. The first sample we have discussed.

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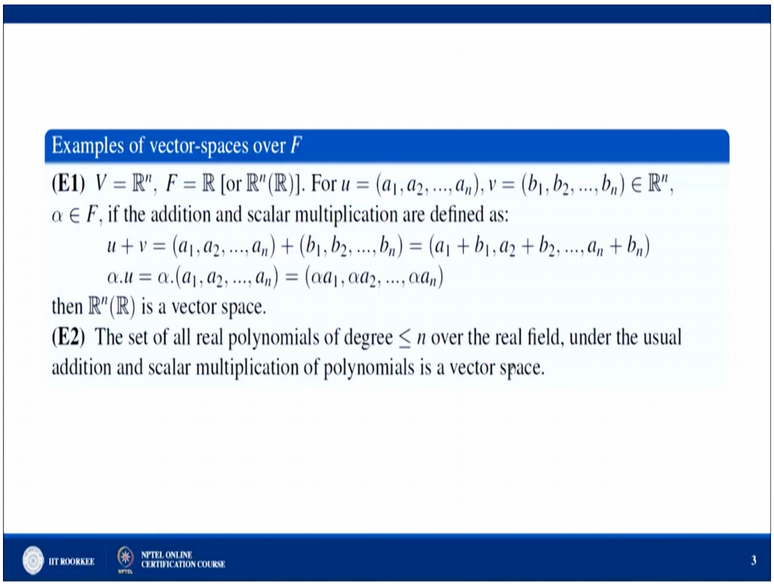

So, say we are having instead of R 2 we are having R n, ok. If you are having R n over the field R or we also re did like this R n over the field R. And how addition scalar multiplication is defined? You take any u which is n R n that is a 1, a 2 of 2 an V is also in R n that is b 1, b 2 up to b n and the addition is defined as a 1 plus b 1, a 2 plus b 2, a n plus b n that is component wise addition and a scalar multiplication result for dot u is alpha dot a 1, a 2 up to an is alpha a 1, alpha a 2 up to alpha n is a vector space. Now, the second this can be proved easily as we did for R 2 over R, ok.

Now, a second example is the set of all real polynomials of degree less than equal to n, over a real field under usual addition and scalar multiplication of polynomials is also a vector space you can see here.

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$V = P_n$ (all polynomials of deg $\leq n$), $F = \mathbb{R}$

$(V, +)$ – $u = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$

$v = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0$

$u + v = (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \cdots + (a_0 + b_0) \in V$

– $(u + v) + w = u + (v + w), \quad \forall u, v, w \in V.$

– $u + v = (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \cdots + (a_0 + b_0)$

$= (b_n + a_n) x^n + (b_{n-1} + a_{n-1}) x^{n-1} + \cdots + (b_0 + a_0)$

$= (b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0) + (a_n x^n + \cdots + a_0)$

$= v + u, \quad \forall u, v \in V$

– $u + 0 = 0 + u = u, \quad \forall u \in V$

– $u + v = v + u = 0 \Rightarrow v = -u, \quad \forall u \in V.$

Now, you are considering V as P n, P n is all polynomials of degree less than equal to n and field is R. And addition is standard addition of 2 polynomials and a scalar multiplication a standard multiplication of a scalar with a polynomial.

So, how can it constitute a vector space? So, this is very easy to show you can simply see that the first is V respect to plus must be an abelian group, in order to show that V plus is an abelian group the first property is closure that means, plus is a binary operation. So, that is simple to show that you take a n x raised to power n plus a n minus 1 x raised to power n minus 1 and so on up to a not, ok.

This is the first polynomial that is u and the v is supposed b n x raised to power n plus b n minus 1 x raised to power n minus 1 and so on up to b naught, ok. So, these are 2 polynomials would degree a less than equal to n. It may be less than n also if these are coefficients are non-zero I mean 0; if it is 0 then degree will be less than n, ok. So, we are not putting any restrictions on the leading coefficients.

Now, if you take u plus v, then if you take component wise addition as defined so this is something like this. Then this is again belongs to V because it is also a polynomial of degree less than equal to n that means, closer with respect to addition is satisfied.

The second property is associative, that is very easy to show if you take u plus v plus w or is also equals to u plus v plus w for every u v w belongs to V, that you can easily show you take the left hand side and then solve it you take the right hand side or you or from the left hand side itself you can easily show the right hand side. You can arbitrary chose u as 1 polynomial of this type, v as this polynomial of this type and w as some c n x raised to power n, c and minus 1 x raised to power n minus 1 and so on.

Next is next is commutative. Commutative is also easy to show you take u plus v which is an which we have already seen that it is a n plus b n x raised to power n plus an minus 1 plus b n minus 1 x raised to power n minus 1 plus and so on a naught plus b naught. And this can be written as b n plus c n because these are real numbers, and real numbers commute respect to addition.

This is b n minus 1 plus an minus 1 x raised to power n minus 1 and this a naught b naught plus a naught, and this can be written as b n x raised to power n plus b n minus 1 x raised to power n minus 1 and so on b naught by the definition of addition vector addition. This is a n x raised to power n plus a naught. So, this is v plus u for every u v belongs to V. So, we can we have shown that it commutes.

Now, identity element is 0 polynomial 0 polynomial is a identity element. If you take u plus 0 it will be 0 plus u equal to u for every u in V, and the inverse of any u is minus u. You can simply say that if you take u if you take u plus v is equals to v plus u equal to 0. So, this implies v is minus u for every u belongs to V. So, for every u there exist a inverse which is additive inverse is satisfied commutative property associative closure. So, this means respect to plus it is an abelian group.

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$$- \ \alpha \cdot u = \alpha \cdot (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0)$$
$$= \alpha a_n x^n + \alpha a_{n-1} x^{n-1} + \cdots + \alpha a_0 \in V$$

$$\to \ 1 \cdot u = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = u \quad \forall u \in V$$

$$- \ (\alpha + \beta) \cdot u = (\alpha + \beta) \cdot [a_n x^n + \cdots + a_0]$$
$$= (\alpha+\beta) a_n x^n + (\alpha+\beta) a_{n-1} x^{n-1} + \cdots + (\alpha+\beta) a_0$$
$$= (\alpha a_n x^n + \alpha a_{n-1} x^{n-1} + \cdots + \alpha a_0) + (\beta a_n x^n + \cdots + \beta a_0)$$
$$= \alpha \cdot (a_n x^n + \cdots + a_0) + \beta \cdot (a_n x^n + \cdots + a_0)$$
$$= \alpha \cdot u + \beta \cdot u, \quad \forall \alpha, \beta \in F, \ u \in V.$$

$$\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v, \quad \forall \alpha \in F, \ u, v \in V.$$

Now, respect to with respect to multiplication also, if you see the multiplication. So, first property is alpha dot u if you take alpha dot with a naught x raised to power n plus an minus for a n x raised to power n a 1 power x raised to power minus 1 and so on a naught which is alpha a n x raised to power n plus alpha an minus 1 x raised to power n minus 1 and so on up to alpha a naught which also belongs to V because it is again a polynomial degree less than equal to n. So, the first property with respect to multi scalar multiplication is satisfied which is closure.

The second is 1 dot u if you take alpha as 1. So, it is simply a n x raised to power n plus substitute 1 here in place of alpha. So, it is a n minus 1 x raised to power n minus 1 and so on a naught which is equal to u for all u in V. So, next property also hold.

Then you take alpha plus beta time dot u. So, it is alpha plus beta dot with u is a n x raised to power n plus and so on a naught. So, it is alpha plus beta times a n by the definition of scalar multiplication. Then it is a n minus 1 x raised to power n minus 1 and so on alpha plus beta times a naught, ok. Now, it is alpha a n plus beta a n, then they can be split into 2 alpha this alpha n minus 1 into x raised to power n minus 1 and so on alpha a naught plus with beta a n x raised to power n plus and so on beta a naught.

And this can be done as alpha dot with a n x raised to power n and so on and so on a naught plus beta dot with a n x raised to power n and so on a naught. So, it is alpha dot u plus beta dot u, for all alpha beta belongs to field and u belongs to u.

And similarly you and similarly we can show the last property that this is equals to alpha dot u plus alpha dot v for all alpha belongs to field and the u v belongs to V. So, in this way we can say that it consecutive a vector space over real field.

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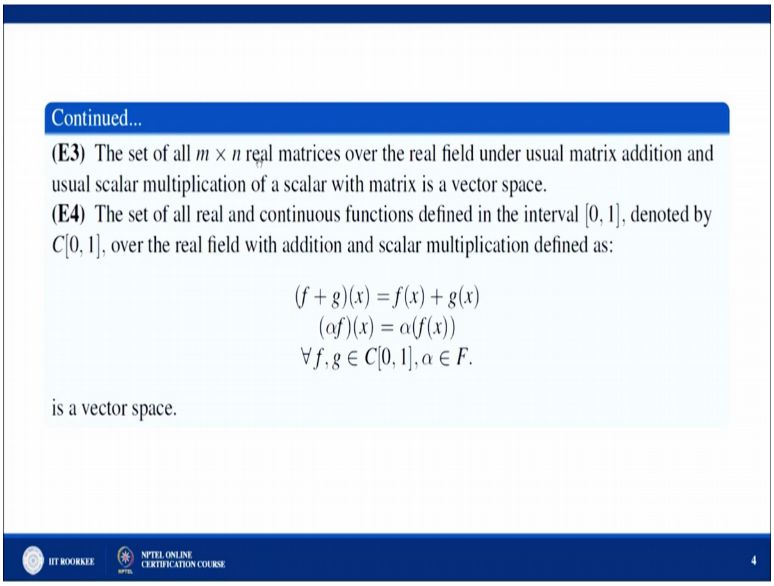

Now, next example is if you consider set of all m cross in mattresses real matrices over a real field under usual addition of matrices and usual scalar multiplication scalar with matrix. It is also a vector space it is also very easy to show that with respect to plus matrix the set of matrices is a vector space sorry is the abelian group and with respect to dot is satisfy all those 4 properties,

Then next is if you take the set of all real and continuous functions defined in the interval 0 1, you take the set of all continuous functions real and contains function which are defined in between 0 and 1 suppose it is denoted by C 0 1 over a real field with a addition defined like this and a scalar multiplication defined like this is also a vector space.

You guys simply see you see with respect to plus the sum of 2 continues function is of 2 continues, so closer is satisfied. Identity element is 0 0 is a continues function which belongs to this set the inverse of any element F in 0 1, C 0 1 is minus f which is also in C 0 1. That is inverse element exist.

And f plus g is equals to F x plus g x which is same as g x plus f x that is g plus f that is commutative combative property hold and this all satisfactivity that means, V with respect to plus is an abelian group.

Now, if you see with respect to dot with respect to dot also you can easily see that you if you multiply as continues function by a non-zero scalar it by a and by any scalar fact it will retrieves also a continuous function then 1 dot f is equals to f that is 1 dot V equal to. And similarly the other 2 properties results alpha plus beta dot v equal to alpha dot v plus beta dot v and the last property is alpha dot u plus v should be equals to alpha dot u plus beta dot b.

So, in this way we can also show that this constitutive a vector space over the field R.

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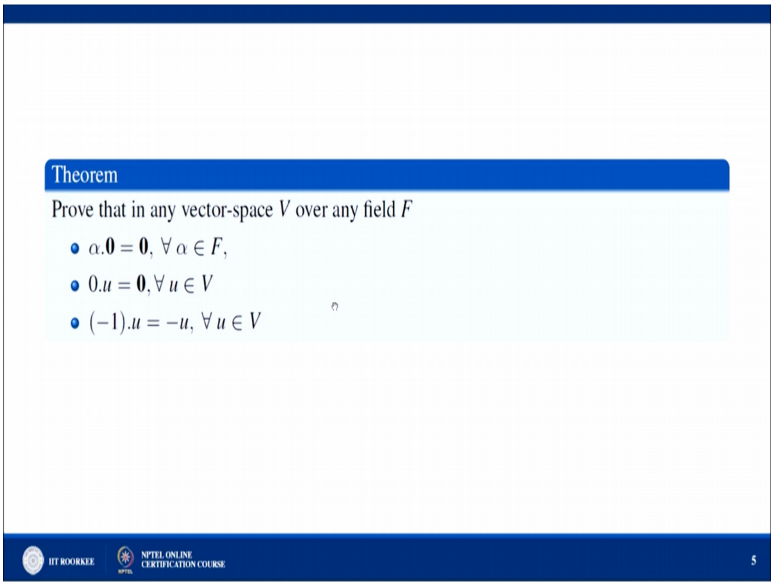

Now, these are the important theorem, that in any vector space over any field F. If it is a vector space over the field F then these 3 properties also hold and it is very easy to prove you can see here.

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$$\alpha \cdot 0 = 0, \quad \forall \alpha \in F$$

$$\underline{LHS} \quad \alpha \cdot 0 = \alpha \cdot (0+0)$$

$$= \alpha \cdot 0 + \alpha \cdot 0$$

$$\text{let } \alpha \cdot 0 = u$$

$$u = u + u$$

$$u + (-u) = u + (u + (-u))$$

$$0 = u + 0$$

$$\Rightarrow u = 0 \Rightarrow \boxed{\alpha \cdot 0 = 0}$$

The first property is and it is for a for any alpha, it is alpha for all alpha in field. Now, this 0 is from field, I mean this 0 is from vector space, this 0 is going to be a usual 0 it is a 0 of a vector space. So, this 0 is additive identity basically and we have to show that if you multiply if you if you take a scalar multiplication of any alpha with this identities it is always 0. So, the proof is 0 take left side it is alpha dot 0 which capital as alpha dot 0 plus 0 which is equals to because it is a vector space. So, alpha dot u plus v is equal to alpha dot u plus beta dot v.

Now, let alpha dot v as 0 as u. So, it is u equal to u plus u. Now, u is a element in vector space if u is in element in vector space then additive inverse of u also exist and it is it belongs to V. So, we can we can take with add minus u both sides, ok. Now, u plus its inverse is additive identity and u plus inverse u plus additive inverse imply u equal to 0 and u is nothing, but alpha dot u which is equal to 0. So, we have put the first part.

Now, what is second? Second is 0 dot u, this 0 is from field this is not additive identity the things which have marked by bold 0 is additive identity. Now, 0 is from field and 0 dot which any u in V any u in V is always additive identity again it is simple to prove you can see.

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$$0 . u = 0 , \quad \forall u \in V$$

$$0.u = (0+0).u$$
$$= 0.u + 0.u$$

let $0.u = v \in V$

$$v = v + v$$
$$v + (-v) = v + v + (-v)$$
$$0 = v + 0 \Rightarrow \underline{v = 0} \Rightarrow 0.u = 0.$$

It is 0 dot u is 0 for all u in V. Then it is 0 plus 0 dot with u you see 0 u 0 dot u can be written in this way this is a real 0 and this 0 is 0 plus 0 is again 0. So, this is 0 dot u plus 0 dot u by the property of vector space. Now, if you take 0 dot u as v then it is simply v equal to v plus v and v is any element in vector space v.

So, its additive inverse exists. So, we can always at its additive inverse both the sides and it is 0 which is equal to v plus 0 and this implies v equal to 0 and v is nothing, but 0 dot u equal to 0. So, we have shown that 0 dot with u is always 0 this is additive identity, ok. The last property is minus 1 which is a real number dot with u is nothing, but additive inverse of u for all u in V. So, this is also simple to show. This is minus 1 dot with u is minus u this is to show.

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$$(-1)\cdot u = -u \quad \forall\, u \in V \qquad (\alpha+\beta)\cdot u = \alpha\cdot u+\beta\cdot u$$

$$0\cdot u = 0 \Rightarrow (1-1)\cdot u = 0$$

$$\Rightarrow (1+(-1))\cdot u = 0$$

$$1\cdot u + (-1)\cdot u = 0$$

$$u + (-1)\cdot u = 0$$

$$\underbrace{(-u)+u} + (-1)\cdot u = -u$$

$$\Rightarrow 0 + (-1)\cdot u = -u$$

$$\boxed{(-1)\cdot u = -u}$$

So, let us write 0 dot u is 0 this we have proved in the second part. Now, this 0 can be written as 1 minus 1. Now, this 1 minus 1 because we know that alpha plus beta dot with u is alpha dot u plus beta dot u because the vector space. So, this can be done as 1 plus minus 1 dot u equal to 0 alpha and beta.

So, it is alpha dot u plus beta dot u equal to 0 1 dot u is u plus minus 1 dot u equal to 0 and if u is in v then its additive inverse exist which is minus u and it is also in v. So, we can always add minus minus u both sides and this process this is additive identity. Now, element with its additive identity is always itself. So, it is minus 1 dot u which is equals to minus u. So, we have roved this is also, ok. So, all the 3 parts we have done.

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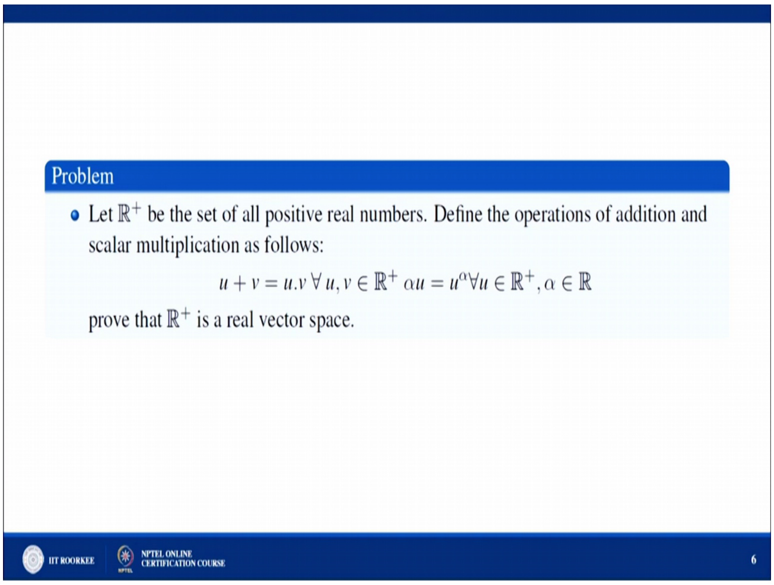

Now, this set of R plus which a set of all positive real numbers and we define the operation of addition is scalar multiplication like this is also a real vector space. So, this is very easy to prove.

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$$u + v = uv, \qquad \alpha . u = u^{\alpha}, \qquad u \in R_+$$

$(R_+, +)$, let $u, v \in R_+$, $u + v = uv \in R_+$

$$-\ (u+v)+w = (uv)+w = (uv)w$$
$$= u(vw)$$
$$= u + vw$$
$$= u + (v+w)$$

$$-\ u + \bar{0} = u$$
$$\Rightarrow u\bar{0} = u$$
$$\Rightarrow \bar{0} = 1$$

$$u + v = \bar{0} \Rightarrow uv = 1$$
$$v = \frac{1}{u} \in R_+$$

$$u + v = uv = vu = v + u, \quad \forall u, v \in R_+$$

You see that if you take. Now, how addition is defined? Addition is simply u into V and alpha dot u is simply u times alpha and for every u belongs to R plus set of all positive real numbers field is real. So, first we see with respect to with respect to addition, with respect to addition is must constitute a respect to addition it must consecutive a abelian group. The first is closer; you take let u v belongs to R plus then u plus v which is defined like u v is also prod of 2 real positive real number also positive real number. So, it is also belong to R plus. So, closer property is satisfied.

Next is associative, if you take u plus v plus w which is equals to u v plus w it is. Now, this plus will again multiplied and u v into w which can be done as u v into w because multiplication it satisfy associative property and this is u plus v w and this is u plus v plus R so that means, associative property hold with respect to addition. The second property hold the first properties here, then it is identity. You can take you can take u plus 0 which is equal to u. So, you plus 0 is u in to 0 this 0 in the usual 0 this is a additive identity u can denoted by a bar also by any other number also. So, this is equals to u by this. So, this implies 0 bar is 1. So, here additive identity is 1.

So, clearly the inverse of any element if you take u plus v should be additive identity. So, this implies u v equal to 1 and this implies v equal to 1 by u which also belong to R plus. So, identity, so inverse of any element is 1 by u additive identity is 1 with respect to addition it satisfy all the properties of course, competitive also hold if you take u plus v it is u v which is equal to vu which is equal to v plus u for every u v in R plus. So, all the property with respect to addition is satisfied. So, with respect to plus it is an abelian group.

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$$\alpha \cdot u = u^{\alpha} \in R_+ \qquad u + v = uv$$

$$- \; 1 \cdot u = u^{1} = u$$

$$- \; (\alpha+\beta) \cdot u = u^{(\alpha+\beta)} = u^{\alpha} \cdot u^{\beta} = u^{\alpha} + u^{\beta}$$

$$= \alpha \cdot u + \beta \cdot u$$

$$- \; \alpha \cdot (u+v) = \alpha \cdot u + \alpha \cdot v$$

Now, if you say with respect to scalar multiplication; if you see for a scalar multiplication it is defined like this. So, of course, closure property holds you take any alpha in real field and any u in R plus then it raised to power alpha is always belong to R plus that means, closer closest to at a scalar multiplication holds. Now, you take 1 dot u which is u raised to power 1 is u satisfied.

Now, you take alpha plus beta dot with u which is u raised to power alpha plus beta which is u raised to power alpha into u raised to power beta which is alpha dot u, ok. So, this is equal to you see this dot is means multiplication means addition u alpha plus u beta, ok. Because u plus v is u into v by that property and this is alpha dot u plus beta dot v by the definition of scalar multiplication.

So, similarly the next property can be shown that alpha dot u plus v is equals to alpha dot u plus v alpha dot v, ok. So, in this way we can show that is a real vector space.

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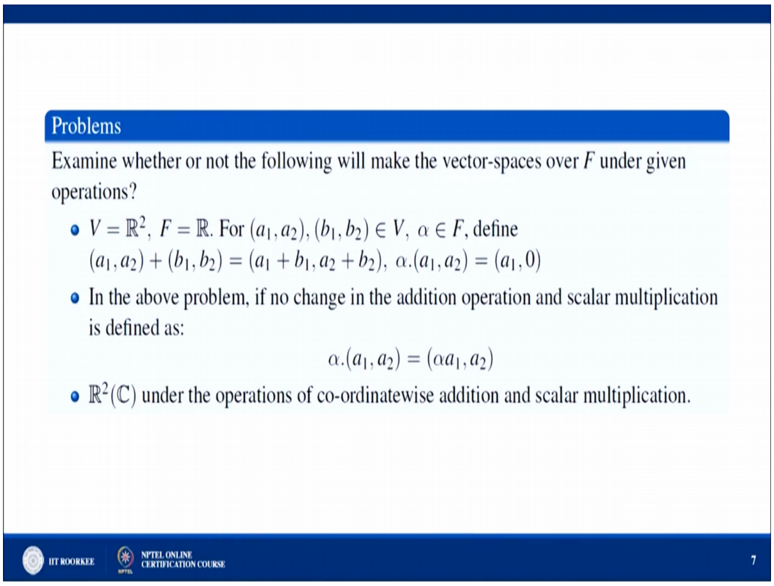

Now, let us discuss few examples which are not vector spaces you see. If you take V equal to R 2 and over a real field and take any 2 arbitrary element a 1, a 2; b 1, b 2 in b alpha belongs to field and you defined addition like this which is a stand addition and dot like this then it will not constitute a vector space. The reason is very simple you see with respect to addition it will be an abilean group no problem, but if we if we see here, so for vector space 1 dot v should be v.

If you take 1 dot a 1 a 2 it must be equal to a 1 a 2 how however, it is equal to a 1 comma 0 which is not equal to a 1 a 2 for a 2 not equal to 0. So, this means all the property vector space are not satisfied. So, it will not a vector space. If you take here a standard addition is the same you see no change the addition operation if you take the scalar multiplication like this you take 1 dot a 1 a 2 is same as a 1 a 2 that is 1 dot v is v.

But if you take if you take the other property say for this, if you take the other properties say alpha dot alpha dot a 1 a 2 is alpha a 1 and a 2 if you take alpha plus beta dot with a 1 a 2. So, it must be equals to, comma it must be equals to alpha dot alpha dot alpha plus beta dot with a 1 and alpha plus beta with a 2 you see.

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$$\alpha \cdot (a_1, a_2) = (\alpha a_1, a_2)$$

$$(\alpha+\beta)\cdot v = (\alpha+\beta)\cdot(a_1, a_2) = ((\alpha+\beta)a_1, \underline{a_2})$$

$$\begin{aligned} &\alpha\cdot v + \beta\cdot v \\ &= \alpha\cdot(a_1,a_2) + \beta\cdot(a_1,a_2) \\ &= (\alpha a_1, a_2) + (\beta a_1, a_2) \\ &= ((\alpha+\beta)a_1, \underline{2a_2}) \end{aligned}$$

$V = P_2$ ($\deg \geq 2$), $F = R$

$u = x^3 - 1$

$v = -x^3 - 1$

$u + v = -2 \notin V$

What I want to say basically, if it is a vector space is must satisfy all the property vector space if it is not satisfying anyone the property of vector space means it is not a vector space, ok. Now, if you take this property alpha plus beta dot with v. So, it is alpha plus beta dot with a 1 a 2 by this property it is by this definition it is alpha plus beta a 1 and a 2.

Now, but for a vector space it must be equals to alpha dot v plus beta dot v which is alpha dot a 1 a 2 plus beta dot a 1 a 2 which is equal to alpha a 1 comma a 2 plus beta a 1 comma a 2 and which is equals to alpha plus beta a 1 by the addition to a 2 and this should not this may not be equal to this if a 2 is not equal to 0. So, for any a 1, a 2 is this side is not equal to this side. So, this means it is not it is not a vector space.

Now, if you take R 2 over a complex field it will also not a vector space. Because if you take a scalars from scalars from a complex numbers and you multiply with a real numbers the resultant is not in R 2 resultant will be in C 2. So, closer with respect to the scalar multiplication is not satisfied. So, this means it will not be a vector space.

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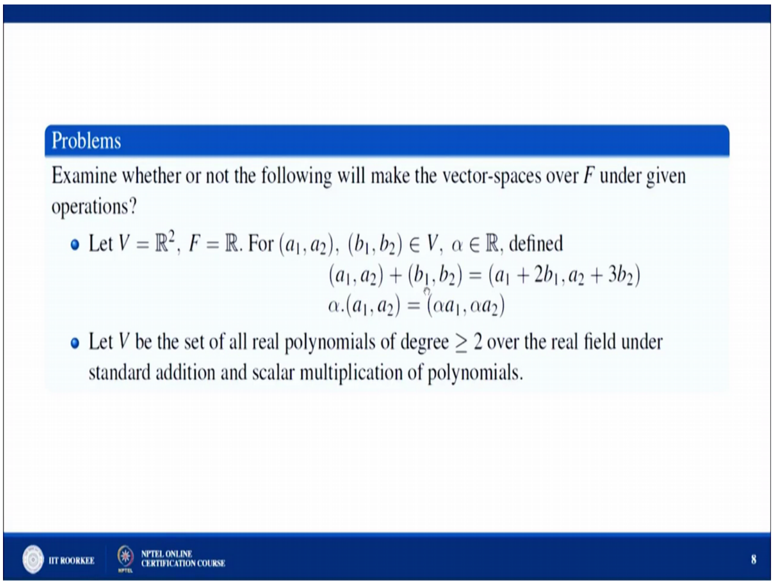

Similarly, you we can easily show that this addition scalar multiplication same standard one. But if you take vector addition, vector addition is simply this thing, and it does not satisfy a associative property you can verify.

Now, if you take the real polynomials of degree greater than equal to 2 over the real field it is also not a vector space you can see. If you take a polynomial you take 1 polynomial of degree greater equal to 2 over a real field. You take 1 polynomial as suppose x square minus 1 and second polynomial as say minus or x cube minus 1 or minus 2 minus 1 then u plus v is minus 2 which does not belongs to v, that is closure property with respect to addition is not satisfied. So, it will not constitute a vector space.

So, in this lecture we have seen that what are vector spaces and what are the standard examples of vector spaces. In the next lecture we will see that what are the spaces are and how we can find out basis and dimension of vector space.

Thank you.