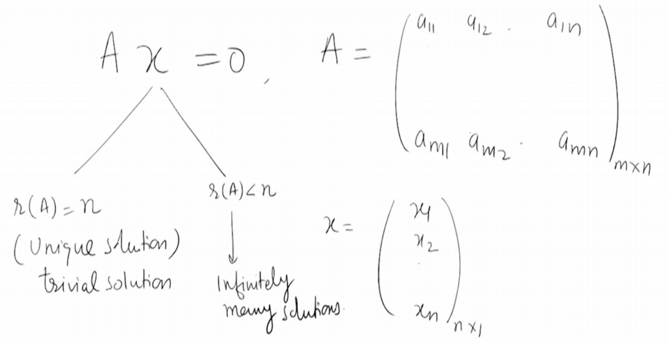
Matrix Analysis with Applications

Prof. Dr. S.K. Gupta
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 05
System of Linear Equations-II

Hello friends, welcome to lecture series on matrix analysis with applications. So, in the last lecture we have seen that how can we solve homogenous system of linear equations. We have seen that if you have a system of equation like this A x equal to 0 where A is A matrix of order m cross n see a 1 1 a 1 2 up to a 1 n and a m 1 a m 2 up to a m n which is A matrix of order m cross n and x is A vector it is x 1 x 2 up 2 x n n cross 1.

(Refer Slide Time: 00:40)



So, this A is called coefficient matrix and this x is called unknown vector which is to find out. So, we have seen that while solving a homogenous system of equation and right hand side is 0. So, this system is always consistent because at least x equal to 0 in the solution, we satisfy this system of equations; linear system of equations. So, this is the always consistent and if it is consistent, so there is only 2 possibilities; either it has a unique solution or it has infinitely many solution.

So, where it will be having unique solution? Unique solution means 0 solution or the trivial solution. When rank of A is equal to number of unknowns, then it will be having unique solution or trivial solution. We also call it trivial solution which is A 0 solution and if rank of A is less than n; that means, number of linearly independent equations are less than number of unknowns, then the system will be having infinitely many solutions ok. So, this we have seen the last lecture that how can we solve system of homogenous linear system of equations.

(Refer Slide Time: 02:36)

$$Ax = b \quad , \quad b \neq 0 \qquad A \rightarrow m \times n \text{ matrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Augumented matrix

$$[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]_{m \times (n+1)} \qquad 0 \neq b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\text{rank}(A|b) \geq r(A)$$

Now, if you are having system of non homogenous equation that is A x equals to b b which is the linear system of equation, but non homogenous. Non homogenous means b vector is not equal to 0 ok. A is the coefficient matrix which is the what m cross n, x is the vector which is to find out x 1 x 2 up to x n and this b is A vector which is b 1 b 2 up to b m and this is not equal to 0 because if it is 0, then it this will be the homogenous system of equations.

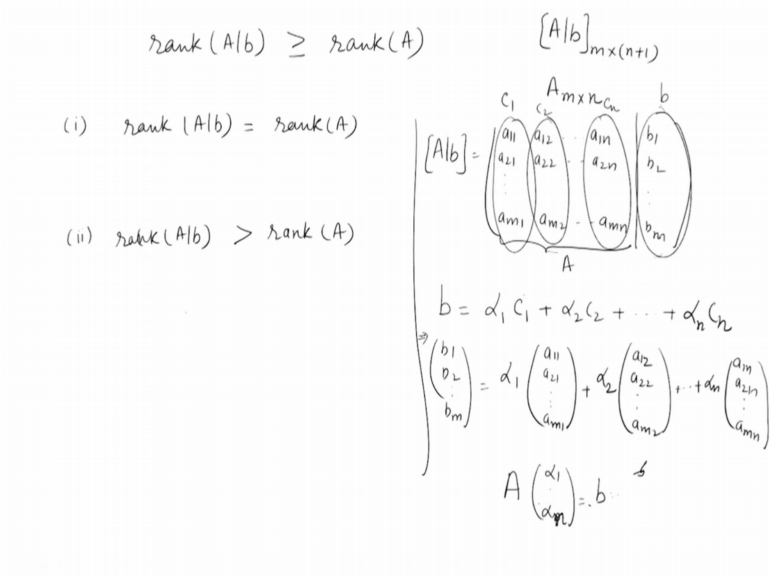
And now we are interested to find out the solution of non homogeneous system of equations. Now to find out the solution of this system, so there are three possibilities; either system will be having no solution or in consistent or unique solution that is only 1 solution or infinitely many solutions. Now how can we see; how can you find out where this system has unique solution, many solution or no solution? So, we first find we first construct a new matrix which we called as Augmented matrix.

So, what is augmented matrix? You see augmented matrix is given by we denoted by A slash b. This matrix is augmented matrix and the augmented matrix means you first write A, A is a 1 1, a 1 2 up to a m 1 m 1 n this a 2 1 a 2 2 up to a 2 n and then a m 1 a m 2 and m n which is a m cross n matrix. This is A matrix. this A matrix and then right hand side right side is A b 1 b 2 up to b m. So, this matrix is called augmented matrix.

Now what is the order of this matrix? Now you see, there are m rows in this matrix and number of columns are n plus 1. So, the order of this matrix is m cross n plus 1. Now of course, if you find the rank of this matrix, rank of augmented matrix; it will be always greater or equal to rank of A, rank of A. Why did is always greater then equal to because rank of because augmented matrix is basically having higher order than the than the order of A.

A is simply of order m cross n and you add one more column in the a matrix to get a augmented matrix. So, it has a order m cross n plus 1 which is of higher order then the order m cross n. So, the rank of A will always rank of augmented matrix will always be greater than or equal to rank of A ok.

(Refer Slide Time: 05:49)



Now, so we have seen that rank of augmented matrix will always be greater than or equal to rank of A ok. So, augmented matrix has the order m cross n plus 1; while A matrix has A order m cross n. Now the now there are 2 possibilities number 1, either rank of A augmented matrix is equal to rank of A or rank of augmented matrix is greater than rank of A greater than means not equal to ok. Now if rank of augmented matrix is equal to rank of A, what does it mean? You see, what is the augmented matrix we are having?

Augmented matrix is a 1 by it is a 1 one a 1 2 a 1 n a 2 1 a 2 2 a 2 n and so on, a m 1 a m 2 a m n and this is b vector b 1 b 2 up to b n b n.

Now we have seen that the first case, we are discussing that what happens is the rank of A augmented matrix is equal to rank of A? What does it imply? Whether it imply that system is consistent or whether it imply the system is inconsistent? How can we say? So, let us discuss this thing. We have first found augmented matrix here. Now this is this is matrix A and the rank of A is same as rank of augmented matrix that is number of linearly independent rows or columns, it is having this matrix A is having the same number of rows and columns.

The same maximum number of rows and columns, the entire matrix which is the augmented matrix have; that means, what? or we can say that linearly independent columns, linearly independent columns of matrix A will be same as linearly independent maximum linearly independent columns of augmented matrix, the full matrix and that is possible only when this column, the extra column can be written as linear combination of these columns.

If you are if you are taking this has column 1, this has column 2 and this has column n and this vector we are taking as b suppose and we know that rank of this A matrix is the same as rank of the entire matrix. This means linearly maximum linearly independent columns of matrix A will be same as the linear maximum linearly independent columns of augmented matrix and this means that this is vector b can be represented as linear combination of c 1 vector, c 2 vector up to c n vector; then only it will be then only it will be possible that rank of augmented matrix will be same as rank of A.

So, what does it mean basically? This means this vector b can be represented by a some linear combinations of these column vectors ok. Now what is b? b is simply b 1, b 2 up to b m alpha 1 times, what is c 1? c 1 is a 1 1 a 2 1 and so, on up to a m 1. What is c 2? c 2 is a 1 2 a 2 2 and so on up to a m 2 and what is c n? c n is a 1 n a 2 n and so on up to a m n. So, this means b 1 will be equal to this means there exist some alpha 1 alpha 2 up to alpha n such that b can be expressed as linear combination of this column vectors.

So, now this implies that b 1 will be equal to alpha 1 time say 1 1 plus alpha 2 times a 1 2 plus alpha n times a 1 n. Similarly b 2 can be written as alpha 1 times a 2 1 plus alpha 2 times a 2 2 and so on plus alpha and times a 2 n and similarly the m'th equation. So, what it is basically? It is it is simply A into alpha 1 up to alpha m is alpha n sorry alpha n and it is equal to vector b; that means, in place of x 1, we are having alpha 1; in place of x 2, we are having alpha 2; in place of x n, we are having alpha n; that means, we have shown the existence of the solution of this linear system of equations. Hence we can say that solution exist, if rank of augmented matrix same as rank of A.

(Refer Slide Time: 11:22)

$r(A|b) = r(A) \Rightarrow$ Solution exist (Consistent)

Unique solution

$r(A|b) = r(A) =$ nos. of unknowns

$r(A|b) = r(A) <$ nos. of unknowns $\longrightarrow$ Infinitely many solutions.

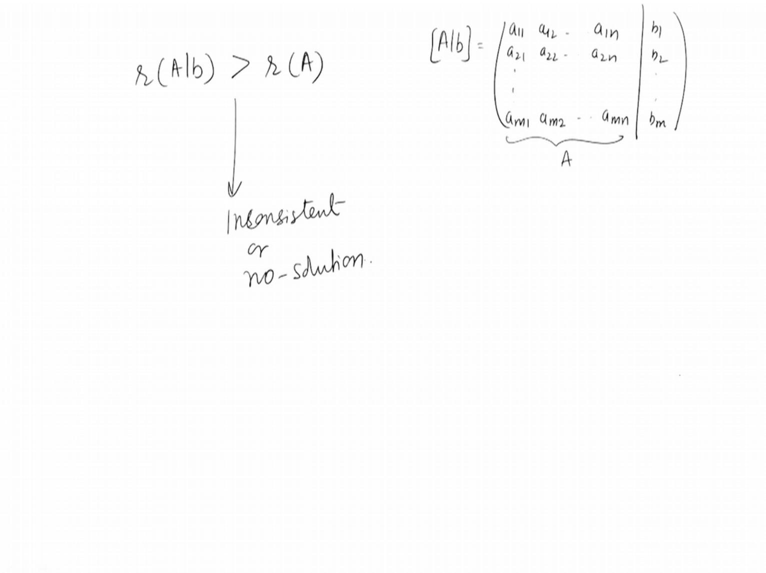
So, we can say that if rank of augmented matrix is equal to rank of A, this implies solution exist or system is consistent we can say.

Now, if system is consistent, there are 2 possibilities; either it has having unique solution or it has having infinitely many solutions. Now if it is having unique solution, so when it will be having unique solution you see? It will be having unique solution only when number of linearly independent equations, linear equations will be same as number of unknowns and that will be possible only when rank of augmented matrix, which is same as rank of A should be equal to number of unknowns.

Then only this means that number of linearly independent equations are same as number of unknowns because rank of A is rank of A means linearly maximum number of linearly independent rows or columns ok. And if it is equal to number of unknowns, this means number of linearly independent equations are same as number of unknowns. This means unique solution.

Now when it will be having infinitely many solutions? If rank of augmented matrix will be equal to rank of A and will be less than number of unknowns ok, then it will be having infinitely many solution. The reason is very simple, you see if rank of rank of A rank of A or rank of augmented matrix are same and it represent number of linearly independent equations and if they are less than number of unknowns this means equation are less and number of unknowns are more; number of linearly independent equations are less and number of unknowns are more, this means infinitely many solutions ok.

(Refer Slide Time: 13:46)



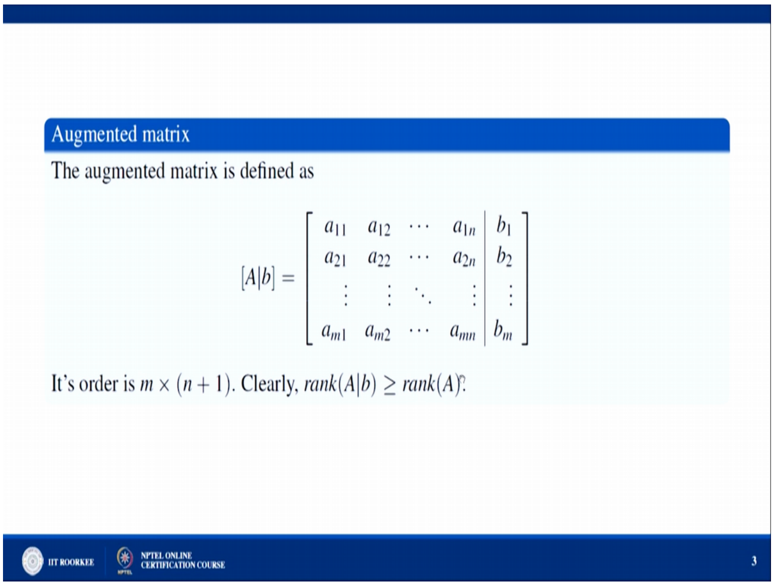
Now, the second case is if the second case is you see, the second case is if rank of augmented matrix is more than rank of A; the second possibility. If rank of augmented matrix is more than rank of A, so what does it mean? You see augmented matrix is what we have discussed. Augmented matrix is a 1 1 a 1 2 and so on up to a 1 n, a 2 1 a 2 2 and so on up to a 2 n. Here it is m'th row is a m n a m 1 sorry a m 2 and so on a m n and here it is b 1 b 2 up to b m.

Now if rank of augmented matrix is not equal to rank of A or rank of augmented matrix is more than rank of A, this means this means in the echelon form of this matrix you see, if rank of this rank of this matrix A is less than rank of this rank of the entire matrix ok. This means this will be possible only when in the echelon form of this matrix A. There is a there is a row which contains all 0 and corresponding v i is not equal to 0 ok. You can you can simply see that if it is 0, the entire row the last row is 0 and this is not equal to 0 then what is the rank of this matrix? Rank will be m for sample and sorry rank will be m minus 1 and here it is m.

So, what I want to say that rank of augmented matrix will be more than the rank of A only when in the echelon form of the augmented matrix, you will be having 1 row entire row having 0 in a and the corresponding b i is not equal to 0 and this means what? This means that that 0 is equal to some non-zero quantity which is not possible. So, system is inconsistent. In this case, system is inconsistent or no solution ok.

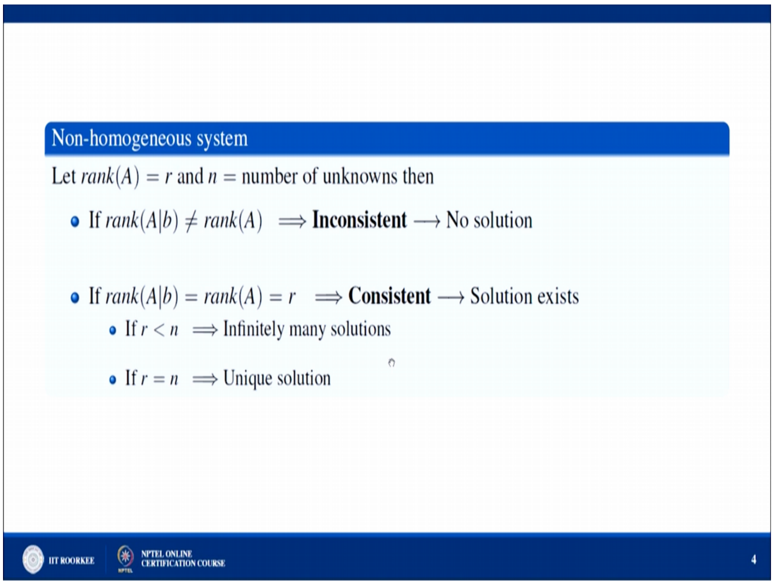
So, let us discuss based on this, let us discuss few things few properties. This we have already discuss that system of non homogenous equation is given by A x equal to b where b is not equal to 0 ok. These are coefficient matrix. This is unknown and this vector is A right hand side.

(Refer Slide Time: 16:28)



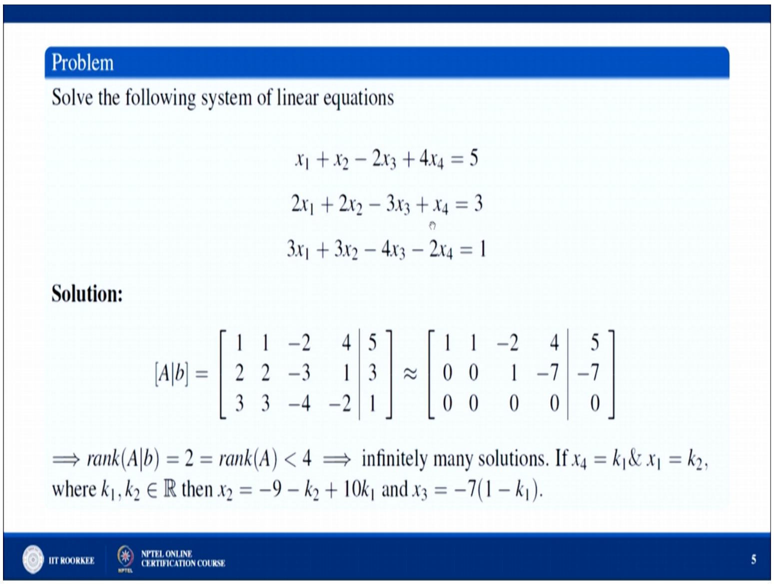
Augmented matrix is defined like this. You see that we first try the matrix A and then the right hand side, this entire matrix is called augmented matrix. Its order is m cross n plus 1 and clearly rank of A plus rank of augmented matrix is greater than or equal to rank of A.

(Refer Slide Time: 16:44)



Now, if you talk about consistency, consistency or inconsistency in the system of linear equations and then we have already discuss that if rank of augmented matrix is not equal to rank of A not equal to me is greater than. This mean system is inconsistent and if rank of augmented matrix is equal to rank of A is equal to r which is r here, then this means consistent and if this r, if this r equal to number of unknowns means unique solution. If this r is less than and this means infinitely many solutions as we have already discussed.

(Refer Slide Time: 17:25)



Now let us discuss few examples based on this ok. The first example is we can see here. We are having here we are having three equations and how many unknowns? We are having we are having 4 unknowns ok. So, x 1, x 2, x 3, x 4; these are the unknowns and these are three equations. So, you first write augmented matrix of this equation ok.

(Refer Slide Time: 17:58)

$$[A|b] = \left(\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{l} x_1 + x_2 - 2x_3 + 4x_4 = 5, \\ x_3 - 7x_4 = -7 \end{array}\right] \sim \left(\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad R_3 \to R_3 - 2R_2$$

$$r(A) = 2, \quad r(A|b) = 2 \Rightarrow r(A) = r(A|b) = 2 < 4$$

So, how can we write? Let us see ok. So, what is the augmented matrix we are having here? You can see, the augmented matrix is basically; here the augmented matrix given by 1 1 minus 2 4. The right hand side is first 5, then 2 2 minus 3 1, then 2, then 3 3 minus 4 minus 2 1. This is the augmented matrix ok.

Now if you see here, the first equation is x 1 plus x 2 minus 2 x 3 plus 4 x 4 equal to 5. The second equation is 2 x 1 plus 2 x 2 minus 3 x 3 plus x 4 equal to 3. Third equation is 3 x 3 3 x 1 plus 3 x 2 plus minus 4 x 3 minus 2 x 4 equal to 1. Now the first important properties is when you apply elementary row operations on the system of linear equations, it will not change the solution. It may change the equations, but the solution say set remain unchanged.

So, you first find the echelon form of this matrix. You see you leave first row as it is. You make 0 in the second row of the first element with the help of the first element, first leading non minus I mean element. So, what a operation you will apply? R 2 r 2 minus twice of r 1. So, so this is 0, this is 0, this minus 2 time this will be 1. This minus 2 time, this will be minus 7. This minus 2 time, this will be again minus 7.

Now in the third row, you will apply the operation third row minus three times first row to make 0 here. So, this minus three time this will be 0. This minus 3 time, this again will be 0. This minus 3 time this that is minus 4 plus 6 is is 2 and this minus 3 time this. So, the it is so, it is minus 2 minus 12 that is minus 14 and this minus 3 time, this is again minus 14 ok.

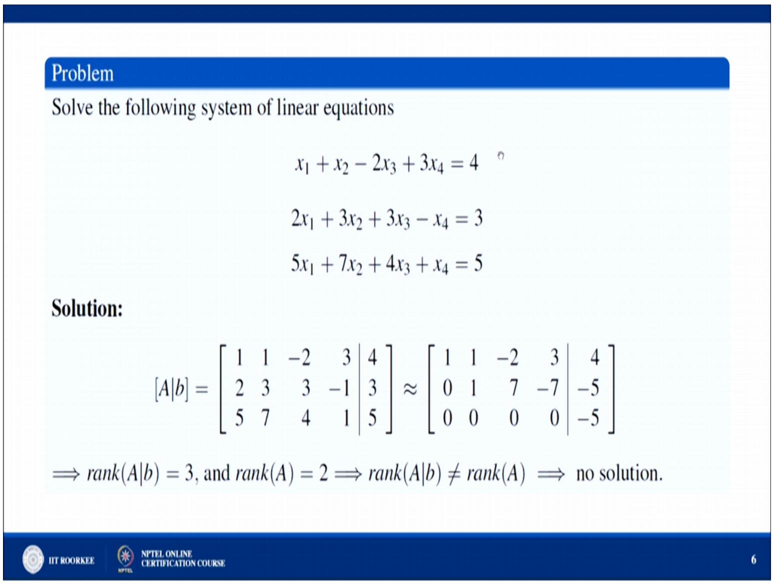
So, now, you will make the first leading non-zero element is second row is 1. You will make 0 below this element. So, how to make 0 here? You will make this minus 2 time this. So, in the next step, the first row remain unchanged. The second row remain unchanged again and the third row, you will take third row minus twice of second row. So, this is 0 0 0 0 and 0.

Now what is the rank of A, rank of this matrix A? This is simply 2; number of non 0 rows is 2 and what is the rank of augmented matrix or the entire matrix? The rank of entire matrix is also 2. So, rank of A this implies rank of A is equal to rank of augmented matrix which is equal to 2. So, rank is equal this means system is consistent. How many unknowns it is having? It is having 4 unknowns. So, it is less than 4. This means it as it is having infinitely many solutions.

Now how you will find the solution? You will arbitrarily because now we are having 2 independent equations. The first question is you can simply see. The first equation from here is what? From here, the first independent equation is x 1 plus x 2 minus 2 x 3 plus 4 x 4 equal to 5. The second equation is x 3 minus 7 x 4 equal to minus 7. So, these are two linearly independent equations and we are having four unknowns.

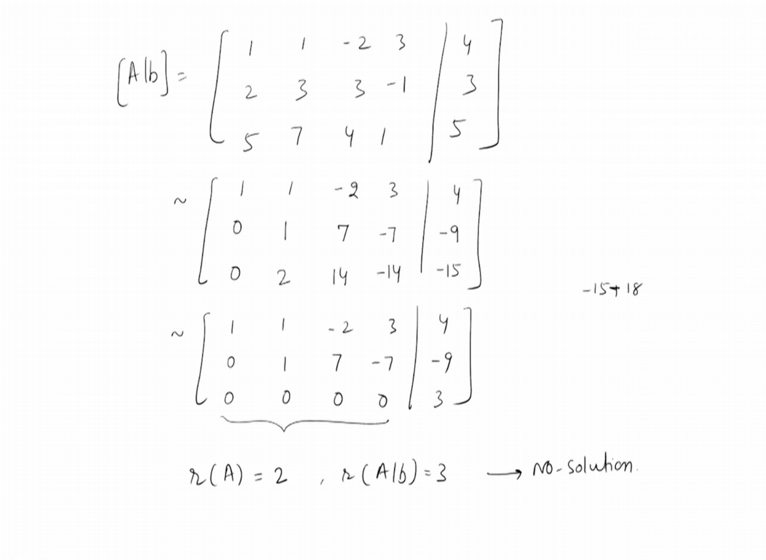
So, you can take 2 variables as arbitrary values and find out the remaining 2 in terms of that ok. So, that is what we have discussed in the solution. You will you can take x 4 as that k 1 x 1 has k 2, then you can find x 2 as this and x 3 as this where k 1 and k 2 are any real numbers

(Refer Slide Time: 22:05)



Now, let us discuss the second problem. The second problem is again having 4 unknowns and 3 equations. So, how can we solve this problem? Again we will find the augmented matrix. Convert the augmented matrix and so, it is echelon form. Find the rank of a rank of augmented matrix and then we can see, whether system is consistent or inconsistent ok. So, let that that is discuss this example also.

(Refer Slide Time: 22:34)

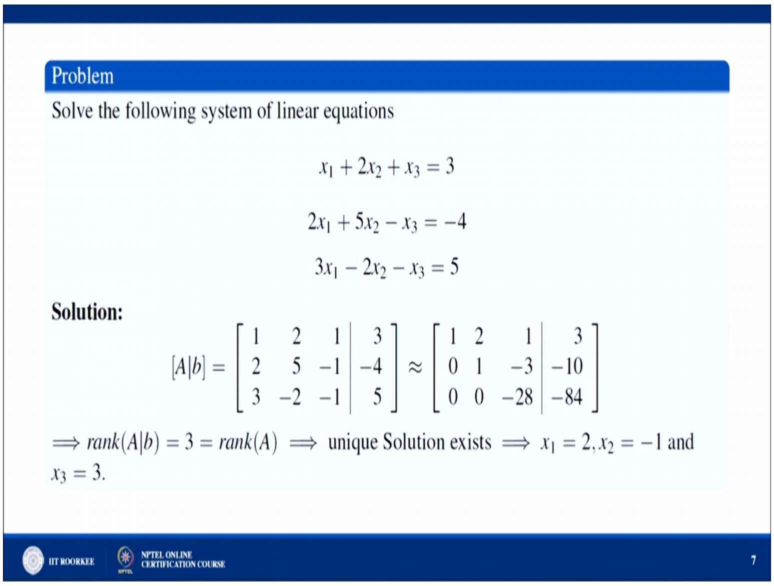


You see, augmented matrix here is what? Augmented matrix here is 1 1 minus 2 3 and 4, it is 2 3 3 minus 1 and 3, it is 5 7 4 1 5. This is the augmented matrix of this problem.

Now you will try to find its echelon form. You leave the first row as it is. Now make 0 here with the help of this. So, this minus 2 times this is 0, this minus 2 times this is 1, this minus 2 time this is 7, this minus 2 time this is minus 1 minus 6 and minus 7, this minus 2 time this is 3 minus 12 3 minus 12 is minus 9 ok. This minus 5 time this so 0. This minus 5 time this is 2, this minus 5 time this is 14. This minus 5 time this that is 1 minus 15 that is minus 14 and this minus 5 time, this is 5 minus 20 is minus 15 ok.

So, what we have obtained here? What we have observed? Now it is not a echelon form. You again you have to make 0 here with help of this. So, how you will make this? Again take this row as it is zero 1 7 minus 7 minus 9. This minus 2 time, this will make 0 0 0 0; however, this is not equal to 0. When you take this minus 2 times this is minus 15 plus 18. So, this is 3. So, here rank of A is 2 ok. This rank is A is 2. How of the rank of the entire matrix is 3? So, rank of A is not equal to rank of augmented matrix, this means no solution ok. So, system is having no solution.

(Refer Slide Time: 24:45)



Now you come to third problem. Here we are having 3 equation with 3 unknowns. Now so, to find out where the system is consistent or inconsistent or it is having many solution or no solution or unique solution, we are first find its augmented matrix ok. This is 1 2 1 3 1 2 1 3 2 5 minus 1 minus 4 2 5 minus 1 minus 4, the second row. The third row is 3 minus 2 minus 1 5 3 minus 2 minus 1 5. Now you convert you find out the echelon form of this matrix which is this. You can easily, you can easily find out the echelon form of this matrix by making 0 here and 0 here and then 0 here ok.

So, clearly the rank of this matrix a, which is this matrix. The rank of this matrix is 3 because number of linearly independent rows it is having r 3 and the rank of the entire matrix is also 3.

So, rank of A is equal to rank of augmented matrix is equal to 3 which is equal to number of unknowns. Unknowns are also 3 so; that means, system is having unique solution. Now how to find that solution? Now to find out that solution is simple. You can simply see you can see, what is the first equation?

(Refer Slide Time: 26:10)

$$x_1 + 2x_2 + x_3 = 3 \Rightarrow x_1 = 2$$

$$x_2 - 3x_3 = -10 \Rightarrow x_2 = -10 + 3x_3 = -1$$

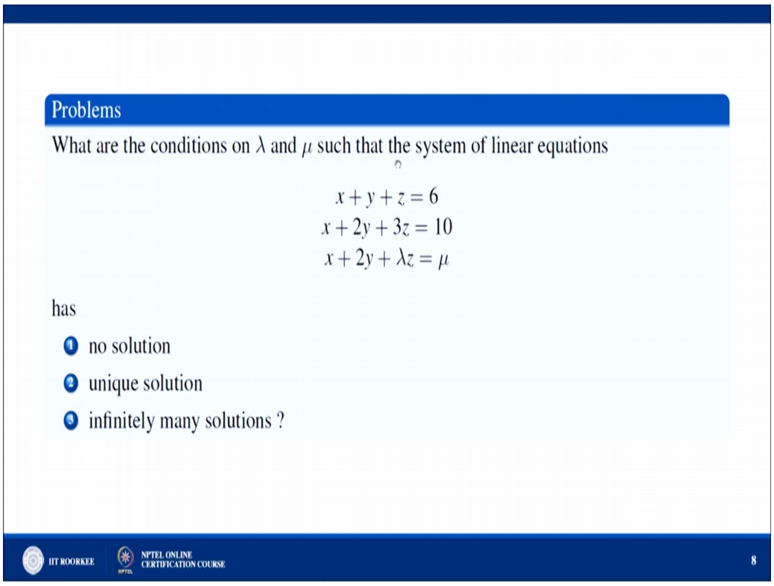
$$-28x_3 = -84 \Rightarrow x_3 = 3$$

The first equation is x 1 plus 2 x 2 plus x 3 equal to 3. We can see the first equation from here x 1 plus 2 x 2 plus x 3 equal to 3. What is the second equation? x 2 minus 3 z 3 equal to minus 10 x 2 minus 3 x 3 equal to minus 10. And what is the third equation? Third equation is minus 2 x x 3 minus 28 x 3 equal to minus 84.

So, minus 28 x 3 equal to minus 84 so; that means, instead of solving from here, we see we solve it from the echelon form of this matrix because we know that when we apply elementary row operation or system of linear equations, it will not change the solution. Solution were remain unchanged. So, what are equations we are having here? The first equation is x 1 plus 2 x 2 plus x 3 equal to 3. The second equation is x 2 minus 3 x 3 equal to minus 10. The third equation is minus 28 x 3 equal to minus 84.

Now we will apply a back substitution. So, how we will apply back substitution? From here we got x 3 as we were simply see x 3 is 3 ok. Now, the substitute x 3 over here; if you put x 3 equal to 3 here, so x 2 will be minus 10 plus 3 x 3 and x 3 is 3. So, it is minus 1. And from here x 1 will be when you substitute x 2 is minus 1; so it is minus 2 minus 2 plus 3 minus 2 plus 3 is 1 3 minus 1 is 2. So, this is the solution of this equation. So, we go to go by back substitution ok. So, this is a solution of this system.

(Refer Slide Time: 28:02)



Now, let us find out, let us solve this problem where we have to find out the conditions on lambda and mu such that system this system is having no solution unique solution or infinitely many solutions; that means, what should be the values of this lambda and mu. So, that this system is having no solution unique or infinitely many solutions. So, how can we see this? How can we find out? So, we again find out the echelon form of this matrix ok.

(Refer Slide Time: 28:38)

$$[A|b] = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \qquad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] \qquad R_3 \to R_3 - R_2$$

(i) No-solution: $r(A) \neq r(A|b)$ or $r(A|b) > r(A)$. If $\lambda = 3$, $\mu \neq 10$, then no-solution.

(ii) Many solutions: $r(A) = r(A|b) = 2 < 3 \Rightarrow \lambda = 3,\ \mu = 10$.

(iii) Unique solutions: $r(A) = r(A|b) = 3 \Rightarrow \lambda - 3 \neq 0$ or $\lambda \neq 3,\ \mu \in \mathbb{R}$.

So, let us le us try to find it. So, what is this augmented matrix of this problem? This will be this is 1 1 1 6, the first equation. The second equation is 1 2 3 10. The third equation is 1 2 lambda and mu ok. Now we have to first find out the echelon form of this matrix. So, how to find the echelon form? You can see it is 1 1 1. First row remain as it is because this is 1, no change I mean this is non-zero.

Now, you will be 0 here with the help of this. So, r 2 in r 2 you will make r 2 minus r 1. So, this is 0, this is 1, this is 2, this is 4. Now you will make 0 here with the help of this. So, in r 3 you will apply an operation that is r 3 minus r 2 oh sorry r 1. So, this is 0 this is 1, this is lambda minus 1, this is mu minus 6 ok. Yesterday it is not a nucleon form of this matrix because it is 1. So, you will be 0 here with the help of this. So, it is 1 1 1 6. It is 0 1 2 4. So, in r 3 you will apply on operation r 3 minus r 2. So it is 0 0 lambda minus 3 and mu minus 10. So, this is the echelon form of this matrix.

Now the first is no solution. For no solution, rank of A should not equal to rank of augmented matrix or we can say that rank of augmented matrix will always be more than rank of A, if it is having no solution. Now the rank of A cannot be less than 2 because we have seen that this is a ok. Now we have seen that there are 2 non-zero rows is it is having. So, rank of A is at least 2 ok. It will be 3, if lambda is not equal to 3 and it will be 2, if lambda equal to 3 ok.

Now you want rank of augmented matrix more than rank of A ok. So, so, if lambda is equal to 3, then rank of this matrix is 2 and we want rank of this matrix as 3. So, this entire matrix rank will be 3, if mu minus 10 is not equal to 0 that is mu should not equal to 10; then only if lambda is equal to 3. This means rank of this matrix is 2 and if mu is not equal to ten; that means, rank of the entire matrix is 3 so; that means, ranks are not same; that means, no solution this is the only possibility ok. So, if this happen, then no solution.

Now second case is many solutions or infinitely many solutions. For many solution rank of A should be equal to rank of augmented matrix and should be less than number of unknowns. Now here unknowns are 3. So, rank must be less than 3 and less than 3 means; either 1 or 2 and we have seen that rank of A is at least 2 so; that means, 2 because you want it less than 3, less than 3 means either 1 or 2 and rank of 2 is at least 2 so; that means, that means equal to 2 ok; that means, equal to 2 or less than 3. This is the only possibility ok. Now this implies, now rank of this is 2; well lambda equal to 3 and rank of this entire matrix is also 2, if mu equal to 10; if mu equal to 10 yeah ok. So, if this happens then, it has it is having many solutions; infinitely many solutions.

Now the third case is unique solution. For unique solution rank of this should be equal to rank of augmented matrix and should be equal to 3. Now rank of A is 3, if lambda minus if lambda minus 3 is not equal to 0 or lambda is not equal to 3. So, if lambda is not equal to 3.

Now if lambda is not equal to 3, the rank of this matrix a is 3 and whatever the value of mu maybe the rank of the entire matrix is also 3 is also 3. So, mu may belongs to r. So, this is the condition on lambda and mu. So, that it is having unique solution. For this values, it is having many solution and for this values it is having no solution. So, this is how we can find out whether the linear system of equations has unique solution, no solution or many solutions ok.

So, in the next class we will see about vector spaces that what vector spaces are what vector spaces are and how they are important for solving some problems so

Thank you very much.