Matrix Analysis with Applications
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Lecture - 40
Polar Decomposition

Hello friends. So, welcome to the last lecture of this course which is on Polar Decomposition. So, again it is a decomposition of a matrix in terms of product of different matrices like in case of singular value decomposition and this particular decomposition is having various applications in different fields of science and engineering. Hence, it is a good way to close this course after introducing you polar decomposition.

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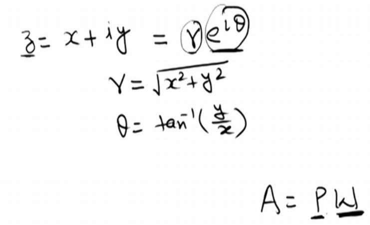

So, what is polar decomposition? From the basic knowledge of complex number you know that if I am having a complex number Z which is having real part as x and then iota times y, y is the imaginary part of this number. Then this number I can also write r into e raised to power i theta, where r is a positive quantity which is square root of x square plus y square and theta is tan inverse y upon x.

So, what I am having I am having a complex number and I am writing this number as the product of these 2 thing r and e raised to power i theta, where r is positive and e raised to power I theta is some sort of rotation type of thing with. So, can we have the same type of thing in case of matrices, means can I write a matrix A as the product of 2 different matrices let us say P and W, where W is some sort of rotation matrix and P is having some sort of positive definiteness or positive semi definiteness type of property. So, this is the idea of polar decomposition.

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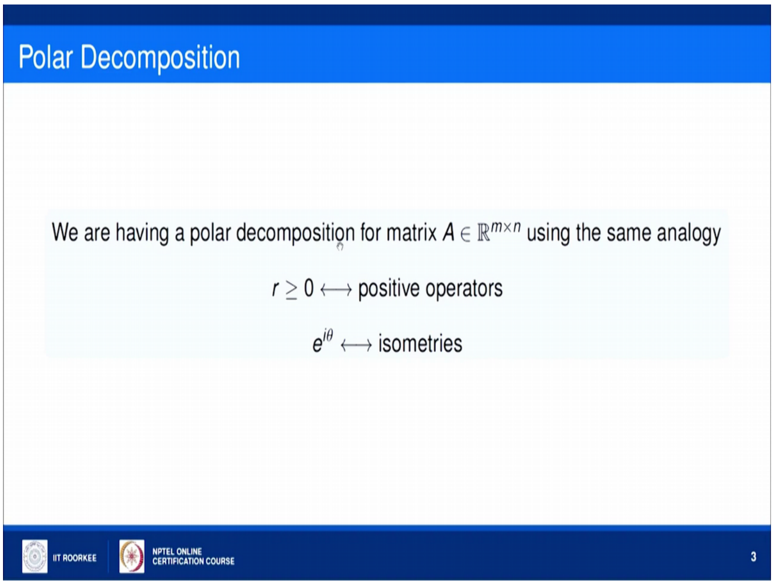

So, what I want to say that we are having a polar decomposition for any matrix a having real entries. Let us say here it is also valid for complex matrix is also of size m by n using the same analogy as we are having in case in case of complex numbers that in case of complex number r is greater than equals to 0 means I should have a positive operators and e raised to power i theta is something similar to isometries or transformations rotational transformations. So, let us learn this.

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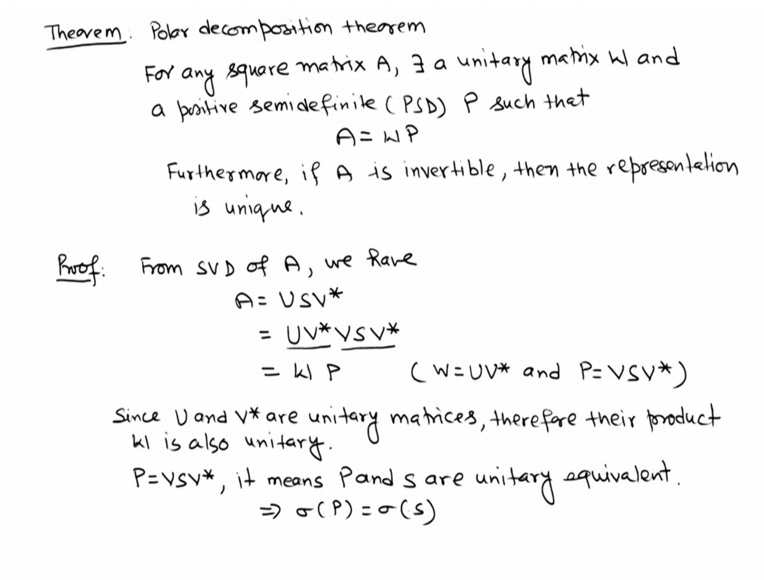

So, let us learn a theorem before going to this and this theorem is called as polar decomposition theorem. So, the statement is something like that let us first restrict up to square matrix then we will discuss the case of rectangular matrix. So, for any square matrix A there exist a unitary matrix W and a positive semi definite matrix, in short let me write it PSD, P such that a can be written as the product of W and P. Furthermore, if A is invertible then the representation or decomposition is unique. Means you can write this unique way as the product of 2 matrices one is positive semi definite and one is unitary.

So, let us see the proof of this. From singular value decomposition of A, we have A equals to USV star, where U and V are unitary matrices and S is a diagonal matrix having singular values at the diagonal, at the diagonal entries. This I can write as U into V star into V because V is a unitary matrix V star V will be identity matrix S into V star. Now, write this U into V star is W and VSV star is P. So, what I am having? W equals to U V star and P equals to VSV star.

Now, since U and V star are unitary matrices therefore, their product W is also unitary. So, we have seen we have saw that the W is unitary what I need to show. Now, that P is a positive semi definite matrix. So, here if you see P equals to VSV star it means P and S are unitary equivalent or in other word I can said they are unitarily similar, they are similar matrices. So, it means the spectrum of P equals to spectrum of S. Means eigenvalues of P is also the eigenvalues of S or vice versa. Now, if you see S, the eigenvalues of S are the singular values of A, and what is that singular values are always non-negative. This implies eigenvalues of P are non-negative and if the eigenvalue of matrix are non-negative it means the matrix is a positive semi definite matrix.

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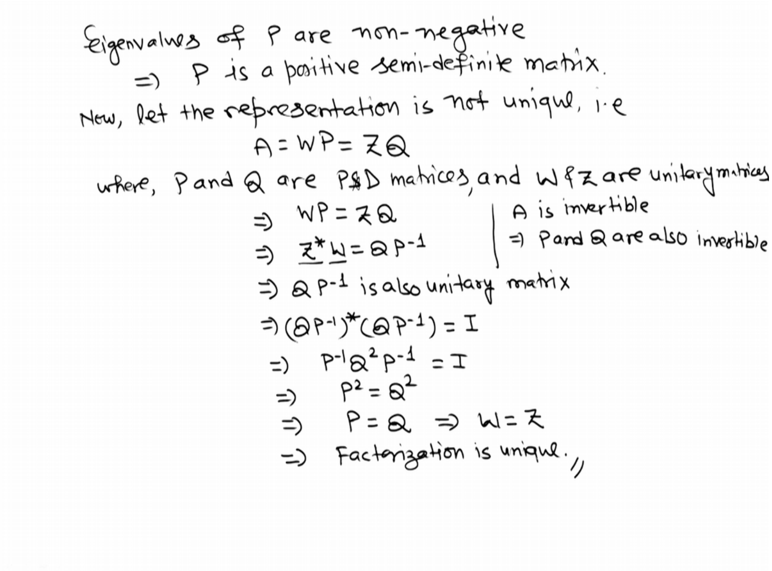

So, in this way we have prove the first part of the theorem that I can write a as the product of the W and P, where W is unitary and P is positive semi definite matrix.

Now, let us do the second that if a invertible then this decomposition is a unique decomposition. Now, let the representation is not unique that is a equals to WP equals to ZQ, where P and Q are positive semi definite matrices and W and Z are unitary matrices. So, what I need to show now, that W equals to Z and P equals to Q then only I can say that representation is a unique representation. Now, here what I am having WP equals to ZQ. So, if I multiply both side by Z star that is complex conjugate of conjugate transpose of Z then what I will be having? Z star into W equals to Q into P inverse, I can write.

Here you just note that, A is an invertible matrix, A is invertible. And what I am having? It means P and Q are also invertible because A is the product of W and P. So, A is invertible means the both the matrices is should be invertible W is unitary. So, it is invertible. So, there P should also be invertible similarly we can say that Q is also be invertible. So, that is why I am saying these means we have put this condition that if A is invertible then only representation is unique.

Now, what I can have? If you see that Z star is unitary matrix, W is also a unitary matrix, it means their product is unitary. So, from here I can say Q into P inverse is also unitary matrix, it means QP inverse conjugate transpose into QP inverse means conjugate transpose of a unitary matrix product itself should be I it means P inverse, if I see this Q square into P inverse equals to I, it means P square equals to Q square.

Now, if you see here P and Q are positive semi definite matrices not even semi definite they are positive definite matrices in this case because A is invertible. So, what I will be having? If P and Q are positive definite matrices then P square equals to Q square implies that P equals to Q. If P equals to Q and P and Q are invertible this implies W also equals to Z, it means the factorization is unique. So, this is the proof of second part of the theorem such kind of factorization of a matrix, that if you can write a matrix if we are writing a matrix as the product of 1 unitary matrix and 1 positive semi definite matrix such kind of factorization is called polar decomposition of a given matrix. So, let us take an example of this.

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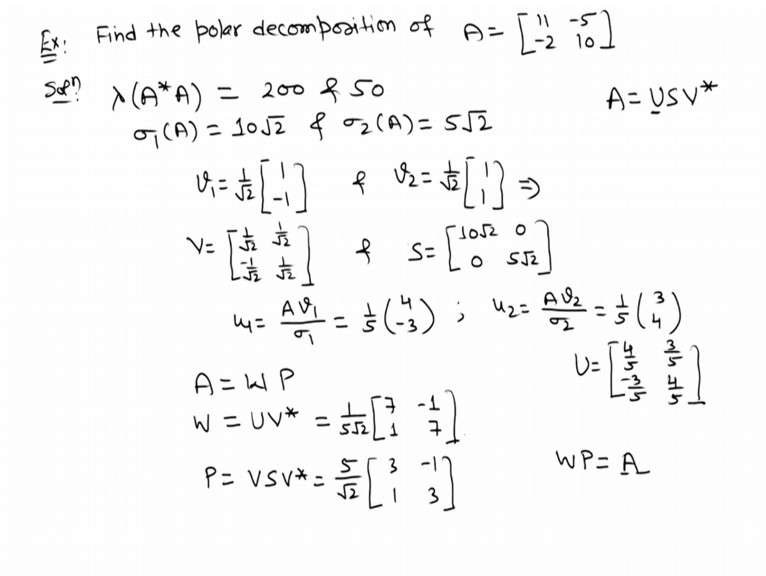

So, find the polar decomposition of A equals to 11 minus 5 minus 2 and 10. So, it means I need to write matrix A as the product of 2 matrices 1 is unitary and another one is positive semi definite.

So, let us first perform the singular value decomposition. So, if I calculate the eigenvalues of A star A these comes out to be 250. So, hence singular values of a the bigger singular value is square root of 200, that is 10 root 2 and square root of fifty means this is the second singular value, so this is 5 root 2. Now, if I calculate the eigenvector of A star A corresponding to eigenvalue 2 hundred what I will find that this eigenvector comes out to be 1 by root 2. So, orthonormal eigenvector, I am writing 1 and minus 1.

Similarly, another eigenvector means eigenvector of a star a corresponding to eigenvalue 50 comes out to be 1 by root 2, 1 1. So, from here I can write the matrix V equals to 1 by root 2, minus 1 by root 2 and then 1 by root 2 and 1 by root 2. So, if the singular value decomposition of A USV star, then here S is sigma 1 that is 10 root 2, 0, 0, 5 root 2.

So, now, what I need to calculate? For completing the singular value decomposition of A, I need to calculate A matrix U. So, my U 1 will become A into V 1 upon sigma 1 and this comes out to be 1 by 5, 4 and minus 3. Similarly U 2 will become A V 2 upon sigma 2 if you can remember it from the singular value decomposition lecture it will be 1 upon 5 3 and 4. So, hence my U becomes 4 upon 5, minus 3 upon 5, 3 upon 5, and 4 upon 5.

Now, I want to perform polar decomposition of A. So, if polar decomposition of A is equal to W into P. Then what in my W according to singular value decomposition these W is U into V star. So, U into V star means U into V conjugate transpose, U is gained by this V star will become transpose of this matrix and this comes out to be 1 upon 5 root 2 into 7 minus 1, 1 and 7; so here W. Hence P will become V into S into V star and this becomes 5 upon root 2 3 minus 1 1 3 and if you multiply W and P. You will get the matrix A, that is the polar this is these are the matrices for the polar decomposition of the matrix A.

So, what we are using? Basically, we are using the singular value decomposition for performing the polar decomposition and as I told you this is the case of square matrices and this is one of the way of doing polar decomposition. Let us talk about general matrices means rectangular matrices.

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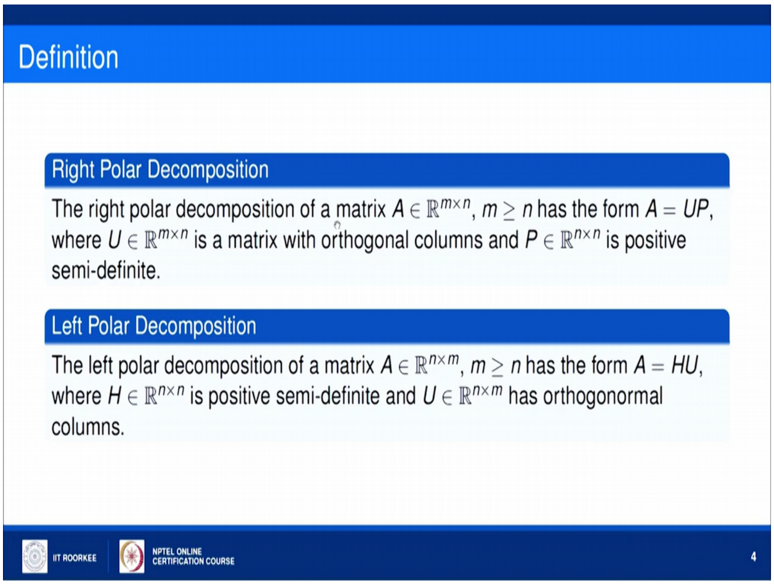

So, there are 2 types of polar decomposition in case of rectangular matrices, the right polar decomposition and the left polar decomposition.

The right polar decomposition of a matrix A which is m by n matrix, where m is greater than equals to n means you are having more number of rows than columns hence the form A equals to U into P, where U is a matrix with orthogonal columns. So, here U is not a square matrix in case of the which we have discussed just earlier, where A is a square matrix both U and P will be square, but if A is a rectangular, A is of size m by n then U will be of size m by n and P will be size of n by n. So, here U is m by n matrix with orthogonal columns means if you take the dot product of each column it will be 0 and P will be a n by n positive semi definite matrix.

In the same way we can define left polar decomposition in case when the matrix is having more number of columns when compared to row. So, if A is a n by n matrix where m is greater than equals to n means you are having more number of columns, then we will be having left polar decomposition and it is H into U, where H is a positive semi definite matrix of size n by n and U is a unitary or orthogonal means U is a matrix having orthogonal columns.

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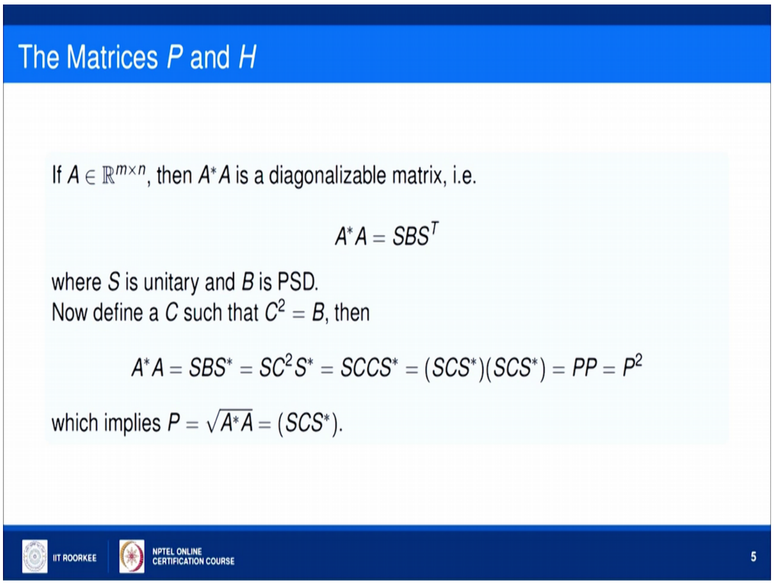

Now, how to perform the polar decomposition of these matrices? So, another way of doing it let me explain to you.

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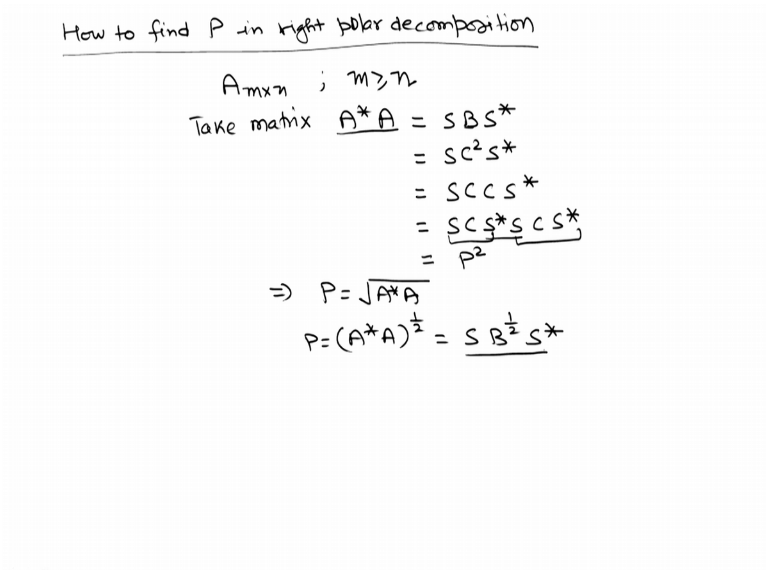

So, how to find P in right polar decomposition? So, what I am having here? I am having a matrix A which is of size m by n and m is greater than n. So, take the matrix A star into a means conjugate product of conjugate transpose of A together with A. Then certainly this matrix will be a diagonalizable matrix and let us write the diagonalization of this matrix as SBS star because it will be a symmetric matrix. So, it will be always diagonalizable. Here B will be a positive semi definite matrix means it will be a diagonal matrix where all entries are greater than equals to 0, it will not contain the negative entries.

So, what I can write if this is the case I can write this B as C square, where C is square root of B means and it is a diagonal matrix. So, it will be square root of each diagonal entries into S star or this can be written as S C C into S star. So, these become S C S star into S into C into S star because S star S star into S is an identity matrix in this case it will be unitary matrices S is a unitary matrix.

So, from here take this as P this will again be P. So, star A will become P square. So, from here I can write P equals to square root of the product of a conjugate transpose with matrix A. How to find out it? You can find out by the diagonalization of A star A because square root of A star A that is P will become raise to power half it will become S into B raise to power half into S star. And here B is a diagonal matrix. So, B raised to power half will be just the square root of each diagonal entry. So, in this way we can find out P in the right polar decomposition. So, let us take an example of this.

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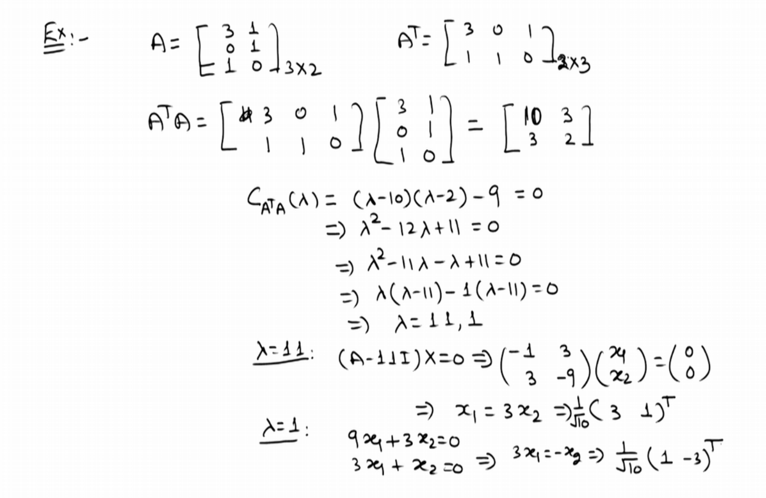

Take A as, let us say take 3 1, 0 1 and 1 0, this 3 by 2 matrix. So, here a transpose will become 3 1, 0 1 and 1 0. So, now, A transpose into A will be a 2 by 2 matrix, so this entry will be 1. So, A transpose is 2 by 3. So, now, a transpose A will become 3 1 0, 1 1 0 into 3 1, 0 1, 1 0.

So, here I will be having like 3 into 0. So, this entry will be 11 as I told you. The product of this row with this column then product of this row with this column will become 3, 3 and then I will be having 2. Let me check one more time this entry will be 10, 3 into 3, 9 plus 1 into 1. So, it is the product of a transpose into A.

Now, if we see the eigenvalues of A transpose A the characteristic polynomial of A transpose A will become lambda minus 10 into lambda minus 2 minus 9. So, this will become lambda square minus 12 lambda plus 11. So, from here what I will be having if this equals to 0, this equals to 0 means what I am having lambda square minus 11 lambda minus lambda plus 11 equals to 0. So, if I take lambda common lambda minus 11 minus 1 common lambda minus 11 equals to 0. So, eigenvalues of A transpose A become 11 and 1.

Now, eigenvector corresponding to 11 will be lambda equals to 11 if I find out the eigenvector it will be A minus 11 I X equals to 0. So, this will give me minus 1 3 3 and then 2 minus 11 will become minus 9 X 1, X 2 equals to 0 0. So, from here what I am having? X 1 equals to 3 X 2. So, from here I got eigenvector as if I take X 2 as 1, X 1 X 2 as 1, so X 1 will become 3 transpose. And if I want to normalize it, so what I have to do? 1 upon root 10 into means 3 upon root 10 and 1 upon root 10. This will be the eigenvector corresponding to lambda equals to 11.

If I calculate it corresponding to lambda equals to 1 it will be 9 X 1 plus 3 X 2 equals to 0 and from second row 3 X 1 plus X 2 equals to 0. So, this gives me 3 X 1 equals to minus X 2. So, from here what I having? The eigenvector is again 1 upon root 10 and if I take X 1 as 1 X 2 will become minus 3 transpose.

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$$A^TA = \underset{S}{\begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix}} \underset{B}{\begin{bmatrix} 11 & 0 \\ 0 & 1 \end{bmatrix}} \underset{S^T}{\begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix}}$$

$$\underline{\underline{P}} = (A^TA)^{\frac{1}{2}} = S B^{\frac{1}{2}} S^T$$

$$= \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \sqrt{11} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix}$$

$$A = W \underline{\underline{P}}$$

$$W = AP^{-1} \checkmark$$

$$\Rightarrow$$

So, what I can write? I can write A transpose A the diagonalization of this matrix is S B into S transpose, where S is given as, where S is given as 3 upon root 10, 1 upon root 10 that is the eigenvector corresponding to lambda equals to 11 and 1 upon root 10 minus 3 upon root 10 which is the eigenvector corresponding to lambda equals to 1. Then at diagonal matrix B 11 0 0 1 into S transpose S transpose will become here 3 upon root 10, 1 upon root 10, then 1 upon root 10 which is similar to S. So, now, P will become a square root of A transpose into A.

So, this will become S into B raised to power half into S star or here I am using transpose, because real matrix is S transpose. So, it will be 3 upon root 10, 1 upon root 10, 1 upon root 10, minus 3 upon root 10 into B raised to power half. So, square root 11 0 0 the square root of 1 will become 1, into S transpose which is 3 upon root 10 1 upon root 10 1 upon root 10 and minus 3 upon root 10. So, this is my matrix P.

Once I will be having my matrix P I am having A, A equals to W into P from here I by making the product of these 3 matrices I will get my P. Then if P is available with me W will become A into P inverse I can calculate my W and in this way I can perform the polar decomposition of the given matrix A. This is the right polar decomposition we cannot make left polar decomposition in this case because if I take in left polar decomposition my matrix A will be written as some Q into Z where Q is the positive semi definite matrix. And in that case this will become 3 by 3 matrix having 1 eigen value 0 and hence the inverse of that positive semi definite matrix will not exist and you cannot calculate the unity matrix like we have done in this case here.

So, this is the alternate way of doing this polar decomposition apart from the singular value decomposition method which we have done in the earlier example. Again we are having couple of example here.

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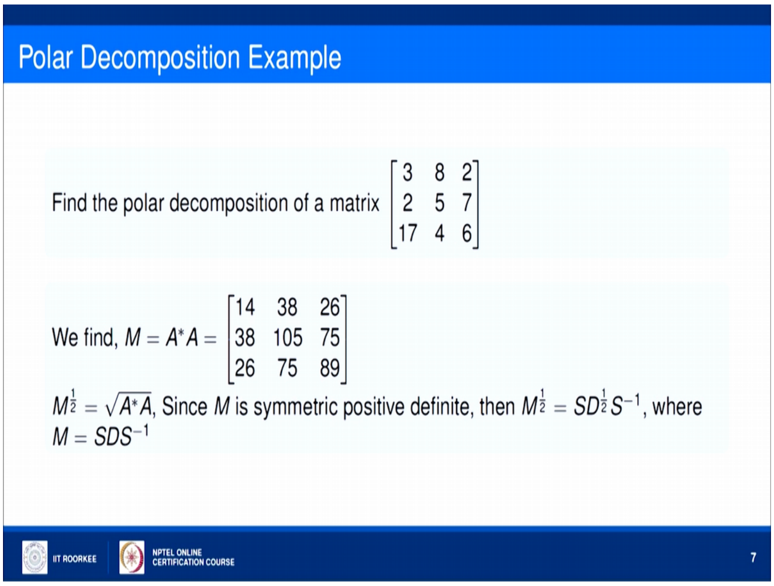

This is again how I have done in the earlier case. This is the matrix, this is A star into A then M half will become A star into A square root and by using the same process I calculate A equals to up, where U is P is given by this one and U is A into P inverse.

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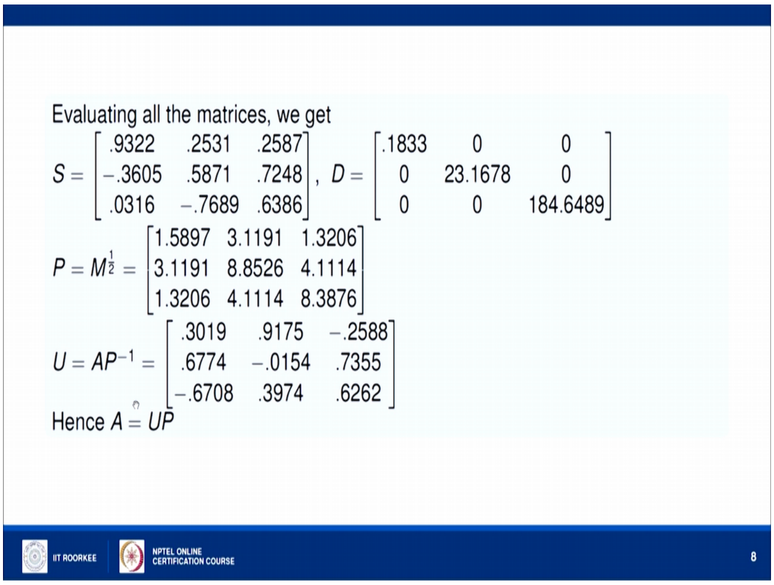

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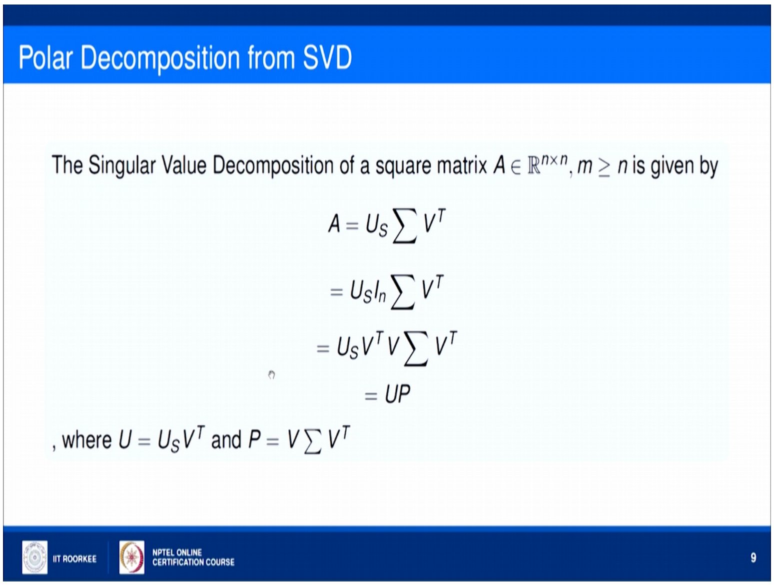

So, another example is like this.

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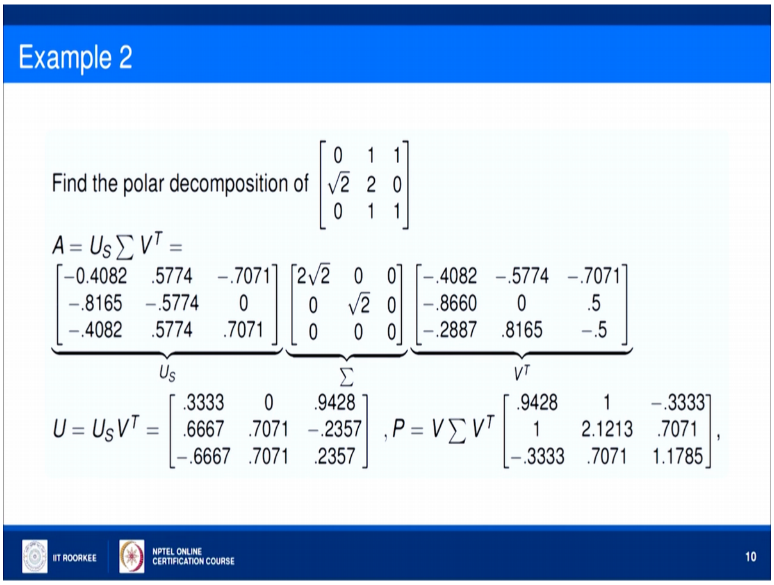

This is again best on the singular value decomposition. So, this is the singular value decomposition of this matrix, then U is us into V star and P is V sigma V transpose. So, by finding these 2 matrices I can perform the polar decomposition of a given matrix.

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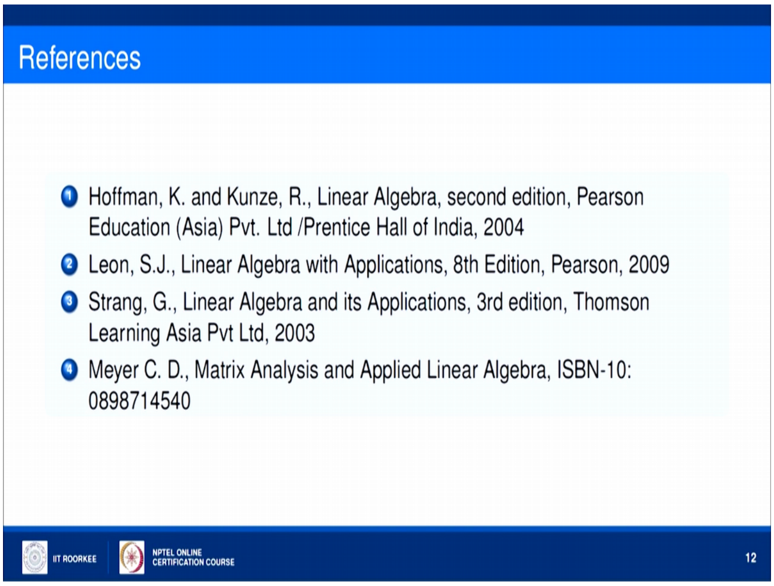

These are the references for this lecture.

Thank you very much.