Matrix Analysis with Applications
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Lecture – 04
System of Linear Equations-I

Hello friends, so welcome to lecture series on Matrix Analysis with Applications. In the last lecture we have seen how we can find out rank of a matrix. I have discussed that rank of matrix is nothing, but number of non-zero rows in that non form of that matrix or it is the maximum number of linearly independent rows or columns of the matrix. We have also seen some of the important properties of rank or the matrix. Now how it is useful to solve system of linear equations we will see in this lecture.

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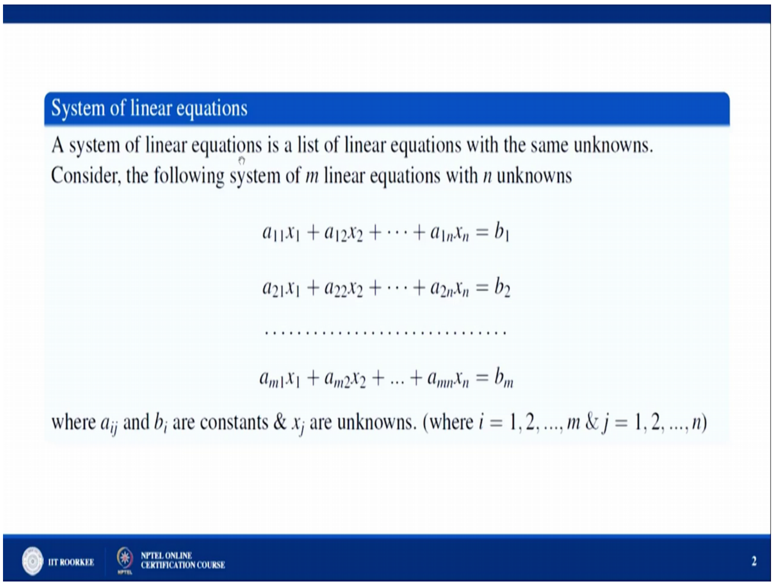

So, what system of linear equations is let us see. A system of linear equations is the set of linear equations or a list of linear equations with the same unknowns. Consider, the following system of m linear equations with n unknowns you see, we are having here we are having m number of equations, our equations are linear having same set of same set of variables x 1 to x n, x 1 to x n ok. So, this system is having n number of unknowns with m equations. So, this here a ijs are known all a ijs are known and the right hand side is also known and we have to find out x js. Here j is varying from 1 to n, these are the unknowns ok.

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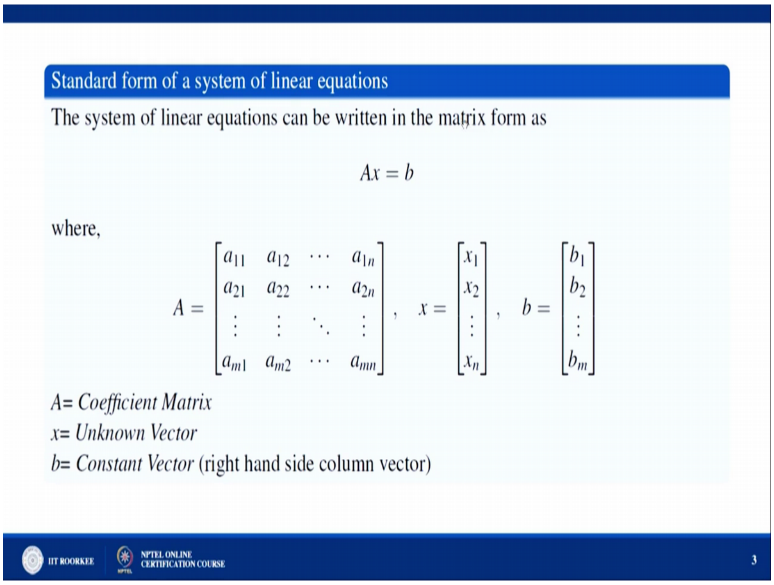

Now this can also be expressed in the matrix form as follows you see, if you write Ax equal to b where A is given by this matrix you see. If you see here the coefficient matrix is a 11 a 12 up to a 1n a 21 a 22 up to a 2n and similarly the m th equation m a m1 a m2 up to a mn.

So, this is the coefficient matrix this x is x 1 x 2 up to x n the unknown vector and this b is b 1 b 2 up to b m this is right hand sight. So, this a is also called coefficient matrix this x is called unknown vector which is to find out, this b is called constant vector or right hand side column vector. So, the matrix representation of a system of linear equation with m unknowns I mean with n unknowns and m equation is given by this expression. Now, system of linear equations are of two types is it is either homogenous or non homogenous.

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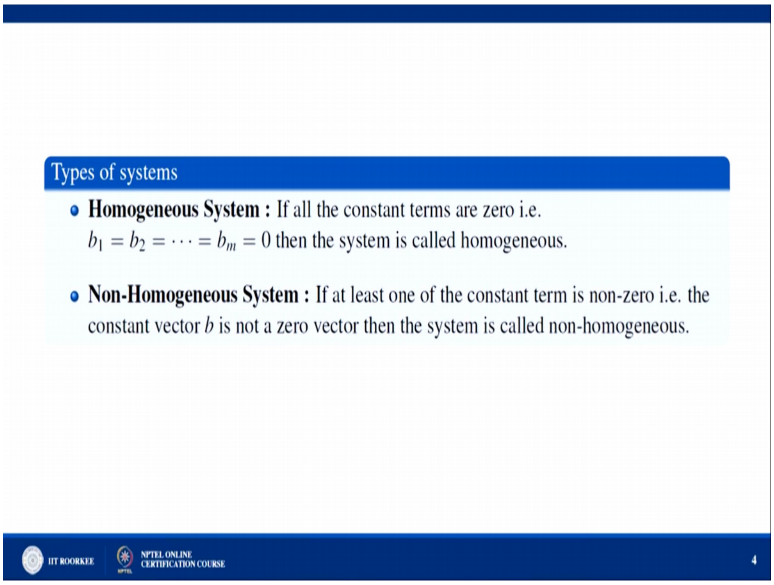

What do you mean by homogenous? You see if the right hand side that is this b, if all b’s are 0 if b 1 equal to 0 b 2 equal to 0 up to b m equal to 0. Then this system of equations are called homogenous system, and if there exists at least one b j which is not equal to 0 then this system of equation is called non-homogenous system of equations.

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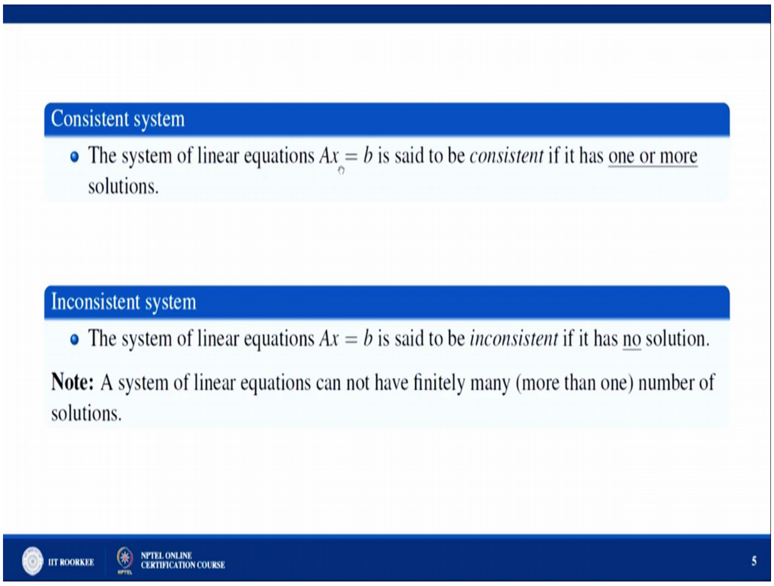

The system of linear equation Ax equal to b is called consistent if it has one or more than one solutions. System of equations are either consistent or in consistent; consistent means it has solution, solution may be one or may be more than one. And inconsistent means no solution ok, no solution. Now, system of linear equation cannot have finitely many more than one number of solutions ok; that means, if you are talking about system of linear equations whether it is homogenous or non-homogenous there are only three possibilities.

The system may either have unique solution that is only one solution, or it has no solution ok, or it has infinitely many solutions. It can never it can never have finitely many solution that is more than one solutions. Why? You can see here you have you have system equation Ax equal to b; where A is the m cross n matrix, x is a vector which is an R n and b is again a vector which is an R m ok.

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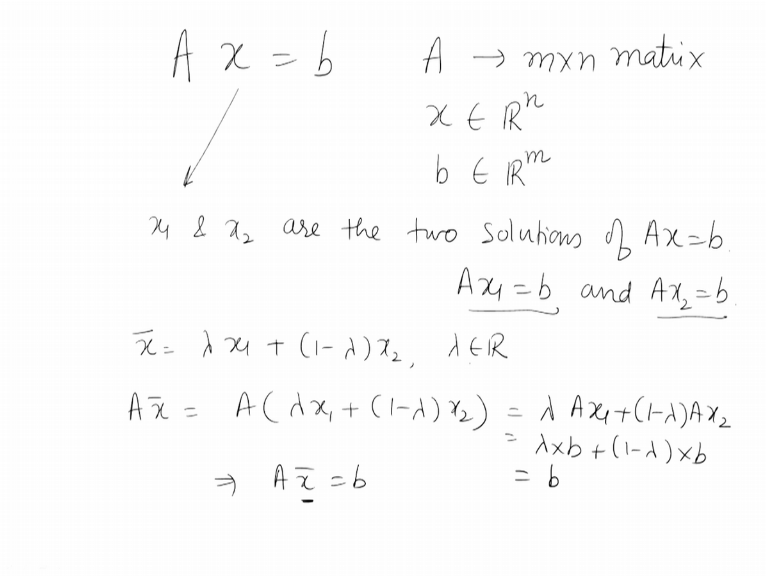

Now, suppose this system has two solutions x 1 and x 2 are the 2 solutions of Ax equal to b. Now, if x 1 and x 2 are the two solutions of this means Ax 1 equal to b and Ax 2 equal to b.

If it is a solution this means it satisfies the system of equations, now if you take any x bar which is lambda x 1 plus 1 minus lambda x 2 where lambda is any real number. If you take any x bar which is defined like this you take Ax bar then Ax bar will be A of lambda x 1 plus 1 minus lambda x 2, which can be equal to lambda Ax 1 plus 1 minus lambda Ax 2 Ax 1 is b here Ax 2 is also b from here.

So, it is lambda into b plus 1 minus lambda into b so it is equal to b. So, this implies Ax bar this implies Ax bar is equal to b; that means, x bar is also the solution of the system of linear equation x equal to b and x bar x bar is simply a lambda x 1 plus 1 minus lambda x 2. So, you vary lambda in R you will get, so many x bar satisfying this equation. So, this equation will be having infinitely many solutions ok. So, it this system can never be having can never have finitely many solutions or more than 1 solutions. It will be having if it has 2 solutions than this may it is having infinitely many solutions ok.

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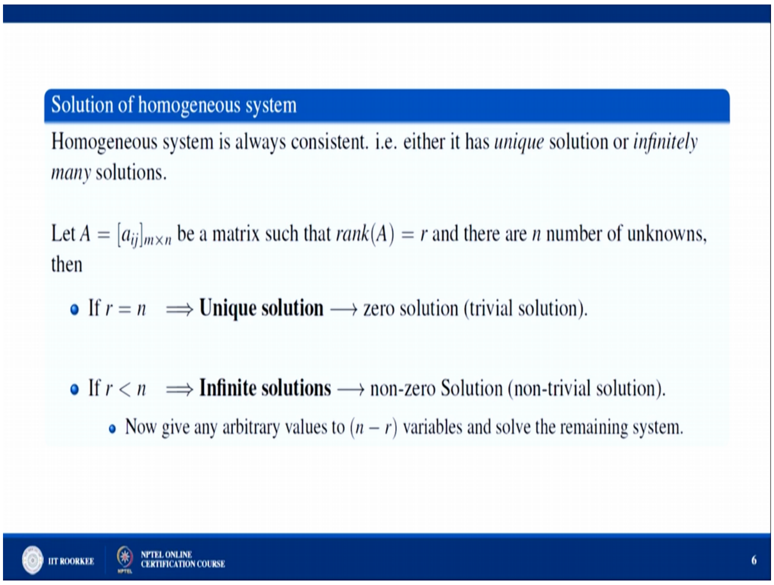

Now, first we will consider how we can find out the solution of homogenous system of equations, homogenous system of equation means the right hand side the 0 ok. Now, the homogenous system is always consistent it will never be having no solution the reason is if you take to Ax equal to 0. So, x equal to 0 which is the trivial solution always satisfy the system of linear equations so; that means, the homogenous system is always consistent. So, only we have only two possibilities either it has unique solution or it has infinitely many solutions ok.

Now, how can we say when we can say that it has unique solution or infinitely many solutions, this unique solution is also called trivial solution which is 0 solution x equal to 0 all x i's are 0 and infinitely many solution is also called non trivial solution, some x i may not be 0. Now how can we find out whether the system has unique solution or infinitely many solutions? So, now, we can use a concept of rank you see let us consider a matrix of order m cross n such that the rank of matrix is r suppose the rank of this matrix is r.

Now, rank of this matrix is r what does it mean, it means that that this coefficient matrix A are having r number of linearly independent rows or columns, because rank is simply means maximum number of linear independent rows or columns. If the rank is r this means this matrix is having r number of maximum r number of linearly independent rows or columns.

Now, here for this coefficient matrix a number of unknowns are n a x equal to b number of unknowns are n. Now, if r is equal to n if r is equal to n; that means, rank of the matrix is n, n is number of unknowns; that means, what that means, you see if you are taking if you are taking Ax equal to 0 the homogenous system, A is the matrix of order m cross n and x is x 1 x 2 up to x n.

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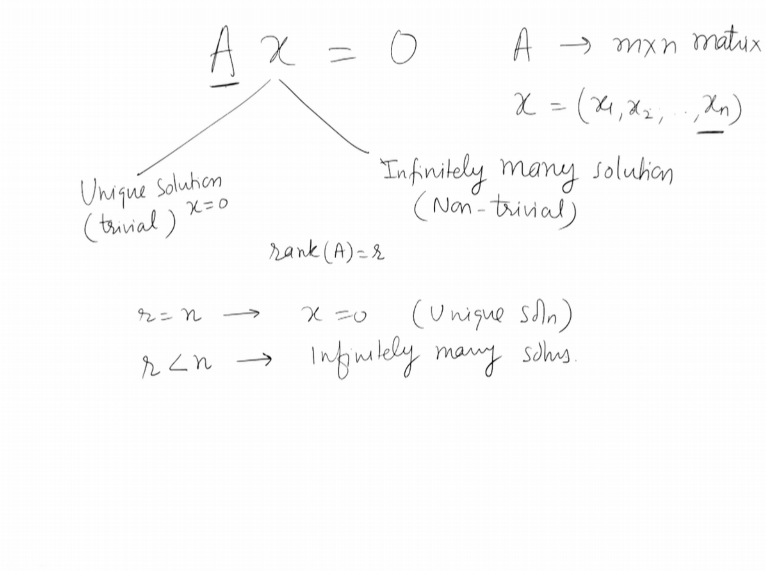

Now, this system is either having unique solution as we have discussed or this system has infinitely many solution. This is also called trivial solution this is also called non trivial solution. A unique solution means x equal to 0, and infinitely many solution means x naught equal to 0 all x are not equal to all xi’s are not equal to 0 ok.

Now, if rank of the matrix is r rank of A is r. So, there are only two possibilities either the rank is equal to n, n is number of unknowns you see n is number of unknowns. Now, if rank is equal to n this means, this means this matrix the rank of this matrix is n this matrix rank is n. This means the maximum number of linearly independent equation r n because the rank is n this means the echelon form of this matrix are having n number of linearly independent rows and; that means, that means we are having n number of linearly independent rows with n number of unknowns. So, it will be having unique solution and which is the 0 solutions or trivial solution.

Now, if r is less than n n r is r is either less than or equal to n because r is less than less than minimum or less than equal to minimum of m or n ok. So, if r is less than n this means numbers of linearly independent equations are less than n that is unknowns are more than number of linear independent equations. So, system will be having infinitely many solutions ok.

So, if r is less than n infinitely many solution that is non zero solution or non trivial solution. Now, how to find out all those infinitely many solutions you see you can put n minus r variables as arbitrary you can choose n minus r variables give arbitrary value to them and then you can solve the remaining system. Now, let us discuss let us see few examples based on this suppose you are having this system ok.

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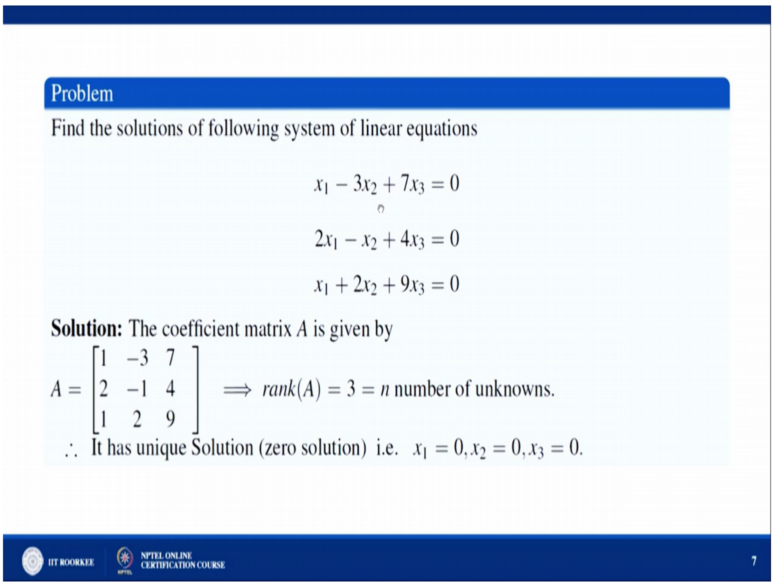

The system of equation is this the coefficient matrix is what it is 1 minus 3 7, it is 1 minus 3 7, second row is 2 minus 1 4 it is 2 minus 1 4, the third row is 1 2 9 1 2 9. Now, how can we see whether this system has unique solution or infinitely many solution because there are only two possibilities because system is homogenous, now this we can find out calculating the rank of this matrix. So, what are the rank of this matrix, so what is this coefficient matrix here is.

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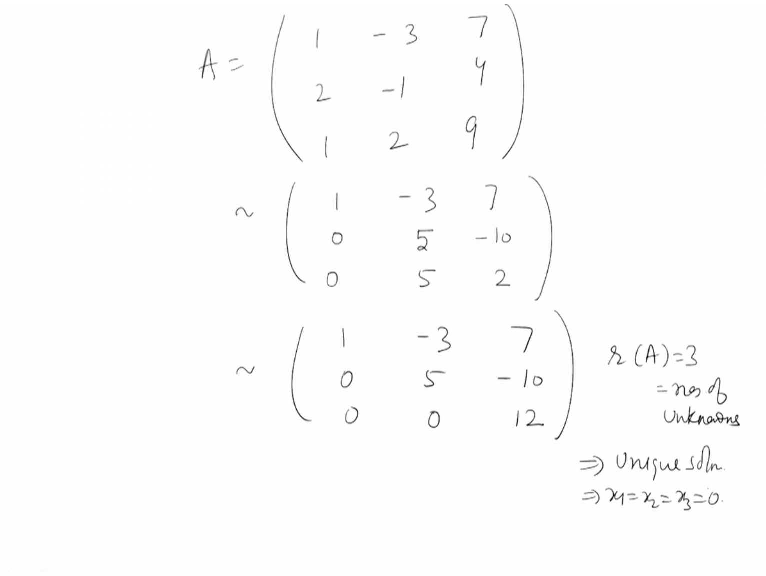

Now, coefficient matrix here is you can see it is 1 minus 3 is 7 it is 2 minus 1 4 1 2 9. Now, let us fine rank of this matrix, convert this matrix and to its echelon form; it is 1 minus 3 7 this minus 2 times this will make 0 here which is 0. This minus 2 time this will be 5, this minus 2 time this will be 4 minus 14 is minus 10 ok. This minus this is 0, this minus this is 5 this minus this is 2. Now, it is 1 minus 3 7 it is 0 5 minus 10, 0 you will make 0 here with the help of this minus this is 0, this minus this is 12. So, the rank of this matrix is 3 which are number of unknowns.

So, this implies unique solution, and unique solution means trivial solution and trivial solution means x 1 equal to x 2 equal to x 3 equal to 0 all x are 0 ok. So, hence we can find out the solution of this system the next problem suppose we are having this problem.

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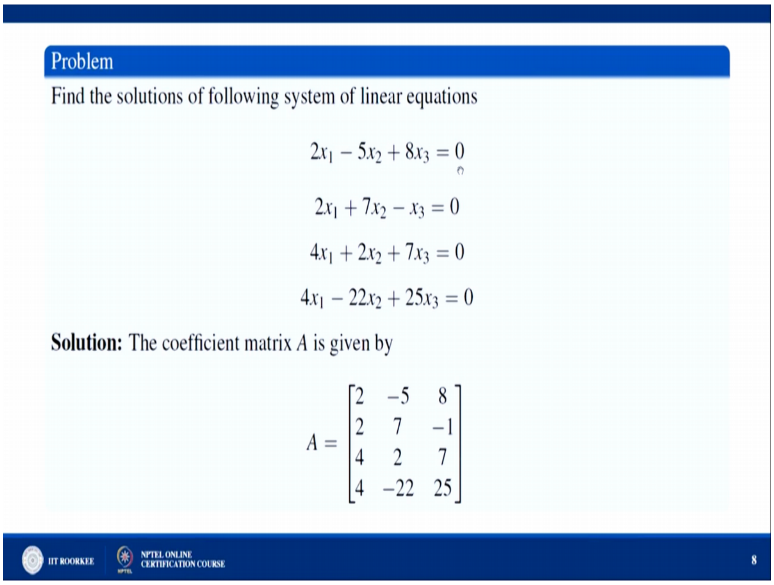

You see here we are having 4 equations with 3 unknowns and we have to see where this system has unique solution or infinitely many solutions. So, you can write out you can write the coefficient matrix, the first row is 2 minus 5 8 the 2 minus 5 8, the second row is 2 7 minus 1 2 7 minus 1 and similarly the other 2 rows; When you find out the rank of this matrix by converting this into its echelon form.

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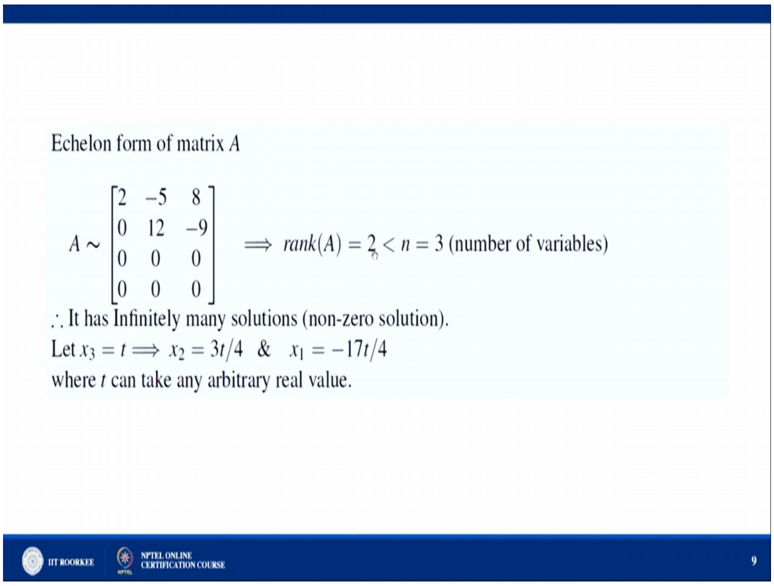

So, rank we can obtain rank as 2, rank as 2 and number of unknowns are 3. So, rank of matrix is less than number of unknowns this means this system has infinitely many solutions, now how can you find out all those solutions you can start with this matrix itself you can start with this matrix what is this matrix is it is 2 minus 5 8.

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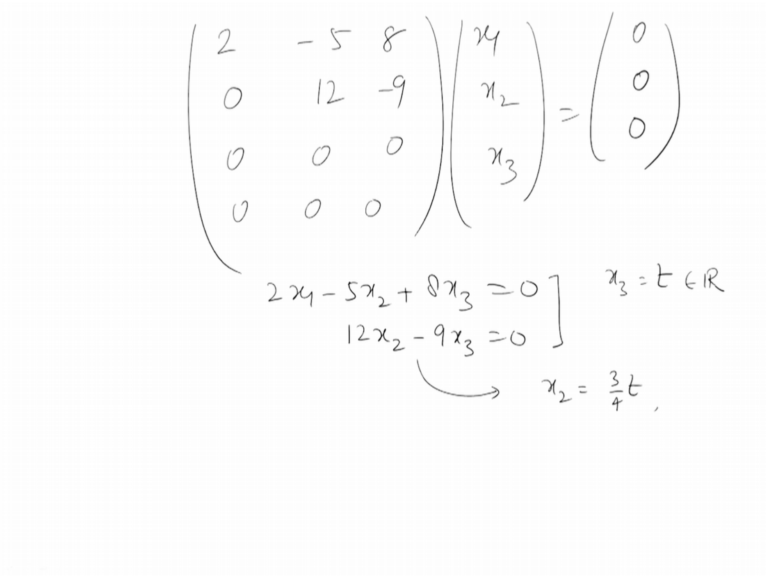

It is 2 minus 5 8 it is 0 and then 12 9 12 minus 9 and 0 0 0 0 0 0, we are having x 1 x 2 x 3 and it is 0 0 0. So, the first linearly independent equation we are having is this equation the second linear independent equation we are having as this equation.

So, you can arbitrary choose we are having 2 equations linear independent equation 3 unknowns. So, 3 minus 2 is 1, so we can put 1 of the variable as give arbitrary value to it and find out the values of other 2 variables. So, let x 3 equal to t, so you can find x 2 x 2 will be simply 3 by 4 times t from here and you can substitute x 3 and x 2 in this equation and find x 1. So, here t is any real number arbitrary value, so in this way we can find out x 1 x 2 and x 3.

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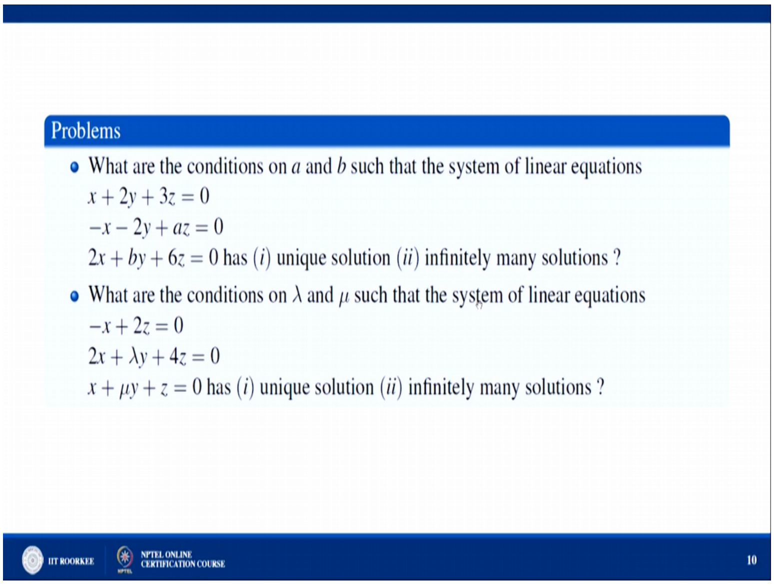

So, now let us solve these two problems what are conditions on a and b such that the system of linear equation this has unique solution or infinitely many solutions. We are to find out the conditions on a and b such that this system has unique solution or infinitely many solutions. So, let us try this problem now what is the coefficient matrix here.

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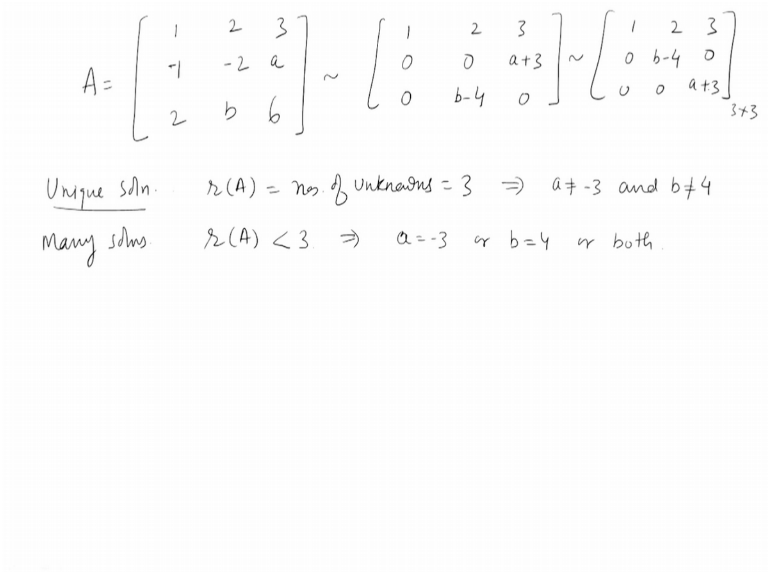

The coefficient matrix is minus it is 1 2 3 it is 1 minus 1 minus 2 a and it is 2 b 6 this is the coefficient matrix. Now to find the conditions on a and b for which the system has unique solution or infinitely many solutions, we first find out the echelon form of this matrix. If there then we can impose the condition on a that when the rank of the matrix is equal to 3 because if rank of the matrix is 3, this means is equal to number of unknowns this means unique solution, and then when the matrix rank of the matrix is less than 3 and that case it is it is having infinitely many solutions.

So, first let us try to find out the echelon form of this matrix. So, you see it is 1 2 3 will remain the same you will make 0 here with the help of this. So, it simply this plus this plus this is 0 the same operation you will apply on the entire row this plus this is 0 this plus this is a plus 3, you will make 0 here with the help of the first row. So, this minus 2 times this will make 0 here 0 this minus 2 times this will be b minus 4 this minus 2 time this will be 0, now it is it is the 0 element it is the 0 element. So, it is better to interchange these 2 rows it this element may or may not be 0 depends on the value of b, but this is always 0. So, better to interchange these 2 row to find out the echelon form of this matrix.

So, it is 1 2 3 it is 0 b minus 4, it is 0 it is 0 0 a plus 3 we have interchange these 2 rows. Now for unique solution for unique solution rank of a must be equal to number of unknowns, number of unknowns there are 3. So, rank of a must be 3 rank of a 3 means for this 3 cross 3 matrix rank will be 3; that means, there is no row containing all 0. So, so this must be non-zero and this also must be non-zero then only this will be this row will not be 0 and this row will not be 0, if anyone become 0 then the rank will be less than rank will be less than 3. So, this implies a should not equal to minus 3 and b should not equal to 4 then only rank of this matrix will be 3.

So, for unique solution we are having these two conditions, now for many solutions infinitely many solutions rank of a must be less than 3. Now rank a less than 3 that mean there exist at least 1 row containing all 0 elements. So, this may be having that this is 0 this is not equal to 0 this is 0 this is not equal to 0 or both of them are 0. So, we can write either a equal to minus 3 or b equal to 4 or both, then the rank then the rank of this matrix will be less than 3.

If both are continuous satisfying the rank is 1, if 1 of the condition satisfy the rank is 2 if both the cases the system is having infinitely many solutions ok. Now, similarly if you if you if you try to find out this system then how can we solve this system let us see what is the coefficient matrix here.

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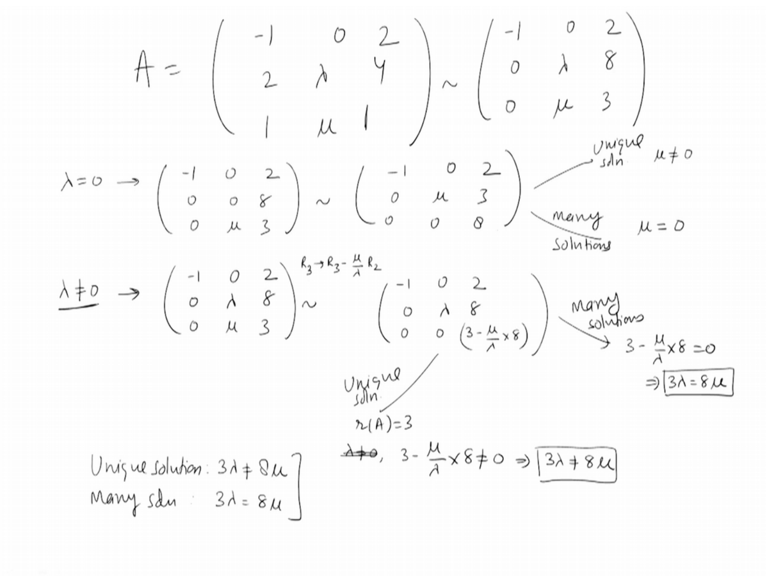

Coefficient matrix here is minus 1 0 2 it is 2 lambda 4 it is 1 mu 1. So first find the echelon form of this matrix again, so remain leave this row as it is this plus 2 times this is 0, this plus 2 times this is lambda this plus 2 times this is 8, this plus this is 0 this plus this is mu this plus this is 3. Now, we have to find out the condition, now if you want to if I want to make 0 here with the help of this.

So, this lambda may or may not be 0, so, I have to take two condition basically. So, basically if lambda equal to 0 then matrix a will be the echelon form will be this 0 to 0 0 8, 0 mu 3 and this will be minus 1 0 2 you can interchange these 2 rows 0 mu 3 0 0 8. Now, this will be having unique solution unique solution, if rank of this matrix is 3 and rank of this matrix will be 3 when mu is not equal to 0 then only then only rank will be 3 ok. So, mu should not equal to 0 and for many solutions, for many solutions rank mu should be 0 because of mu is 0 if mu is 0.

So, you can make 0 here with the help of this. So, rank will become 2 and if mu is not equal to 0, so rank will be 3. So, one condition is over now if lambda is not equal to 0 then it will be minus 1 0 2 0 lambda 8 0 mu 3. Now, you can apply 1 row operation here you can replace R 3 by R 3 minus mu by lambda times R 2.

So, what you will obtain it is minus 1 0 2 it is 0 lambda 8, it is 0 0 it is 3 minus mu by lambda times 8, because you want to convert this into its echelon form. Now, for unique solution how you will get the unique solution. If rank of this matrix is 3, if rank of this matrix is 3 number of unknowns, a rank will be 3 if lambda is not equal to 0, and this is not equal to 0 rank lambda is not equal to 0 is already here. So, we do not need this condition. So, 3 minus mu upon lambda into it should not be 0, so this implies 3 lambda should not equal to 8 mu and for many solution in this case.

For many solution this must be 0 because we want rank less than 3 ok. So, 3 minus mu upon lambda into 8 must be 0. So, this implies 3 lambda should be 8 mu, now if lambda equal to 0 the first condition if lambda equal to 0, you see for unique solution mu should not equal to 0. If a substituted here lambda equal to 0 then mu is not equal to 0 that is coming from this condition itself.

And if lambda equal to 0 for many solution mu should be 0 that is coming from here. So, we can we can combine these two conditions and you can simply say that for unique solution, 3 lambda should not equal to 8 mu and for many solution infinitely many solution 3 lambda should be equal to 8 mu.

So, if this way we can find out the conditions on lambda and mu such there are system will equations are having unique solution infinitely many solutions ok. So, in this lecture we have seen that how can we solve homogenous system of equations using the rank approach. You find the rank of the matrix, if you see that rank of the matrix is equal to number of unknowns this mean system is having, the homogenous system is having unique solution. If rank or matrix is less than n less then number of unknowns this mean system is having infinitely many solutions. In the next lecture we will see that how can we solve system of non-homogenous equations.

Thank you.