Matrix Analysis with Applications
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Lecture - 38
Introduction to Positive Matrices

Hello friends. So, welcome to the 38th lecture of this course. So, title of this lecture is Introduction to Positive Matrices. So, in past 6 lectures we have talk about the different iterative methods for solving linear systems, but this lecture is different from the previous thread. Here, we will see a special type of matrix and we will see the relation between that matrix and we will try to find out that what are the natures of its eigenvalues and eigenvectors. So, let us first define the positive matrix.

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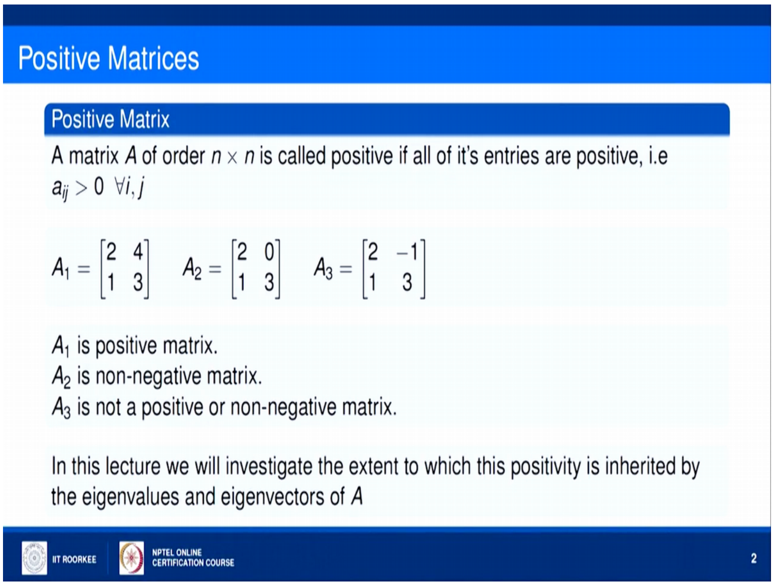

So, a matrix A of order n by n is called positive if all of its entries are positive that is a ij is greater than 0 for all i j. So, here we are having 3 different matrices. So, if you see A 1 all the entries of A 1 are positive. So, it is satisfy this definition and here a 1 is a positive matrix

If you see A 2 the 3 entries of A 2 are positive, but this entry is 0. So, hence it is not a positive matrix, but if all the entries of a matrix are non-negative like in this case then the matrix is called non-negative matrix. So, what we can say every positive matrix is also a non-negative matrix, but reverse is not true. If you see the third matrix A 3 here you can see this entry is minus 1. So this matrix is neither positive nor non-negative. So, basic aim of this lecture is to investigate the extent to which this positivity is inherited by the eigenvalues and eigenvectors of positive matrix A.

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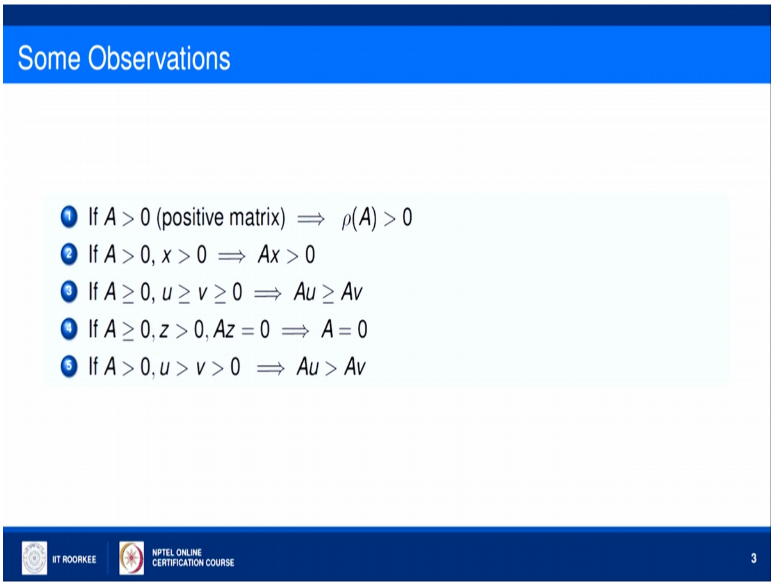

So, now, let us discuss some properties of positive matrix. So, the first property is if A is a positive matrix then the spectral radius of A will be greater than 0, means if it is a positive matrix then spectral radius cannot be 0.

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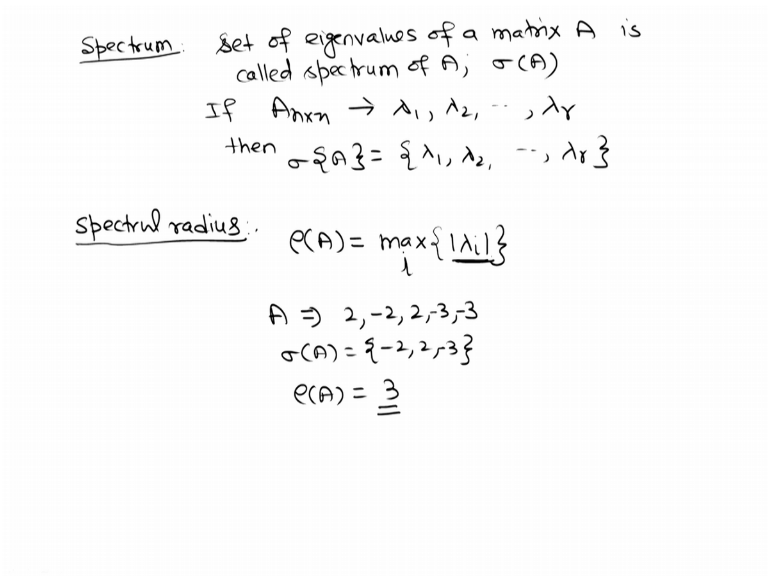

And the simple justification of this is there are 2 things one is spectrum. So, set of eigenvalues of a matrix A is called spectrum of A and it is denoted as sigma of A. So, if A n by n matrix is having eigenvalues as lambda 1 lambda 2 lambda r, why it is up to r because r is less than n because some of the eigenvalues may be repeated then the spectrum of A is simply the set lambda 1, lambda 2 up to lambda r, some of lambda r maybe negative.

The other thing is spectral radius the spectral radius of A matrix is denoted by row of A and it is maximum among all I such that absolute values of the eigenvalue. So, for example, if a matrix A is having eigenvalue 2, minus 2, 2, 3, 3 or then it is a 5 by 5 matrix then spectrum of A is simply minus 2, 2, 3 and here spectral radius of A is 3. So, if it is having entries like minus 3 and minus 3 then the spectrum is minus 2 to n minus 3 and, but the spectral radius will remain 3 because here we are taking the absolute value. So, this is all about the definitions of spectrum and spectral radius because these 2 terms we will use quite frequently in this lecture.

So, first thing is if A is positive, if A is a positive matrix then spectral radius of A will be greater than 0. So, a simple justification of this is if the spectrum of A will contain only the 0 eigenvalues or a spectrum of contain only 0 then only the spectral radius of A can be 0 otherwise it is not because we are taking the absolute value.

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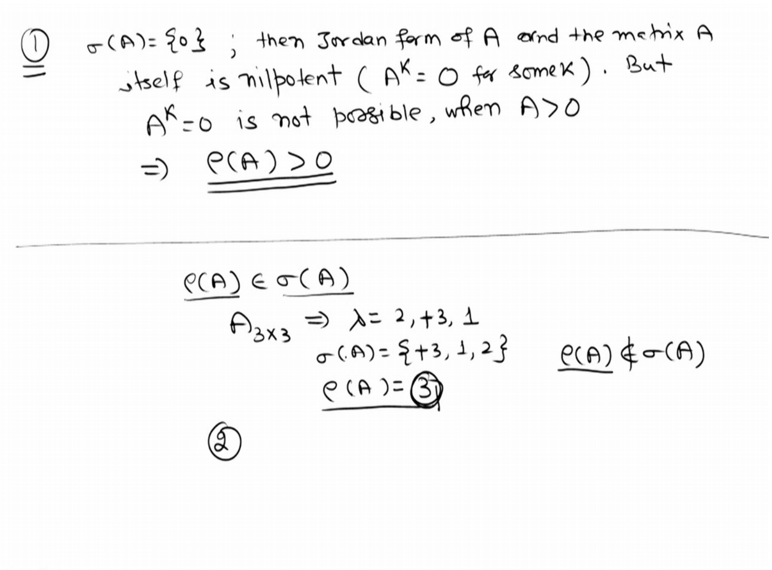

So, if it is this case then Jordan form of A and the matrix A itself is nilpotent, nilpotent means a raised to power K is a 0 matrix for some positive K. So, it is K it is called nilpotency of index K. But a raised to power K equals to 0 is not possible, when A is a positive matrix because all the entries are positive they are multiplying with positive entries. So, they will be grow they cannot be 0 hence spectral radius of A cannot be 0 and it will be positive only for the positive matrices. So, this is the justification for first formula.

My second remark is or second result is if A is a positive matrix and x is a positive vector then Ax will be a positive vector. The third result is if A is a non-negative matrix and u is greater than equals to v greater than equals to 0 u and v are the vectors then Au will be greater than equals to Av.

The next result is if a is non-negative and Z is a positive vector and Az equals to 0 will be only in the case when A is 0. The next result is if A is greater than 0 u is strictly greater than v are 2 positive vectors then Au will be strictly greater than Av. So, these are some basic results we will use further.

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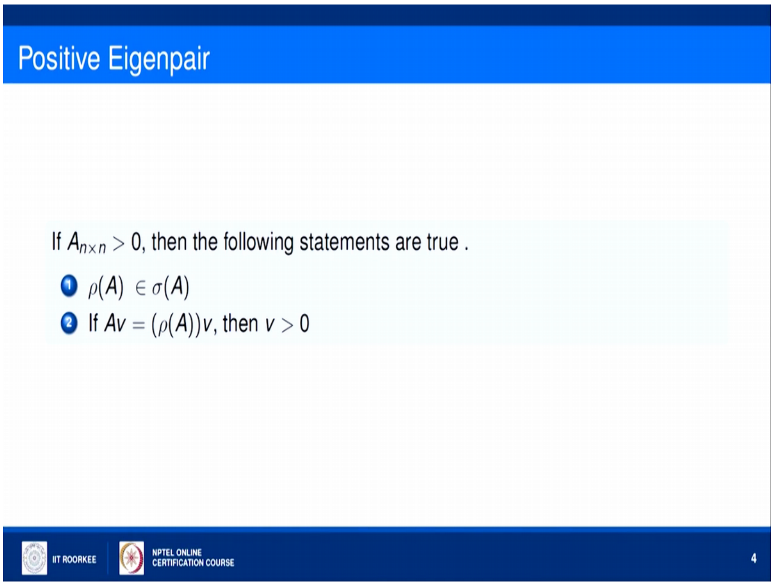

Now, let us talk about positive eigenpair. So, if A be a n by n positive matrix then the following statements are true number one, the spectral radius of A will lie in the spectrum of A and the second is if Av equals to the spectral radius of A into v then v will be a positive vector means the eigenvector corresponding to the spectral radius of A will be a positive vector. So, it means the first condition here first result is spectral radius of A in spectrum of A if let me justify it or let me explain it we are saying that a spectral radius of A belongs to spectrum of A. So, consider a matrix A of order 3 by 3 having eigenvalues as lambda equals to 2 minus 3 and 1 in this case spectrum of A is minus 3 1 2 and spectral radius of A is 3 the maximum of the absolute values of eigenvalues.

So, for this matrix A spectral radius of A does not belongs to spectrum of A, but from that result we are saying that the spectral radius of A will be always in spectrum of A means whatever will be the spectral radius that will be a that will be an eigenvalue of A means like in this case if eigenvalues are 2 3 1 then spectrum is 3 and spectral radius is 3. So, here in this case this line is spectrum. So, this is the explanation of that particular thing.

And the second condition is saying that the eigenvector corresponding to this eigenvalue which is the spectral radius of A will be a positive vector. So, let us prove it.

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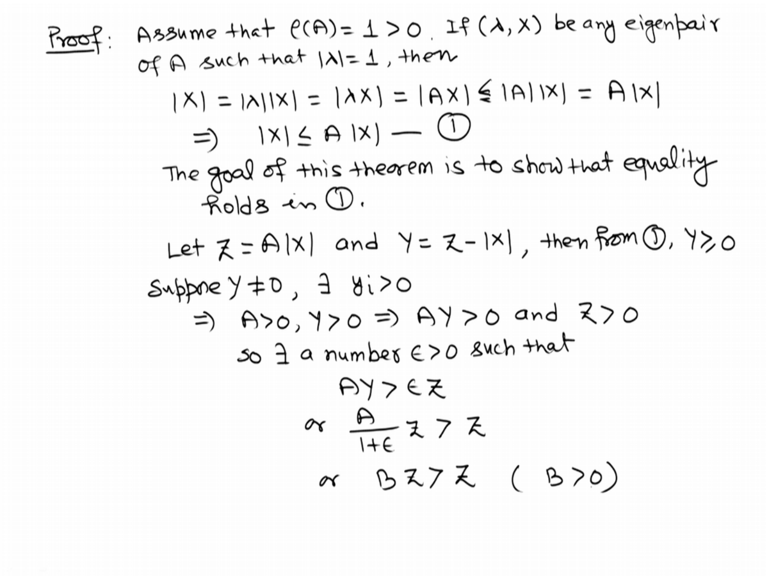

So, for simplicity assume that spectral radius of A is 1 which is positive because here we are taking the positive and that we can have because if a spectral radius of A is r which is a positive number then spectral radius of A upon r will become 1 and A upon r will be again a positive matrix. So, here we are taking spectral radius as fix that is 1.

Now, if lambda X be any eigenpair of a such that the absolute value of lambda is 1 then what we will be having? Mod X equals to mod lambda into mod X because absolute value of lambda is 1 this equals to mod of lambda X this equals to AX because lambda X equals to AX since lambda A and X is an eigenpair of a and this is less than equals to this quantity. And since A is a positive matrix, so I can write in this way and mod of X. So, this gives me that mode of X is less than equals to a into mod of X. Let us write it equation number 1.

Now, the goal of this theorem is to show that equality holds in 1. So, for showing this let Z be a vector which is A into mod X that is the right hand side of 1 and define Y is the difference of Z minus mod X then from 1 because Y equals to Z minus mod X. So, Y should be greater than equals to 0.

Now, here we have to show that equality holds. So, we have to show that Y can be 0. Now, suppose Y is not equals to 0. So, we are proving it by taking some method of contradiction. So, Y is not 0 means there exist Y is not a 0 vector. So, there will be some component y i which is positive. Now, there is a positive matrix Y is a positive vector this implies from the results given in previous slide that AY will be a positive vector and if AY is a positive matrix vector Z will also be a positive vector.

So, there exist a number, in fact, positive number epsilon such that AY will be greater than epsilon of Z, epsilon may be any positive number greater than 1 or less than 1 because these are 2 positive vectors. So, we can find such a number. Or A upon 1 plus epsilon into Z is greater than Z or let us write this matrix A upon 1 plus epsilon as B which is again a positive matrix because 1 upon 1 plus epsilon is a positive entity number and you are dividing each entry of a by this. So, B Z will be greater than Z and as I told you B is again a positive matrix.

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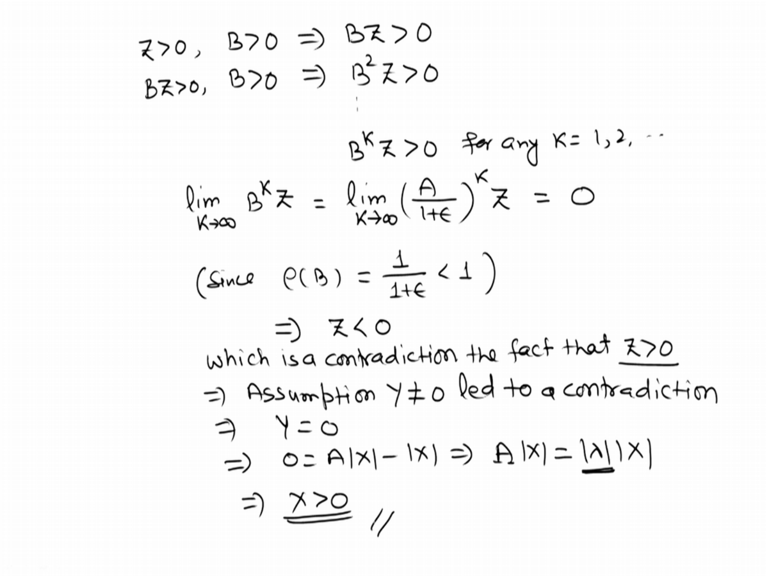

Now, Z is greater than 0 B is greater than 0 this implies the vector B Z is a positive vector. Now, B Z is a positive vector B is a positive matrix this implies B square Z is also a positive vector. In this way we can have that B K Z will be a positive vector for any K equals to 1 2 and so on.

Now, if I take limit K tending to infinity B K Z which is limit K tending to infinity a upon 1 plus epsilon raised to power K into Z because B equals to a upon 1 plus epsilon and this equals to 0. Why it is 0? Because if you recall that the spectral radius of A is 1 and so spectral radius of B equals to 1 over 1 plus epsilon which is less than 1. And so, it is a number which is less than 1. So, raised to power K, where K is tending to infinity will be 0.

So, if this is the case then what we are getting here that Z is less than 0 which is a contradiction the fact that we assumed that Z is greater than 0. This means assumption Y not equals to 0 led to a contradiction. It means Y can be 0 means equality holds that was the thing which we need to prove. If Y is 0 it means 0 equals to A mod X minus mod X which implies A into mod X equals to mod of lambda into mod of X, this is the spectral radius because it is equal lambda equals to 1 it means the eigenvector corresponding to spectral radius of A positive matrix will be a positive vector. So, this is the proof of this particular result.

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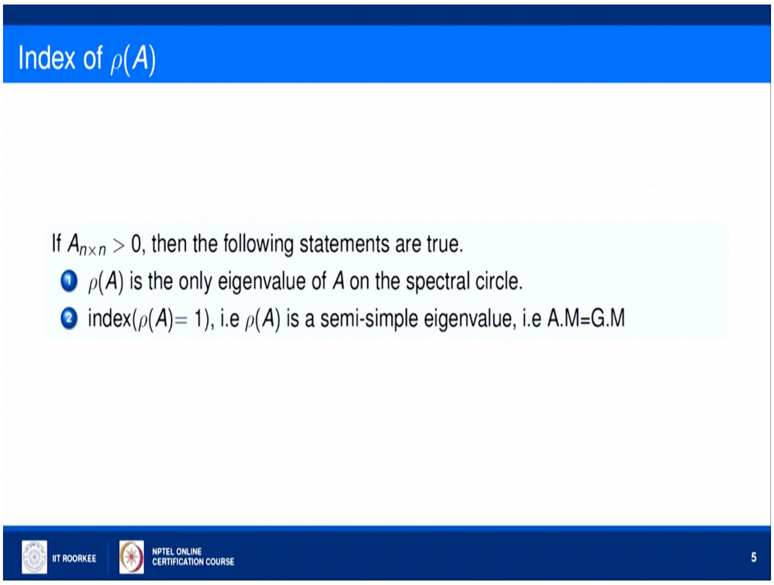

My next result is on the index of a positive matrix. So, if A be a n by n positive matrix then the following statements are true. The first one is the spectral radius of A is the only eigenvalue of A on the spectral circle. Spectral circle means gershgorin circle the second result is index of spectral radius of equals to 1, means whatever eigenvalue be the spectral radius the index of that is 1. And index means here in the Jordan canonical form of a the size of largest block corresponding to that particular eigenvalue. So, that is called a index. So, if lambda is an eigenvalue the index of lambda is the size of the; or dimension of the largest block in the Jordan canonical form of A.

The other one is the spectral radius of A is a semi simple eigenvalue. An eigenvalue is called semi simple if its algebraic multiplicity equals to geometric multiplicity. So, we are not taking prove of proof of these 2 results, but we will take an example on these results.

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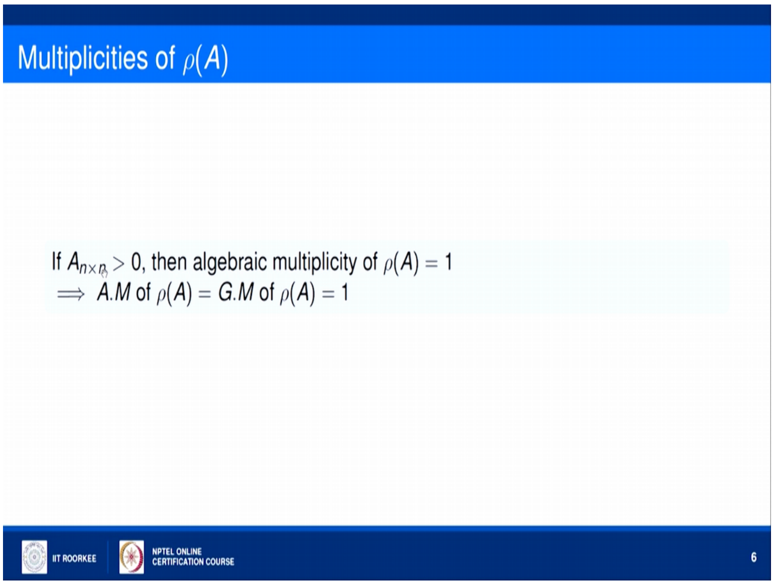

The other results on the positive matrices are if A is a square matrix of order n which is a positive matrix then algebraic multiplicity of spectral radius of A equals to 1. And means its geometric multiplicity is also 1 because it is a semi simple eigenvalue hence algebraic multiplicity of spectral radius of A equals to geometric multiplicity of spectral radius of A and this is equals to 1. So, let us take an example of that.

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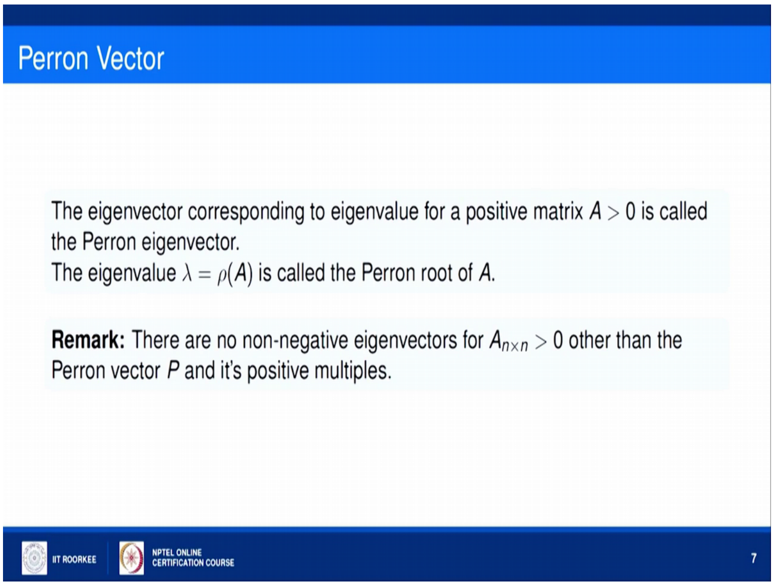

Before that, let us define the Perron vector.

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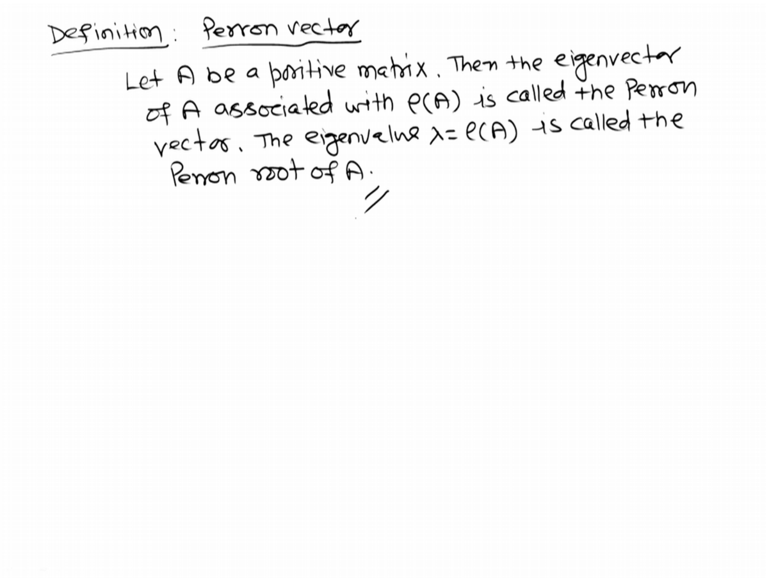

So, I am having a definition here. Let A be a positive matrix then the eigenvector of A associated with the eigenvalue which is the spectral radius of A is called the Perron vector. We have already shown that this Perron vector will be a positive vector. The eigenvalue lambda equals to spectral radius of A is called the Perron root of A. So, this is the definition of Perron vector.

Now, if A is a positive matrix there are no non-negative eigenvector for A, other than the Perron vector means all other vectors will not be non-negative, there will be some negative component in the column vector those are the eigenvectors.

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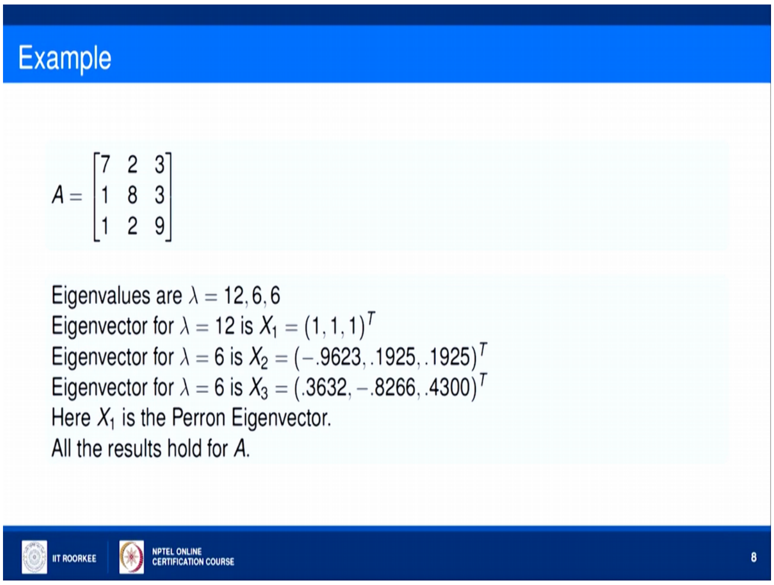

So, now let us take an example to explain this in a better way. So, it is a 3 by 3 matrix which is a positive matrix you can see all the entries are positive. So, it is a positive matrix now if I calculate the eigenvalue of this matrix then these are 12, 6 and 6. So, here the spectral radius of A is 12.

Now, the eigenvector for lambda equals to 12 is 1 1 1 and here you can see this vector is Perron vector for matrix A and as we have already proved that this particular vector is a positive vector. Moreover, in the last slide I have told you that no other eigenvector will be non-negative. So, if you see the other 2 eigenvectors X 2 and X 3 both of them will be having at least 1 negative component. So, they are now non-negative.

The other result is that the only spectral radius of A will lie on the spectral circle only Perron root will lie on the spectral circle. So, here Perron root is 12 and if you draw the spectral circle of A, then we are having A equals to 7 2 3, 1 8 3, 1 2 9.

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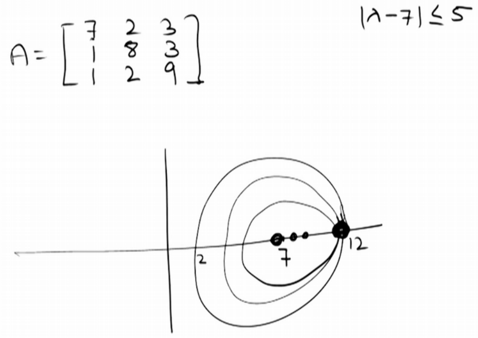

So, if I draw the spectral circle of a then the first circle is lambda minus 7 is less than equals to 5. So, this is the diagonal pivot element. So, it is here less than equals to the absolute value sum of rest of the element in first row. So, this is basically a circle having centre at 7 and radius as 5, so 7. So, 12 and 2 here, and this is the centre at 7.

The other circle the second will be centre at 8 and radius is 4. So, it will go like this. And the third circle is centre at 9 and radius is 3, so it will be like this. So, this all circle will intersect here at 12 and which is our spectral radius or Perron root we satisfy the claim that the only Perron root will lie on the spectral circle.

So, in this lecture I have talked about positive matrices and some of their properties in terms of eigenvalues and eigenvectors.

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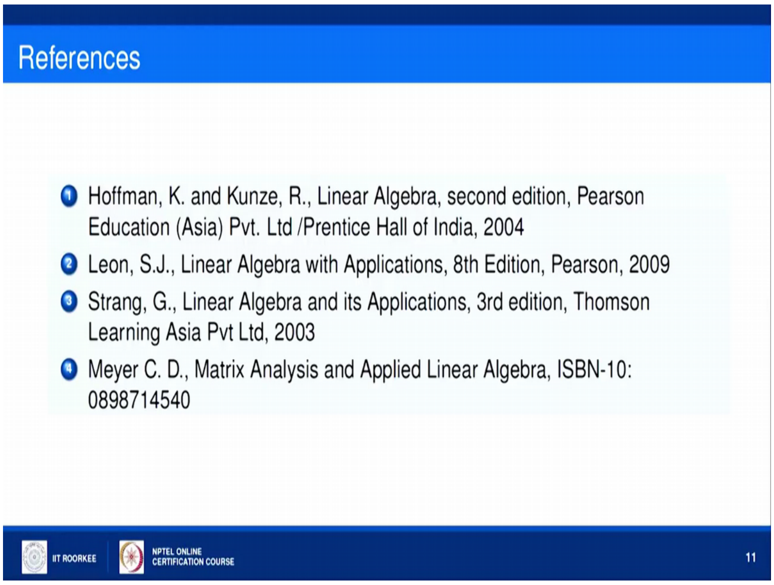

These are the references. In the next lecture we will talk about non-negative matrices.

Thank you very much.