Matrix Analysis with Applications
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Lecture – 34
Non Stationary Iterative Methods: Steepest Descent I

Hello, friends. So, welcome to the lecture on Non Stationary method Iterative Methods. As you know, in the previous couple of lectures we have discussed stationary iterative methods, those includes the Jacobi method, then Gauss-Seidel method and successive over relaxation.

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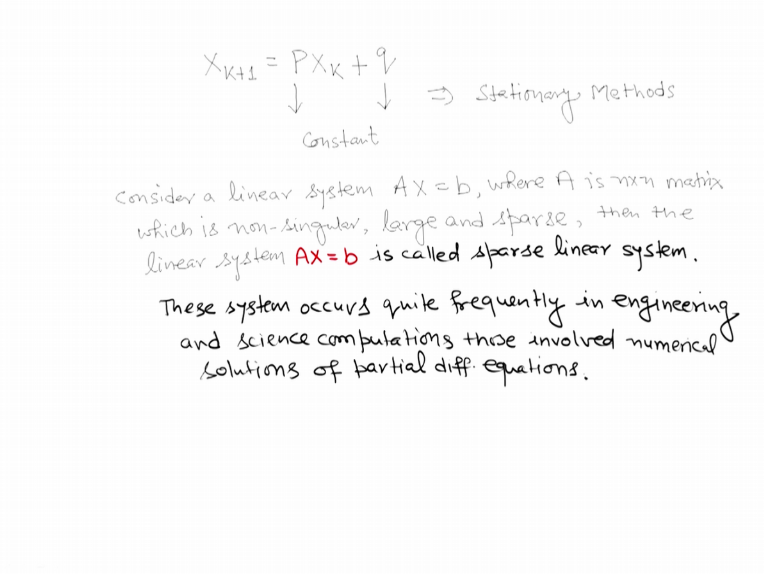

Why we say them the stationary method because the iterative equations for those methods was given like this X K plus 1 equals to an iteration matrix P which is of the same size as the coefficient matrix into X K means the value of the non vector at k-th iteration plus q. So, in Jacobi, Gauss-Seidel as well as successive over relaxation method we have seen that these two means the iteration matrix and the column vector q both are constant throughout the iterations, means we have calculated them once and then we did not change them. So, that is why we told them stationary method.

Today we are going to discuss non stationary methods. In general we use these non-stationary methods for solving large and sparse linear system. So, consider a linear system AX equals to b, where A is a n by n matrix which is non-singular large means n is quite large and sparse I will tell you what we mean by sparse matrices then the linear system AX equals to b is called sparse linear system. The stationary method non-stationary methods those we are going to discuss in next couple of lectures are quite useful for solving such type of sparse system.

Basically, these systems these means large and sparse occurs quite frequently in engineering and science computations those involved numerical solution of partial differential equations, means especially when you are applying finite difference method.

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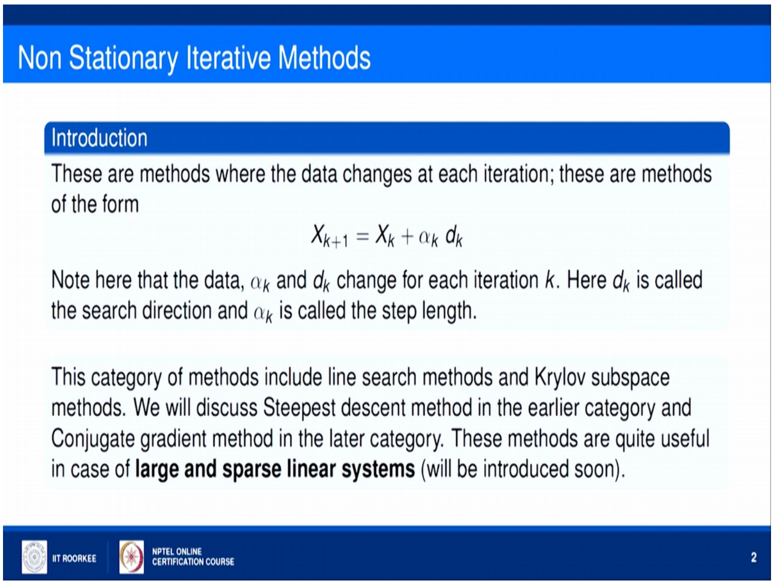

Now, if we talk about non stationary iterative methods then these are methods where the data changes at each iteration; means not like the stationary method where the iteration matrix p and the column vector q were fixed.

These are method of the form X k plus 1 means the vector which you have to calculate a vector of a non variable at k plus 1 iteration equals to the vector at k-th iteration plus alpha k d k. So, here the data alpha k and d k change in each iteration k. That is why I have put here suffix k in both of them. Here d k is called the search direction and alpha k is called the step length.

This category of methods includes line search methods; we will discuss two methods in this category and Krylov subspace methods. We will discuss steepest descent method in the earlier category and then conjugate gradient method. These methods are quite useful in case of large and sparse linear system.

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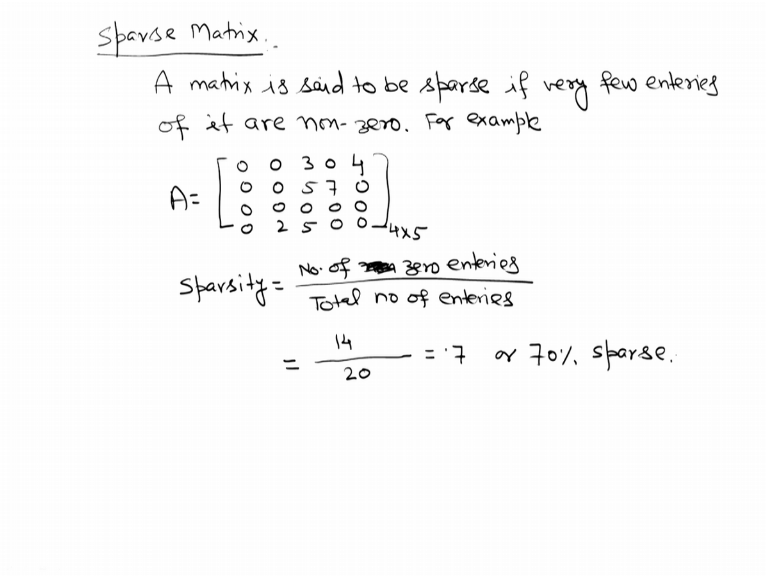

So, let us define now sparse matrix. A matrix is said to be sparse if very few entries of it are non zero; it means most of the entries of the matrix are zero valued. For example, if we take a matrix A which is given as 0 0 3 0 4, 0 0 5 7 0, then 0 0 0 0 0 and finally, 0 2 5 0 0. Then it is a 4 by 5 matrix which is a sparse matrix, because here only six entries are non zero. So, if I define the sparsity of this matrix then sparsity is number of non zero entries upon total number of entries. So, if I take this matrix the sparsity of this matrix is there are total fourteen non sorry number of zero entries it will be zero entries. So, there are total 14 zero entries here and 20 is the total number of entries in this matrix.

So, it is 0.7 or I can define that the matrix is 70 percent sparse. The opposite of the sparse is dense matrix. So, a dense matrix is a matrix in which most of the entries are non zero. In the similar way we can define the density of a matrix. So, density will be number of non zero entries upon the total number of entries. So, for a given matrix density plus sparsity will be equals to 1.

Now, the direct method like Gaussian elimination is not computationally efficient for solving if the system is large and sparse. Why, because if you take a n by n system and you use Gaussian elimination method for solving that there will be total order of n cube operations. If most of the entries are zero then using n cube operations on those zero entries are not a wise way of solving such systems. Moreover you know in the sparse system most of the entries are zero, but if you apply elementary row operations on that matrix in Gaussian elimination method then zero entries will become non zero. So, in that way we can by these two facts we can say the system which is large and sparse are cannot be solve efficiently using the direct method like Gaussian elimination.

So, what is the alternative? Alternative is iterative methods and in this category means non stationary methods first we are going to discuss; sorry in this category yeah in non stationary methods category first we are going to discuss the gradient methods.

(Refer Slide Time: 10:35)

Gradient Methods:

Consider a quadratic form

$$q(X) = \frac{1}{2} X^T A X - X^T b$$

Here, A is a $n \times n$ matrix and $X \in \mathbb{R}^n$, $b \in \mathbb{R}^n$

Our aim is to minimize $q(X): \mathbb{R}^n \to \mathbb{R}$ for some given $b \in \mathbb{R}^n$.

The gradient of q can be treated as residual and computed as $\nabla q(X) = AX - b$ —— (1)

Moreover, the Hessian matrix of q is given by the Jacobian of ∇q, i.e.

$$H(X) = J(\nabla q) = \underline{A}$$

if A is positive definite, then solution of (1) will be the point of minima for $q(X)$.

Thus, the functional $q(X)$ has a unique minima at X^*, the stationary point of $q(X)$ which is

So, consider a quadratic form q x because you know already about quadratic form. So, half X transpose AX minus X transpose b. So, you have seen this form earlier also in positive definite matrices lecture and what we are going to do let us try to find out the minima of this method. So, our aim is to minimize this method. Further here A is a n by n matrix and which is a symmetric matrix because you know we can always associate a symmetric matrix with the given quadratic form and X is a unknown vector having n component b is also column vectors.

Now, our aim is to minimize q X, which is a function from R n to R for some given b. The gradient of q can be treated as residual and computed as gradient of the functional q X equals to AX minus b here. Moreover the Hessian matrix of q is given by the Jacobian of the gradient; means Jacobian of del q, that is, Hessian matrix of which will be a function of X it is Jacobian of gradient of q and this comes out to be A because you have to find out one more time.

And, if A is positive definite then solution of one will be the point of minima. Why because, A is positive definite. So, for second order derivative is positive here for the functional q X. So, what we can claim? Thus the functional q X has a unique minima in case when A is positive definite.

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$$\nabla q(X^*) = AX^* - b = 0 ,$$

Hence

For a symmetric and positive definite (SPD) matrix A, solving $AX = b$ is equivalent to find the minimum of $\underline{q(X) = \frac{1}{2} X^T A X - X^T b}$.

<u>Steepest descent method</u> :-

This method is based on a greedy algorithm in that it chooses the search direction d_K as the local direction of steepest descent, i.e.

$$d_K = -\nabla q(X_K) = r_K$$

And, if this minima at X star the stationary point of q X which is del of q X star which equals to AX star minus b and this equals to 0, because X star is a stationary point. Hence for a symmetric and positive definite in short let me write it as SPD; S for symmetric and PD for positive definite, matrix A solving AX equals to b is equivalent to finding to find a minimum of q X equals to half X T AX minus X T b.

So, what I want to say that in gradient methods for a given system AX equals to b I will write such a functional q X and I will find out the minima of that function that will be automatically a solution of AX minus b and exact solution if it is not then it will be an approximate solution having the minimum residual error because this is defining the residual.

So, in this category the first method we are going to discuss is steepest descent method. So, this method is based on a greedy algorithm in that it chooses the search direction d k in that at the iteration K as the local direction of steepest descent, that is d K equals to minus del q X K and this I define as the residual r K in K-th iteration.

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Algorithm (steepest descent for solving $AX = b$)

1. Choose an initial solution as $X_0 \in \mathbb{R}^n$
2. For $K = 0, 1, 2 \cdots$ do

$$r_K = b - AX_K$$

$$\alpha_K = \frac{\langle r_K, r_K \rangle}{\langle r_K, Ar_K \rangle}$$

3. Iterate the sequence of solution as

$$X_{K+1} = X_K + \alpha_K d_K$$
$$= X_K + \alpha_K r_K$$

So, algorithm of steepest descent is given in this way. So, it is so, algorithm for steepest descent for solving AX equals to b. So, our first step will be choose an initial solution is X naught which is n n dimensional column vector. Now, for K equals to 0, 1, 2 and so on, do calculate the residual which is as I told you minus gradient of q X so, it will become b minus AX K because gradient of q equals to AX minus b. Compute the search direction is the inner product of r K with r K upon inner product of r K with A r K. So, in each iteration we will calculate this search direction.

Once you are having r K and alpha K in Kth iteration you can update your solution in K plus 1 iteration by using the iterative equation. So, iterate the sequence of solution as X K plus 1 equals to X K plus alpha K d K and as you know that d K equals to r K. So, this I can write in this way also X K plus alpha K into r K because in steepest descent r K equals to d K. So, this is the algorithm for steepest descent method let us take an example on it.

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Ex:- Consider a linear system $AX = b$, where

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix}$$

(a) Can we use steepest descent method for solving the above linear system?

(b) If yes, compute first three iterations by SD method starting with $X_0 = (0,0,0)^T$.

Sol^n:- (a) $A = A^T \rightarrow A$ is symmetric.

$a_{11} = 3 > 0$, $\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8 > 0$; $|A| = 20 > 0$

Hence, A is a PD matrix and we can apply SD method for solving $AX = b$

Consider a linear system AX equals to b where a is given as 3 minus 1 1, minus 1 3 minus 1, 1 minus 1 3 and b equals to minus 1 7 minus 7. Now, can we use steepest descent method for solving the above linear system this is my first question and the second part of this is if we can use, if yes, compute first three iterations or first three sequence of approximation by steepest descent method. So, in short I am writing it as SD starting with X 0 is 0, 0, 0 transpose means this is the initial solution.

So, let us try to solve it. So, as we have discuss we can apply the steepest descent method for solving a linear system in case when the matrix A is symmetric as well as positive definite. So, if I check it the given matrix A is a symmetric matrix which is give[n]- here because A equals to A transpose, it means A is symmetric.

Now, if I check for positive definiteness so, there are different types of criteria for checking positive definiteness of a given matrix. So, here it is a 3 by 3 matrix. So, I can apply the principle minor test. So, here a 1 1 is 3 which is greater than 0. Now, if I check the next principle minor here then it will be 3 minus 1 minus 1 and 3. So, this comes out to be 8, which is again positive and then determinant of a comes out to be 20, which is again positive.

So, hence A is a positive definite matrix and we can apply steepest descent method for solving AX equals to b. Now, let us apply this method.

(Refer Slide Time: 27:01)

(b) $\underline{K=0}$, $X_0 = (0,0,0)^T$

$$r_0 = b - AX_0 = \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -7 \end{bmatrix}$$

$$\alpha_0 = \frac{\langle r_0, r_0 \rangle}{\langle r_0, Ar_0 \rangle} = \frac{r_0^T r_0}{r_0^T A r_0} = \frac{99}{423} = 0.2340$$

Here

$$X_1 = X_0 + \alpha_0 r_0 = \begin{bmatrix} -0.2340 \\ 1.6383 \\ -1.6383 \end{bmatrix}$$

$$r_1 = b - AX_1 = \begin{bmatrix} 2.9787 \\ 0.2128 \\ -0.2128 \end{bmatrix}; \quad \alpha_1 = \frac{r_1^T r_1}{r_1^T A r_1} = 0.3667$$

$$X_2 = X_1 + \alpha_1 r_1 = \begin{bmatrix} 0.858 \\ 1.7163 \\ -1.7163 \end{bmatrix}$$

$$r_2 = b - AX_2 = \begin{bmatrix} -0.1418 \\ 0.9929 \\ -0.9929 \end{bmatrix}; \quad \alpha_2 = \frac{r_2^T r_2}{r_2^T A r_2} = 0.2340$$

So, let us say first iteration K equals to 0. So, for K equals to 0, X 0 is given as 0, 0, 0 transpose. Now, I will calculate r 0 which is b minus AX 0. So, it comes out to be b is given as minus 1, 7, minus 7 minus A. So, here A is 3 minus 1 1, minus 1 3 minus 1, 1 minus 1 three into 0 0 0. So, this will become minus 1 7 minus 1 which is same as your X 0.

Now, compute the step length alpha 0. So, according to algorithm it is inner product of r 0 with r 0 in denominator you are having inner product of r 0 with A r 0. So, this will be r 0 T r 0 upon r 0 T A r 0 and this comes out to be r 0 T r 0 will be 99 and our denominator will give you 423. So, it is 0.2340. So, here X 1 comes out to be X 0 plus alpha 0 r 0 and it comes out to be minus 0. 2340, then it will be 1.6383 and then minus 7 into this. So, minus 1.6383 now, calculate r 1 r 1 will become b minus AX 1 which comes out to be 2.9787, 0.2128, and finally, minus 0.2128. So, from here we calculate alpha 1 which o[ne] is r 1 t into r 1 upon r 1 transpose A into r 1 and it comes out to be 0.3667.

So, then the second approximation means in the second iteration become X 2 which is X 1 plus alpha 1 into r 1 and it comes out 0.858, 1.7163 and then minus 1.7163. Then once you are having X 2 I can calculate r 2 residual in the second iteration. So, b minus AX 2 and this comes out to be minus 0.148, 0.9929 and minus 0.9929, once we are having r 2 we can calculate alpha 2, which is r 2 transpose into r two upon r 2 transpose A r 2 and this comes out to be 0.2340.

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$$X_3 = X_2 + \alpha_2 r_2$$
$$= \begin{bmatrix} 0.8250 \\ 1.9487 \\ -1.9487 \end{bmatrix}$$
$$r_3 = b - AX_3 = \begin{bmatrix} 0.4225 \\ 0.0302 \\ -0.0302 \end{bmatrix}$$

The solution is converging (slowly) towards the exact solution $\bar{X} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

So, X r 2 as well as alpha 2 r available with us. So, we can calculate X three the approximation in third iteration which will become according to the generally scheme or iterative equation of the non stationary method X 2 plus alpha 2 r 2 which comes out to be 0.8250, 1.9487, minus 1.9487.

If I calculate r 3 here it will become b minus AX three which is 0.4225 0.0302 and minus 0.0302. The solution is converging and slowly I will say here towards the exact solution which is given as X bar equals to 1, 2 and minus 2. So, in this way we can apply the steepest descent method for a given problem.

So, this method is quite simple, in each iteration you have to calculate the residual or I will say the search direction which is residual only gradient of minus of the gradient of q and the step length which can be a calculated from the search direction and the matrix A. However, the drawback of this method is, this method is quite slowly in quite slow in terms of convergence.

In the next lecture we will learn how can we apply this method when the given matrix A is not positive definite or symmetric or both. And, how can we update or how can we increase the convergence of this method means in what way we should take our initial solution so that we can have a faster convergence.

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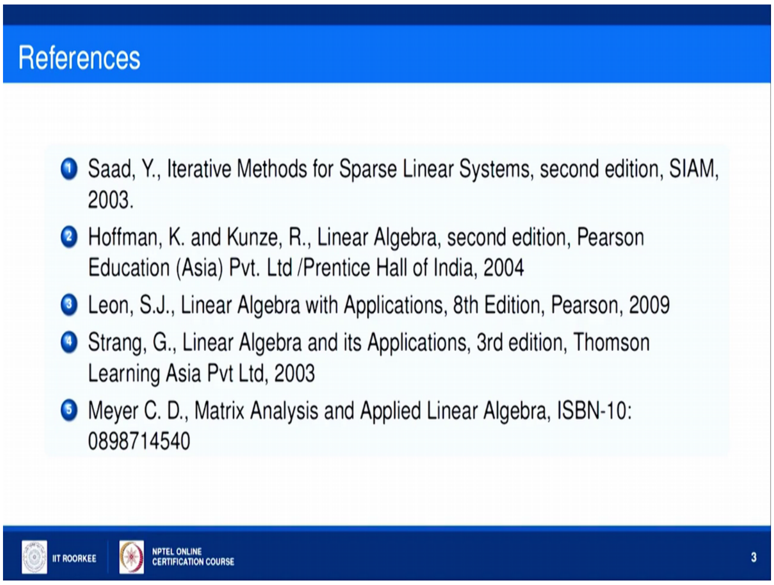

So, these are the references for this lecture in particular reference one is quite important for all these line search method and Krylov subspace method when you are solving large and sparse linear systems.

Thank you very much.