Matrix Analysis with Applications
Dr. Sanjeev Kumar
Department of Mathematics
Indian Institute of Technology Roorkee

Lecture - 33
Linear Systems: Iterative Methods II

Hello friends, so welcome to the second lecture on the Iterative Methods for solving linear systems, so in the previous lecture we have learnt two schemes.

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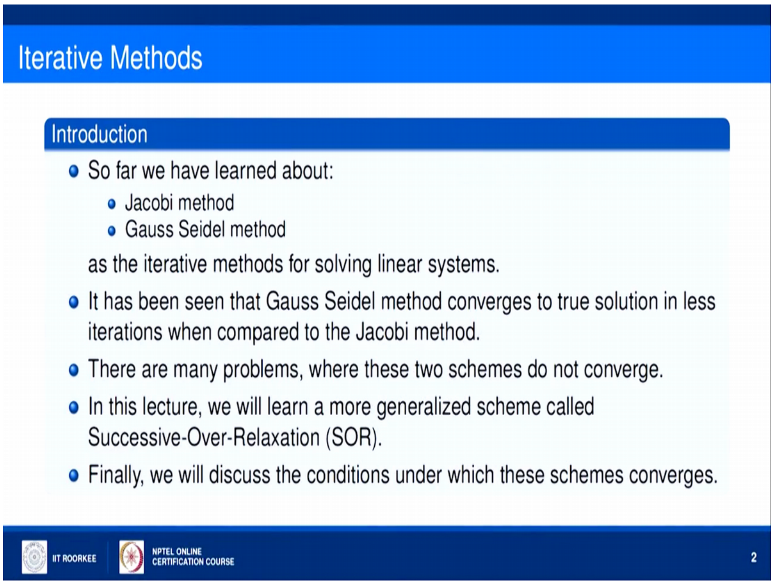

One is Jacobi and another one is Gauss Seidel, for solving the linear systems, it has been seen that Gauss Seidel method converges towards the exact solution in less iterations when compare to the Jacobi scheme. there are many problems where these two schemes do not converge at all in this lecture we will learn a more generalized scheme that is called successive over relaxation finally, in the last page of this lecture we will discuss the conditions under which these iterative schemes means Jacobi Gauss Seidel and successive over relaxation converges.

So, as I told you in the previous lecture these schemes are called non stationary iterative methods means these are of the form s k plus 1 equals to a iteration matrix p times s k plus q, so these are the stationary method.

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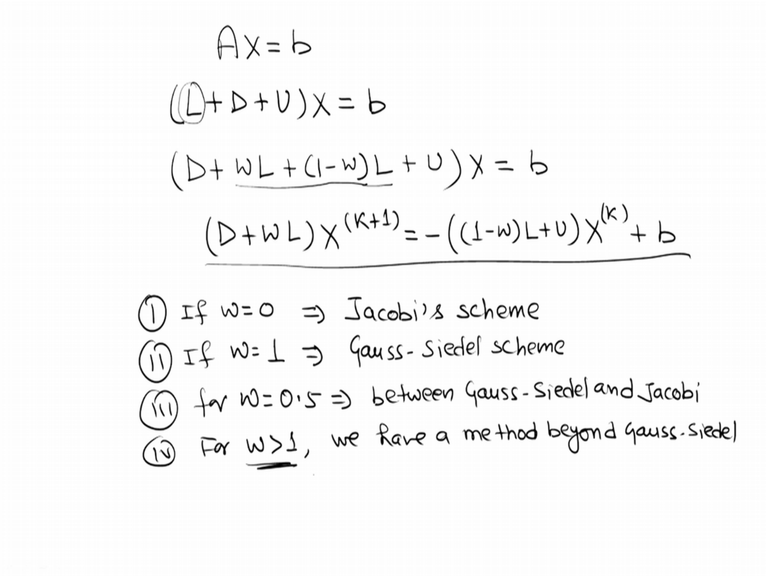

So, now consider A x equals to b write a as the sum of 3 matrix is L D and U times x into b. Now, what you do you write D plus omega, omega is where semi scalar times L plus 1 minus omega L plus U into x equals to b. So, if you observe this is just omega L plus L minus omega l. So, this is just this L only now what you do you write D over iterations like this D plus omega L x at k plus 1 iteration equals to minus 1 minus omega L plus U x at kth iteration plus b.

Now, if in this equation if I take omega equals to 0 then what will happen. So, it will become D x k plus 1 minus 1 L l plus U s k plus b. So, this becomes jacobis scheme, if I take omega equals to 1 then it becomes Gauss Seidel scheme for omega equals to 0.5, this is somewhere between Gauss Seidel and Jacobi for omega greater than 1 we have a method beyond Gauss Seidel. So, in this way we can say we are having some sense of over relaxation in this case and for certain problem it turns out to be highly effective method.

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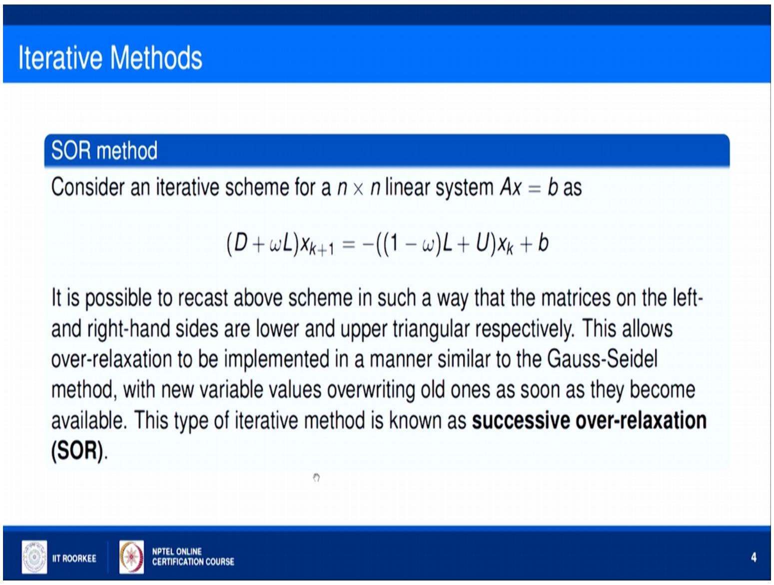

Now, consider an iterative scheme for a n by n linear system a x equals to b as D plus omega L at x k plus 1 iteration equals to minus 1 minus omega L plus U x at kth iteration plus b.

Now, it is possible that is the same thing which I have discussed in the last slide. So, it is possible to recast the above scheme in such a way that the matrices on the left and right hand sides are lower and upper triangular respectively. So, it is lower and it is upper this allows us to use the concept of successive displacement, and over relaxation can be implemented in a manner similar to the Gauss Seidel method with new variables values overwriting old ones as soon as they become available.

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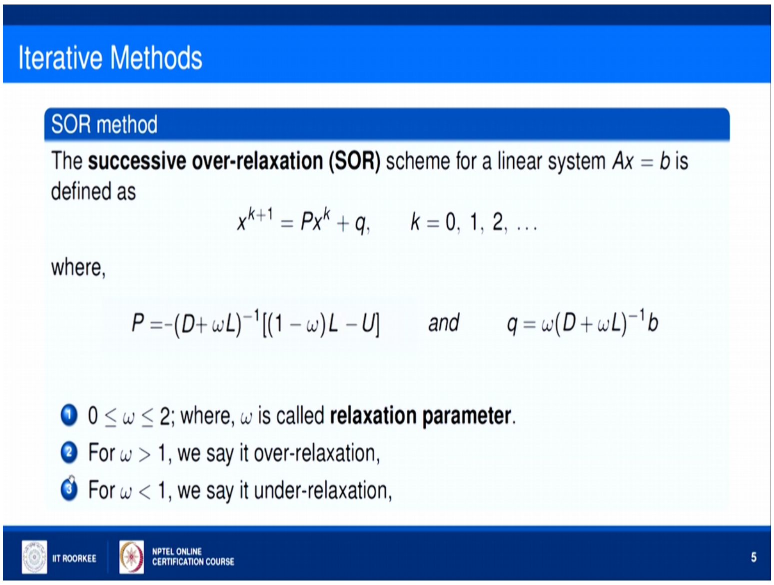

This type of iterative method is known as successive over relaxation or in short SOR. So, the iterative process of SOR method is the same as the other stationary methods that x k plus 1 equals to the iteration matrix p times x at kth iteration plus the column vector q where omega is between 0 to 2, where omega is called a relaxation parameter for omega greater than 1 we say it over relaxation for omega less than 1 we say it under relaxation.

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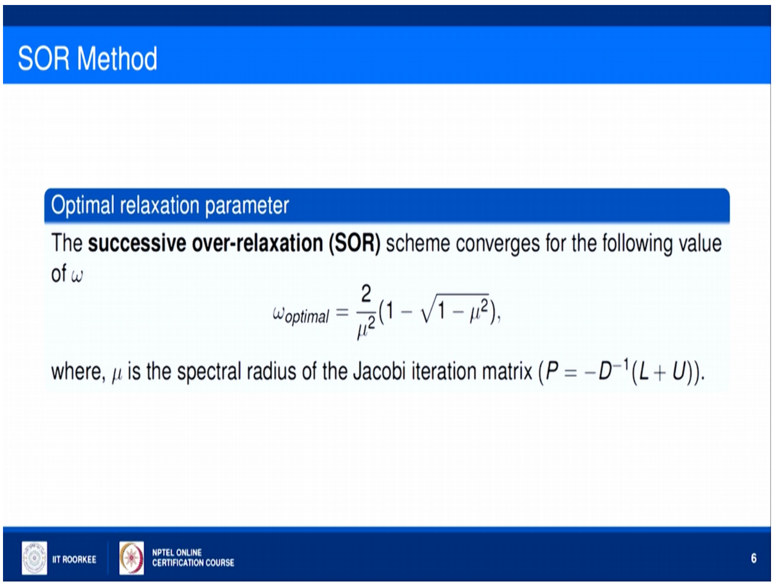

One more thing that how to compute the optimal value because omega will be between 0 to 2. So, what will be the optimal value for a given problem, so that the schemes converge faster?

So, that optimal value of the omega for the SOR scheme can be given by this 1. So, omega optimal will be 2 upon mu square multiplied with 1 minus square root of 1 minus mu square, where mu is the spectral radius of the Jacobi iteration matrix and if you can recall Jacobi iteration matrix is minus D inverse into L plus U. So, you calculate all the eigenvalues of this and the eigenvalue bigger in the magnitude will be the value of mu. That is the spectral radius of this iteration matrix p and once you are having mu you can calculate the optimal value of omega using this particular formula.

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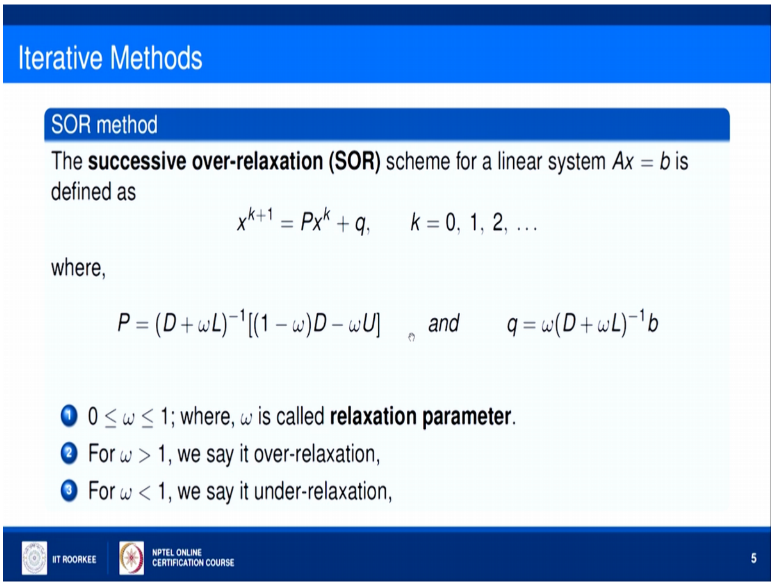

Once you are having omega then you can put the value of omega here and then you can calculate the iteration matrix p as well as column vector q, and you can write your iteration equation iterative equations of SOR method. So, for this let us take this particular example.

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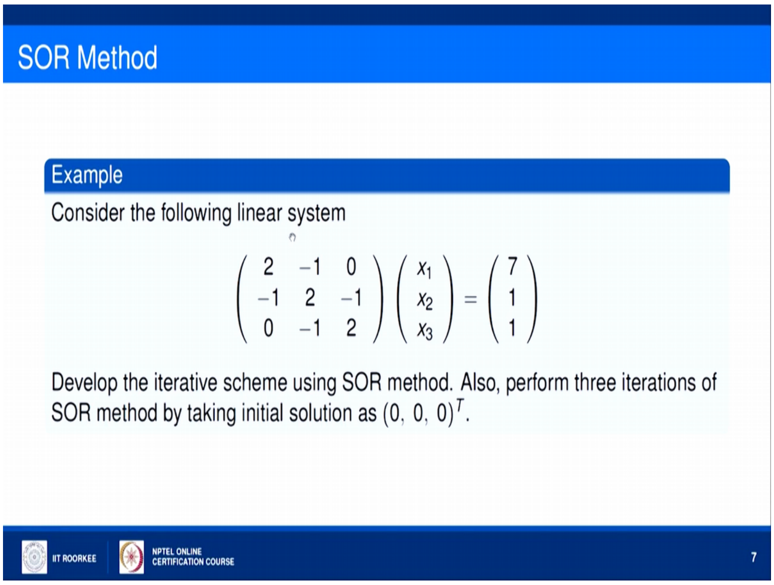

So, consider the linear system 2 minus 1 0 minus 1 2 minus 1 0 minus 1 2. So, this is the coefficient matrix a multiplied with column vector of a non-variable x 1 x 2 x 3 equals to right hand side vector b as 7 1 1, develop the iterative scheme using the successive over relaxation method also perform 3 iterations of this method by taking an initial solution as 0 0 0, so x 1 is 0 x 2 is 0 and x 3 is 0.

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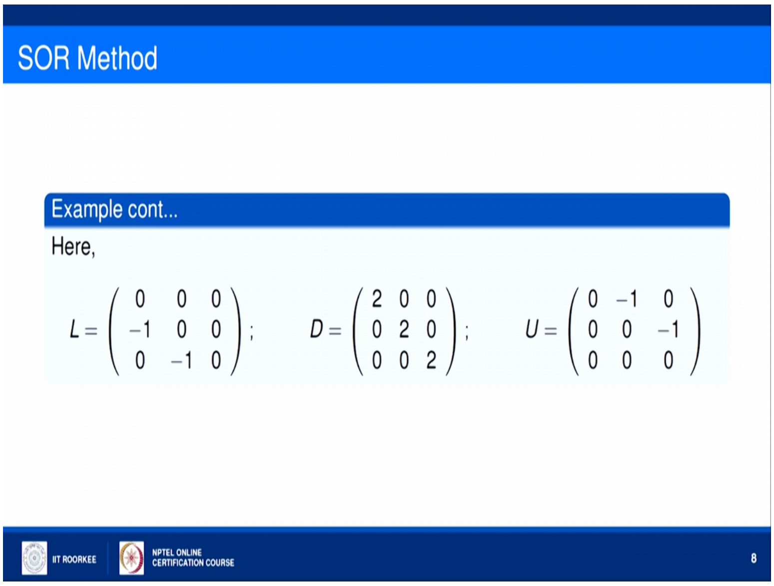

So, if we see now the coefficient matrix a can be written in terms of a lower diagonal matrix lower triangular matrix L a diagonal matrix D and an upper triangular matrix U. So, for this matrix a L will be this D will be like this and U will become this matrix.

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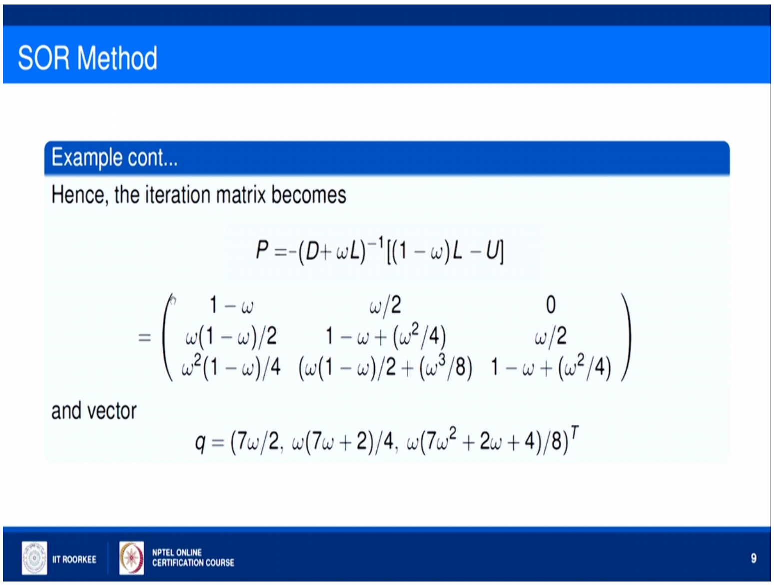

Now, as we know the iteration matrix for successive over relaxation is given by this particular scheme. So, if I calculate here the iteration matrix in terms of omega comes out to be in this form. So, first element is 1 minus omega second is omega by 2 third 1 is 0, then in the second row first element is omega into 1 minus omega by 2 second element is 1 minus omega plus omega square upon 4 and the third element in the second row is a omega by 2.

Then in third row first element is omega square into 1 minus omega upon 4 omega 1 minus omega upon 2 plus omega cube upon 8 and the third element is 1 minus omega plus omega square upon 4. So, this is the matrix which is the iteration matrix for successive over relaxation method and the right at the column vector q comes out in this form. So, now, what we need to do we need to calculate the optimal value of the omega and we have to substitute that value here to get the iteration matrix p in the form, so that we can perform the iterations. So, as I told you the optimal value of the omega will be a function of mu where mu is the spectral radius of the Jacobi iteration matrix.

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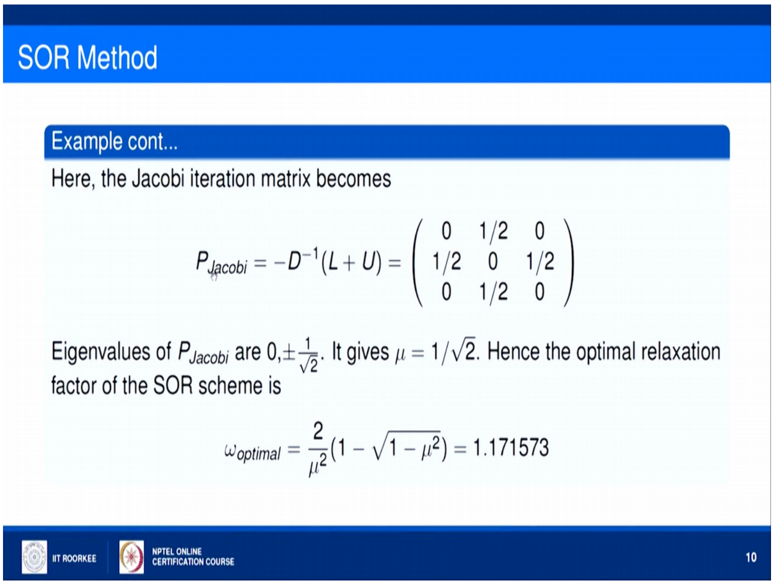

So, if I calculate the Jacobi iteration matrix for this problem it comes out to be minus D inverse into L plus U and in this way, the eigenvalue of this matrix are 0 plus 1 upon root 2 and minus 1 upon root 2. So, it gives the spectral radius mu as 1 upon root 2 and hence the optimal factor for SOR scheme is omega optimal is 2 upon mu square 1 minus square root 1 minus mu square by using this formula and it comes out to be 1.171573. After substituting this value of omega into this iteration matrix and the column vector q.

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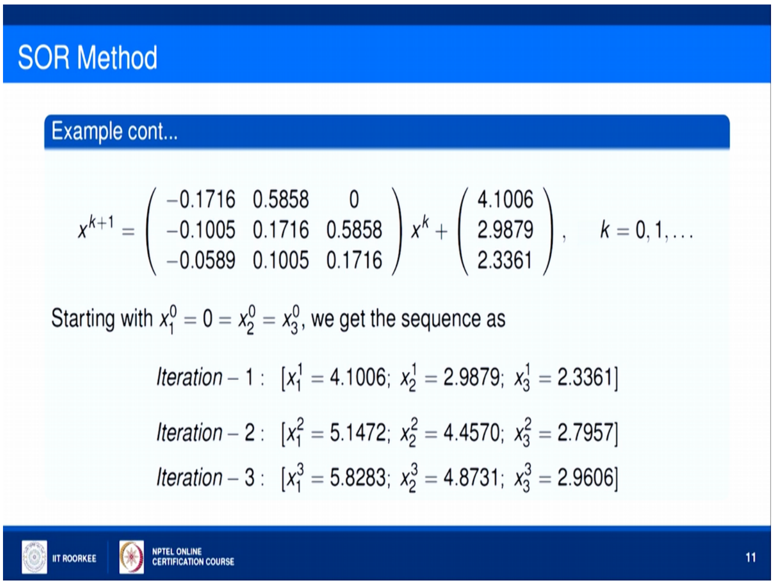

We obtained this matrix p as the iteration matrix and this column vector as the q now take x 1 x 2 x 3 as 0 in the initial solution. So, in the first iteration we obtained x 1 as 4.1006 x 2 as 2.9879 and x 3 as 2.3361.

These are the values of x 1 x 2 x 3 in the second iteration and in the third iteration we obtained x 1 as 5.8283 x 2 as 4.8731 and x 3 as 2.9606. So, these are the 3 iterations of SOR method.

So, this is how we can implement the successive over relaxation scheme, first write down the iteration matrix in terms of omega from the formula of iteration matrix as well as the column vector q, find out the optimal value of omega substitute there and from there you will get the iterative equations for the successive over relaxation method.

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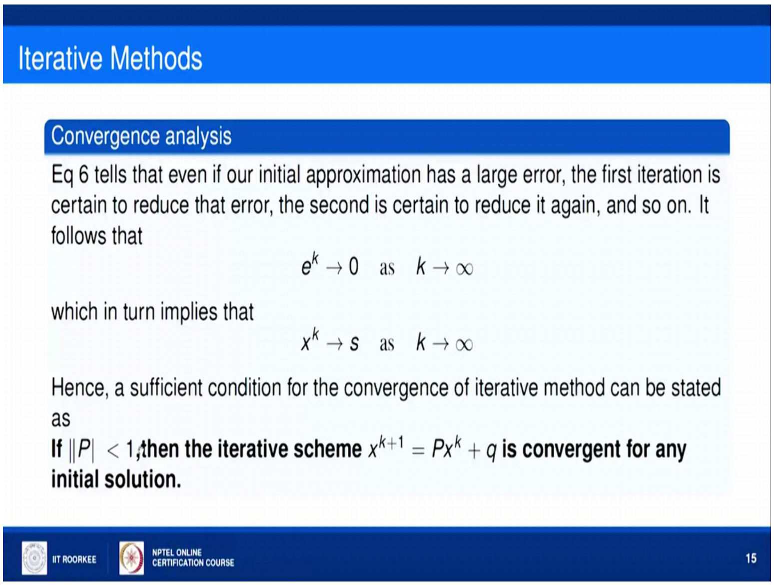

After this implementation we will talk about the convergence analysis for all these iterative methods.

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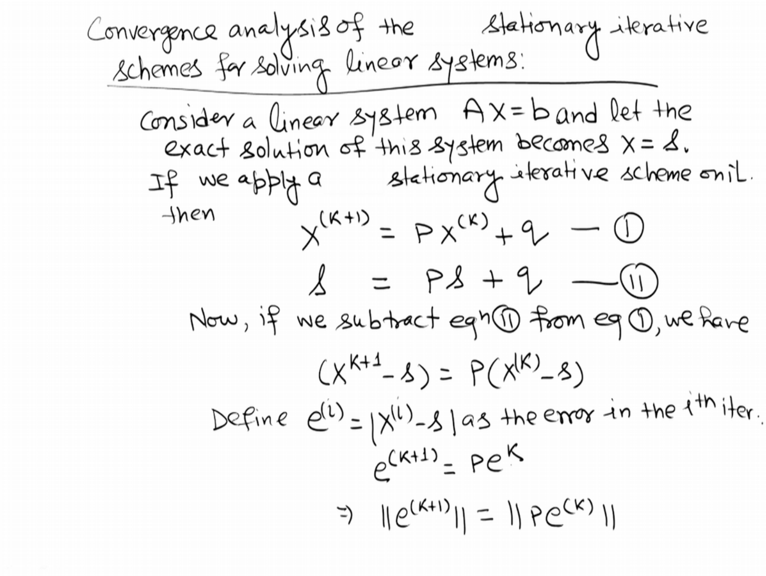

So, we are talking about convergence analysis of the stationary iterative schemes for solving linear systems. So, consider a linear system a x equals to b where a is a n by n matrix and let the exact solution of this system becomes x equals to s.

So, if we apply stationary iterative scheme on it, then we are having the scheme of the form x at k plus 1 iteration is p at x at kth iteration plus q where p is the iteration matrix and q is column vector now since s is the exact solution. So, what we are having s equals to p times s plus q.

Because s is the exact solution and once you obtain the exact solution then whatever iteration you perform you will get the same solution that we have seen in case of the example of Jacobi as well as Gauss Seidel in the previous lecture where in case of Jacobi in forty fifth and forty sixth iterations you are getting the same solution and in case of Gauss Seidel you are getting the same solution in ninth and tenth iteration.

So, now let us say equation 1 and this is my equation 2, now if we subtract equation 2 from equation 1. We have x k plus 1 minus s equals to p x k minus s ok, define e i equals to x i minus s as the error in the ith iteration. Then what we are having we can write this equation as e that is error in k plus 1 iteration equals to p times error in kth iteration.

Or this I can write norm of error because a this error will be a vector for different variables x 1 x 2 x 3 up to x n this can be written as norm of the error vector in kth iteration. So, what we are having we are having the error vector in k plus 1 iteration equals to the norm of error vector p times error vector in kth iteration and this can be written as norm of p into norm of e k.

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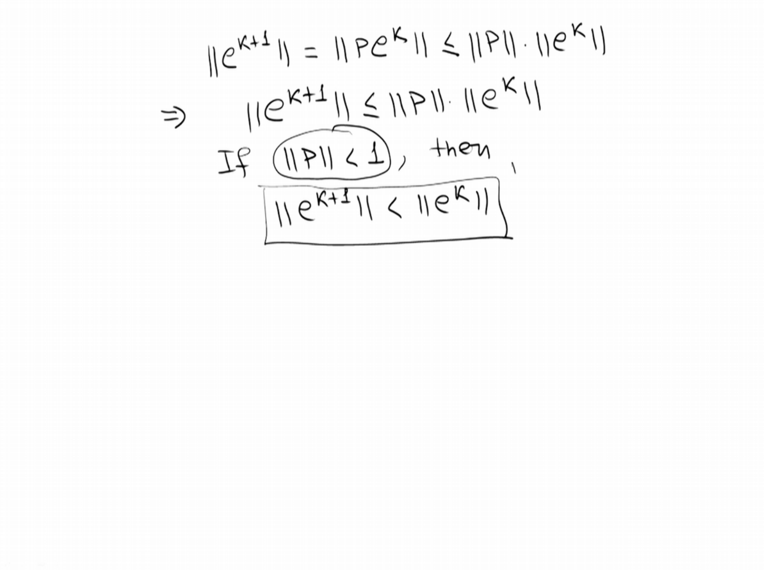

So, what we are having e k plus 1 less than equals to p, that is norm of p into norm of error vector in kth iteration. Now if this norm of p becomes less than 1 then what we are having the error in k plus 1 iteration is less than error in kth iteration. Now this is quite important this tells us that whatever error you are having in first iteration in the second iteration you will be having the less error whatever error you are having in second iteration in the third iteration you will be having the less error means you are going towards the exact solution and this is happening due to this condition.

So, based on this what we can say they we can write the sufficient condition for the convergence of an iterative scheme, and that sufficient condition becomes that even if our initial approximation has a large error the first iteration is certain to reduce that error the second is certain to reduce it further because in each iteration you are having less and less error and so on.

It follows that the error in kth iteration will be 0 when k is standing to infinity which implies that the sequence of approximation will move towards a will tend to exact solution as k tending to infinity, because this error is tending to 0 hence a sufficient condition for the convergence of iterative equation can be stated as if the norm of matrix p is less than 1 then the iterative scheme x k plus 1 equals to p x k plus q is convergent for any initial solution, because whatever large error you are having initially it will reduce in each subsequent iteration.

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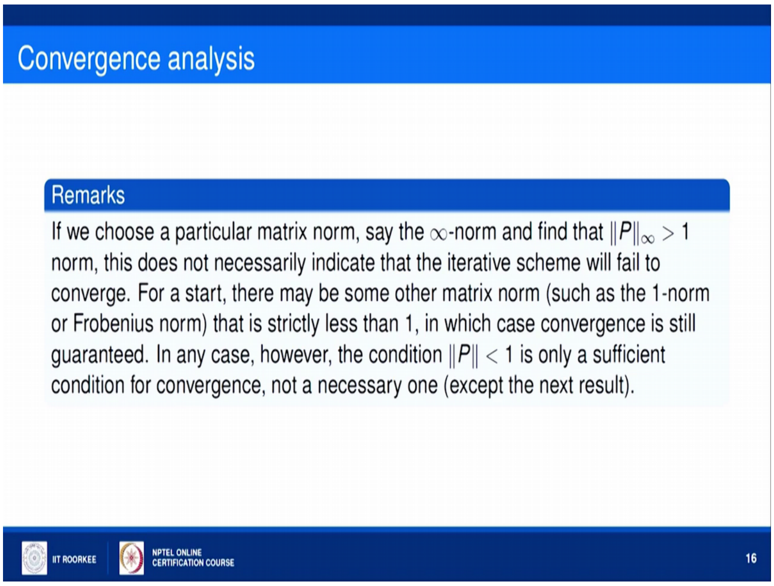

Here one important remark I would like to mention because there are different types of matrix norms. So, if we choose a particular matrix norm say the infinity norm and if we find that the this infinity norm for the iteration matrix p is greater than 1 this does not indicate that the iterative scheme will fail to converge, because it is a sufficient condition not necessary first of all.

And moreover there may be some other matrix norm such as 1 norm Frobenius norm that is strictly less than 1 in which case convergence is still guaranteed. So, there should be 1 norm is less than 1 that will do our job then guaranteed we can say that we can say that the iterative scheme will be convergent for any initial solution. So, in any case the condition the norm of iteration matrix p is less than 1 is only a sufficient condition for a convergence not a necessary 1, now we will write the necessary and sufficient condition.

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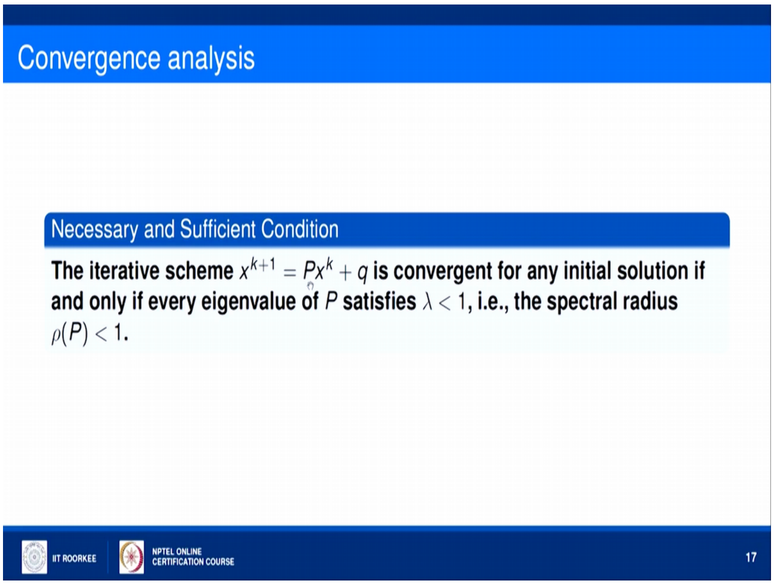

The iterative scheme x k plus 1 equals to P of x k plus q for solving a linear system a x equals to b is convergent for any initial for any initial solution, if and only if every eigenvalue of P satisfy lambda is less than 1 means every eigenvalue of P is less than 1.

Or in other way we can say it the spectral radius of p is less than 1, so this is the necessary as well as sufficient condition for the convergence means if scheme is convergent then the every eigenvalue of p will be less than 1 and if every eigenvalue of p is less than 1 then the scheme will be certainly convergent.

These conditions I have spoken they are on the iteration matrix b can we say some condition on the coefficient matrix a because the coefficient matrix a is directly available to us from the problem p we need to calculate. So, here I am telling one more condition which is which can be implemented on the coefficient matrix a and that sometimes a guarantee of convergence can be established by direct inspection of the coefficient matrix a.

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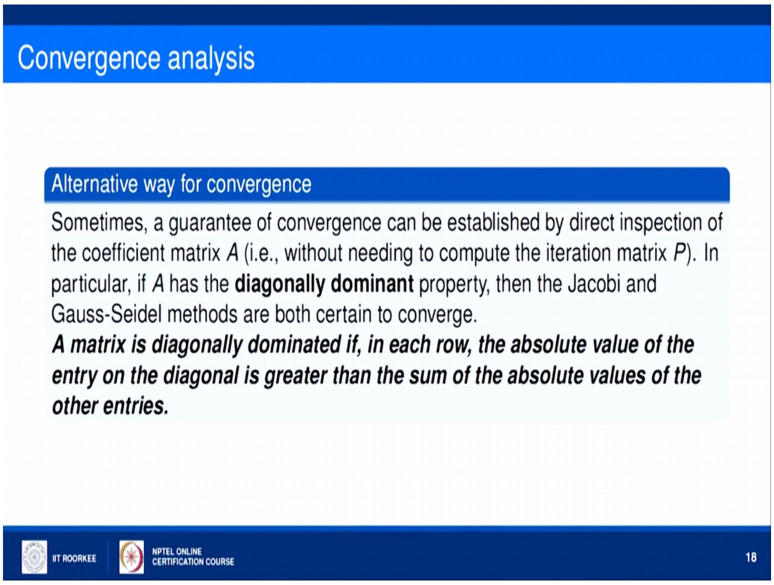

That is as I told you without needing to compute the iteration matrix p in particular if a has the diagonally dominant property. Then the Jacobi and Gauss Seidel methods are both certain to converge, and a matrix is set to be diagonally dominated if in each row. aThe absolute value of the entry on the diagonal is greater than the sum of the absolute values of the other entries.

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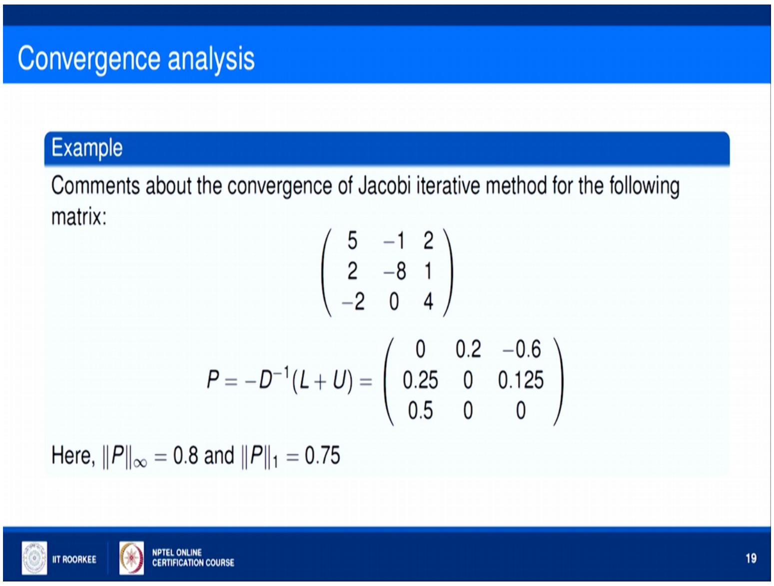

So, consider this particular example, in this example comments about the convergence of Jacobi iterative method for the following matrix. So, here this is my coefficient matrix.

So, if you see the coefficient matrix here the diagonal entry in first row is 5 and some of the absolute value of the other entries are 3. So, 5 is greater than 3 here 8 is greater than 2 plus 1 and here 4 is greater than 2 plus 0.

So, it is a diagonally dominant matrix strictly diagonally dominant and hence the scheme will converge, moreover if you want to check with the iteration matrix if I calculate iteration matrix in case of Jacobi method. It comes out to be in this way and if I find out the norm of this the infinity norm is 0.10 and 1 norm is 0.75, both are less than 1 and hence these are the sufficient are these guaranteed the scheme we converge.

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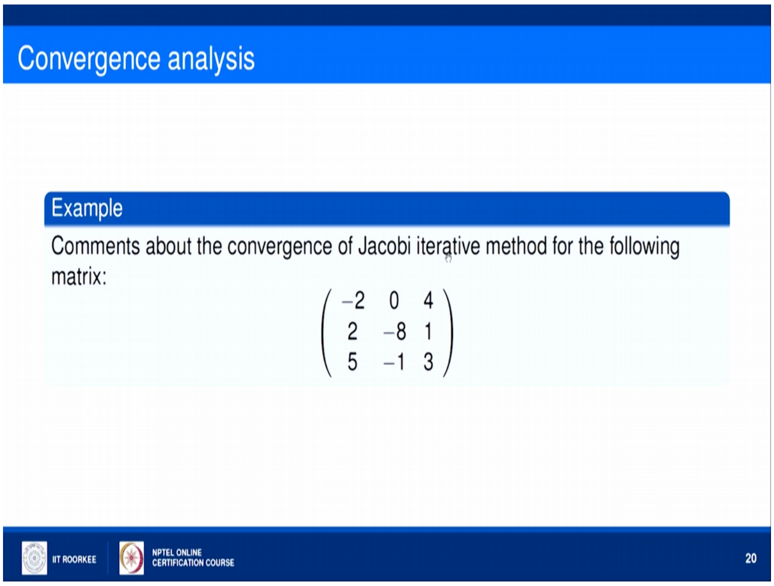

Jacobi scheme will convergence comments about the convergence of Jacobi iterative method for the following matrix, now this is a an interesting example why I am saying interesting let me describe it.

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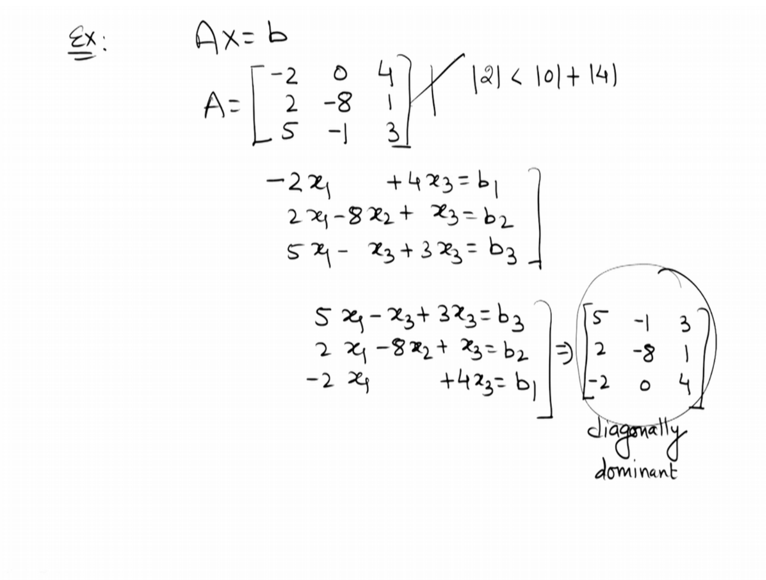

So, here I am having a system a x equals to b where the matrix A is minus 2 0 4 2 minus 8 1 and 5 minus 1 3. If you see this matrix in first row the diagonal entry which is 2 it is less than 0 plus 4. So, this matrix is not diagonally dominant in this form; however, if I write the equations the first equation becomes minus 2 x 1 plus 4 x 3 equals to b 1 the second equation is 2 x 1 minus 8 x 2 plus x 3 equals to b 2 and 5 x 1 minus x 3 plus 3 x 3 equals to b 3.

Now what you do you just interchange first and third equations. So, if I interchange it my system will become 5 x 1 minus x 3 plus 3 x 3 equals to b 3, the second equation will be remain like the earlier and this 1 4 x 2 4 x 3 equals to b 1. Now these 2 systems are same they will be having the same solution just I have interchange the order of equations.

Now, if you see the coefficient matrix of this system it will become 5 minus 1 3 2 minus 8 1 minus 2 0 4. So, what will happen if I take this coefficient matrix it is diagonally dominant, because in each row diagonal vector diagonal element is having the bigger value when compare to the sum of the absolute value of other 1.

In terms of absolute value and what you do apply the Jacobi scheme on this matrix instead of this one, because you will get the same solution only and your scheme will be warrantedly converge.

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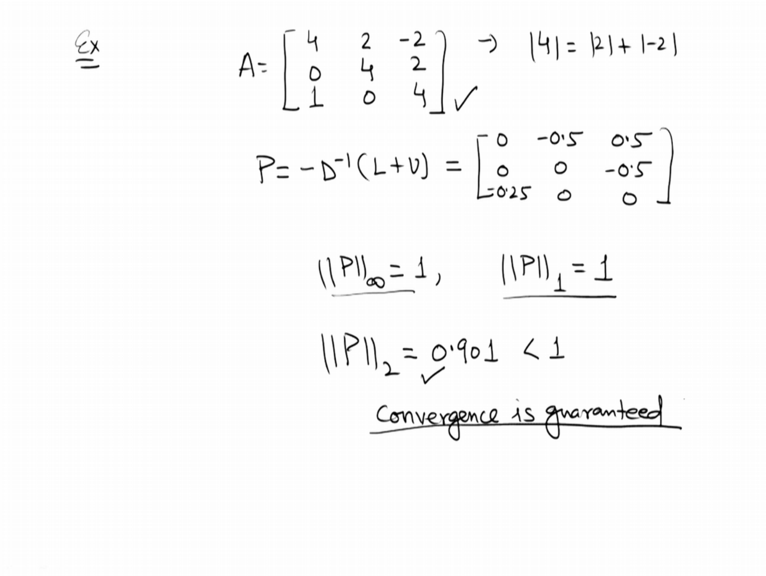

Take one more example let A becomes 4 2 minus 2 0 4 2 and 1 0 4 here matrix is not strictly diagonally dominant, because if you see the first row 4 equals to 2 plus minus 2, so which are same.

So, what we are having here we cannot comment even though we change the order of the equations it will not it will become more worst. So, we cannot comment anything after seeing the coefficient matrix. So, if we talk about Jacobi method and let me calculate the iteration matrix for Jacobi method. So, it will become minus D inverse into L plus U and it comes out to be 0 minus 0.5 0.5 0 0 minus 0.5 minus 0.25 0 0.

If I calculate this norm it comes out to be 1, and if I calculate this norm this is also 1. So, what will happen not diagonally dominant this norm is 1 this norm is 1. So, what is the other choice left with us calculate this also, because best on this I cannot say whatever information I have calculated I cannot say anything about the convergence. So, calculate frobenius norm and fortunately it comes out to be 0.901, which is less than 1 and hence since one of the norm is less than 1, then convergence is guaranteed it satisfy the criteria of sufficient condition.

In this lecture we have learned about the successive over relaxation scheme then we have seen the convergence conditions for different stationary schemes. In the next lecture we will talk about non-stationary iterative scheme so.

Thank you very much.