Matrix Analysis with Applications
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Lecture - 32
Linear System: Iterative Methods I

Hello friends. So welcome to the lecture on Iterative Methods. So, in past few lectures we have learnt about singular value decomposition, least square approximation and their analysis in terms of singular values.

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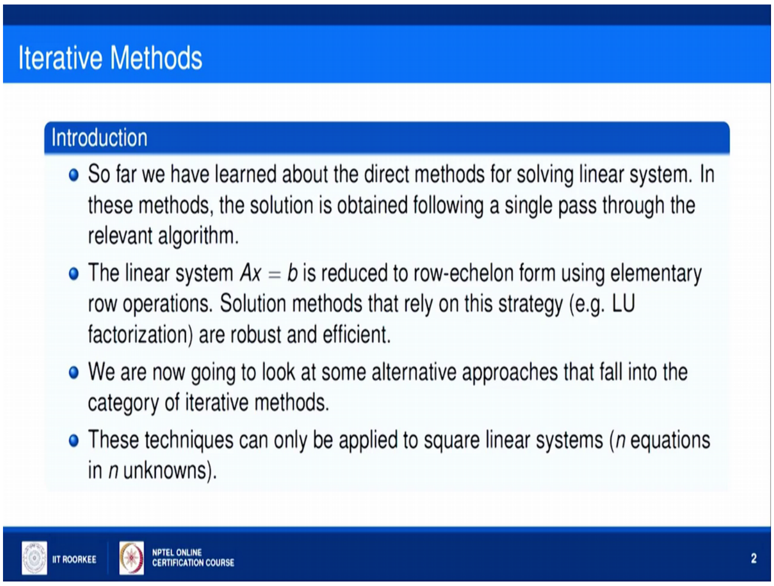

So, now in next few lectures we will learn another method for solving linear system of equations, but by using iterative methods. So, in the beginning of this course we have learnt direct method for solving linear systems. Some of them are like Gaussian elimination and then LU decomposition. like in LU decomposition we write the coefficient matrix a in terms of 2 matrices product of 2 matrices L and U, where L is a lower triangular matrix and U is an upper triangular matrix. And then by using the forward substitutions and then backward substitutions we solve the system of equations. Now in the category of iterative methods we can solve the square linear system means we are having n equations in n unknown.

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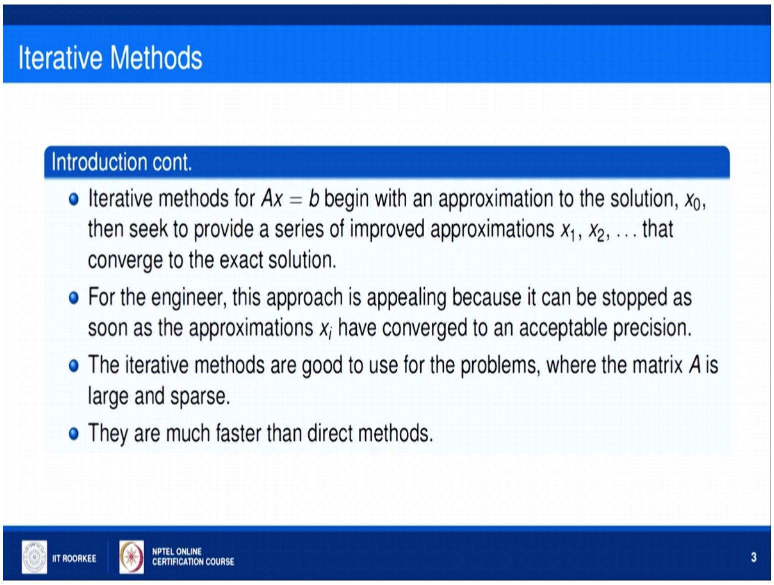

So, iterative methods for Ax equals to b begin with an approximation to the solution let us say x 0, which is the initial solution and then seek to provide a series of improved approximations like x 1 x 2 up to like in this way that converts to the exact solution or somehow we reach near to the exact solution. These iterative methods are quite popular in engineering for engineers because we are looking for the approximations and once we are having a required precision means a marginal error between the exact solution and the sequence of approximation then we can stop.

So, these are in this way we can save the time or computation for solving the linear system. Moreover when we are solving especially partial differential equation using some numerical methods, we encounter the large and sparse systems. We will discuss what we mean by the sparse system. And these iterative methods are quite useful, when we are solving the large and sparse system when compared to the direct method.

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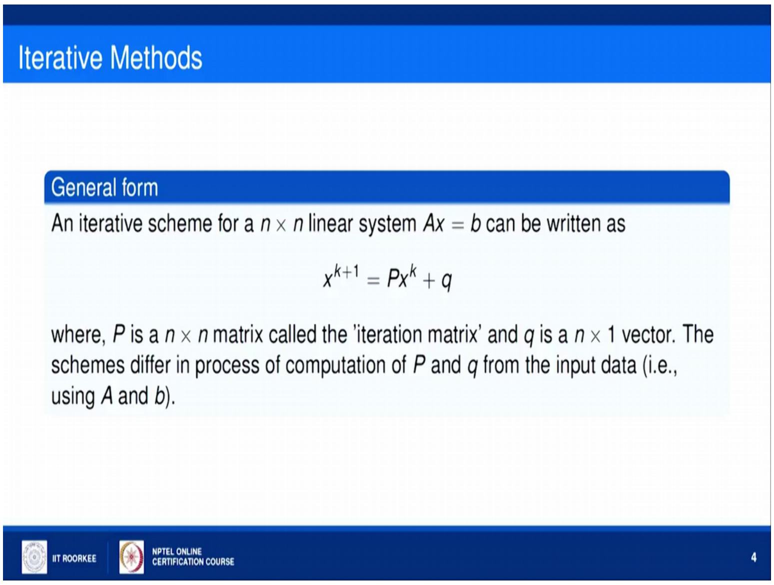

A general form of an iterative methods for solving a n by n linear system Ax equals to b can be seen in this way x k plus 1 means the approximation of the solution in k plus 1 iteration equals to a matrix p which will be n by n matrix into the solution in k-th iteration plus a n by n vector n by 1 vector q.

So, here the matrix p is called a iteration matrix, there are different schemes next couple of lectures we are going to discuss three different schemes and they differ in the process of computing the iteration matrix and the vector q from the input data that is the coefficient matrix a and from the right hand side vector b. So, from these data a and b we will compute our iteration matrix p and the vector q and then using this iteration process we will find out the sequence of approximations.

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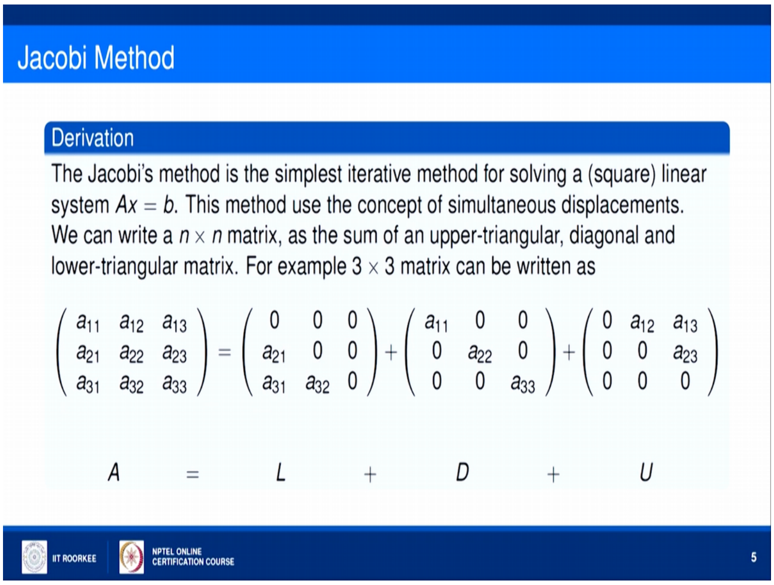

So, first we will take the Jacobi method, so the Jacobi method is the simplest iterative method for solving a square linear system Ax equals to b square means our matrix a is a square matrix this method use the concept of simultaneous displacements.

We can write a n by n matrix a as the sum of a lower triangular matrix let us say L a diagonal matrix D and an upper triangular matrix u. So, if you are you take this 3 by 3 matrix then I can write this equals to this matrix L which is a lower triangular matrix plus a diagonal matrix D plus an upper triangular matrix U.

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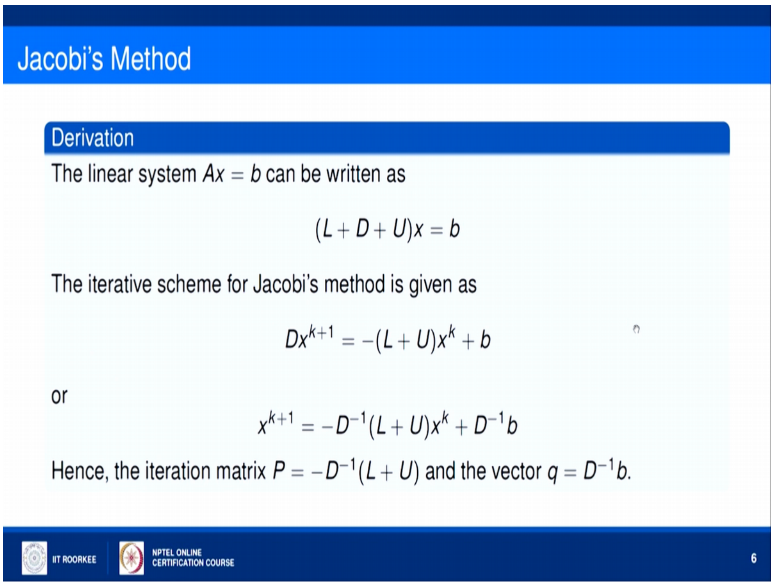

Now, how to drive the Jacobi iterations?

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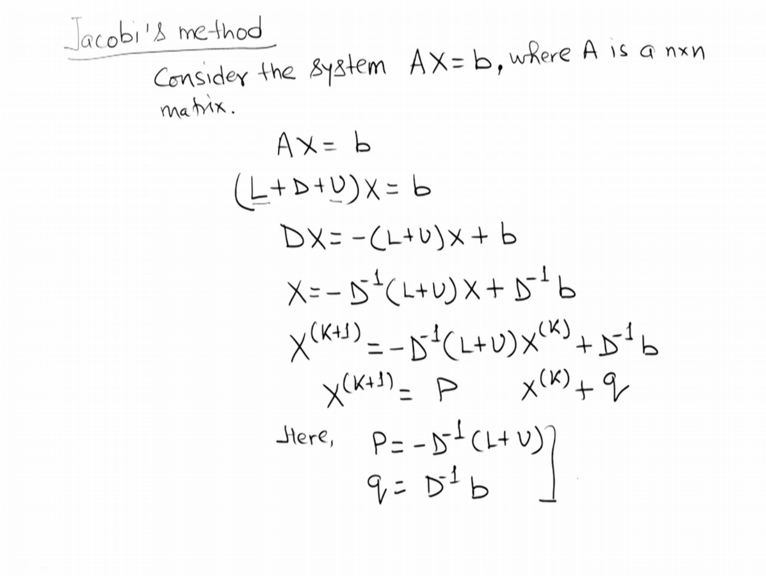

So, I am talking about Jacobi method consider the system Ax equals to b where A is a n by n matrix. Now, you are having Ax equals to b write A as the sum of 3 matrices L, D and U lower triangular triangular lower triangular diagonal and upper triangular into x equals to b or here write it like DX equals to minus L plus U X plus b. So, what I have done I have taken this L and U in the right hand side, or I can write x equals to minus D inverse L plus U x plus D inverse into b and set your iterations in this way.

So, x at k plus 1 iteration can be given by minus D inverse L plus U x at k-th iteration plus D inverse b, which is equals to the general iterative system p into x k plus a column vector q. So, here iteration matrix P is minus D inverse L plus U and the column vector q is D inverse into b. So, this is our Jacobi method if we take a means how to solve a system.

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Consider a 3x3 linear system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$a_{11}x_1 = b_1 - a_{12}x_2 - a_{13}x_3$$
$$a_{22}x_2 = b_2 - a_{21}x_1 - a_{23}x_3$$
$$a_{33}x_3 = b_3 - a_{31}x_1 - a_{32}x_2$$

$$\Rightarrow \quad x_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right)$$
$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)}\right)$$
$$x_3^{(k+1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)}\right)$$

So, consider a 3 by 3 system a 1 1 x 1 plus a 1 2 x 2 plus a 1 3 x 3 equals to b 1 a 2 1 x 1 plus a 2 2 x 2 plus a 2 3 x 3 equals to b 2 a 3 1 x 1 plus a 3 2 x 2 plus a 3 3 x 3 equals to b 3. So, now, by the Jacobi method I can write a 1 1 x 1 equals to b 1 minus a 1 2 x 2 minus a 1 3 x 3. So, this I am writing from the first equation from the second equation I can write a 2 2 x 2 equals to b 2 minus a 2 1 x 1 minus a 2 3 x 3 and from the third equation I can write a 3 3 x 3 equals to b 3 minus a 3 1 x 1 minus a 3 2 x 2.

Now, for setting the iterations what I will do the same system I can write x 1 equals to 1 upon a 1 1 provided a 1 1 is not 0, b 1 minus a 1 2 x 2 minus a 1 3 x 3. Similarly, I can write from the second line x 2 equals to 1 upon a 2 2 b 2 minus a 2 1 x 1 minus a 2 3 x 3 and then x 3 equals to 1 upon a 3 3 b 3 minus a 3 1 x 1 minus a 3 2 x 2.

So, now set this at left hand side at k plus 1 iteration and in the right hand side take everything at k-th iteration. So, this is the iterative equations for Jacobi method for a 3 by 3 system. When I introduce you about Jacobi method I told you this method is a method of simultaneous displacement, why I told it, because here you can see that for finding k plus 1 for each component x 1 x 2 x 3 for each variable I am using the value of these variables from the previous iteration. So, what I am doing simultaneously for getting the value at k plus 1 iteration, I am using the value of k-th iteration. So, now, we will take one example and we will solve this example.

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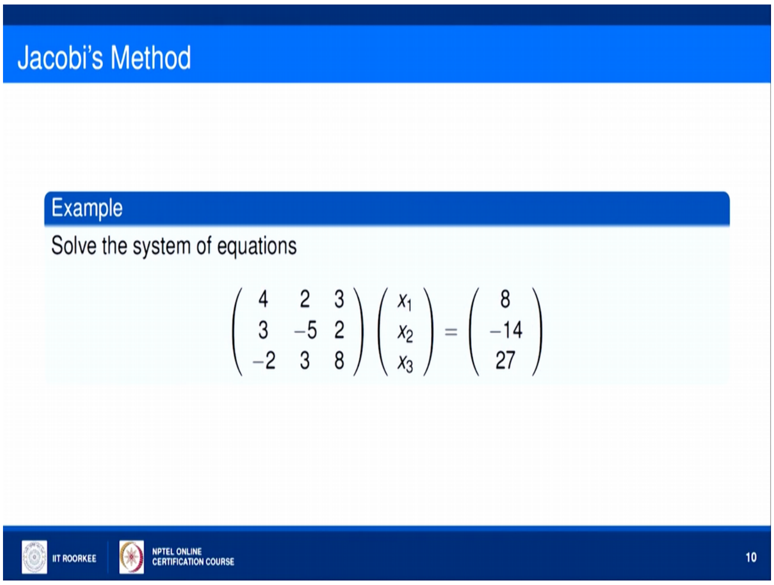

Using the Jacobi method; so let us take an example.

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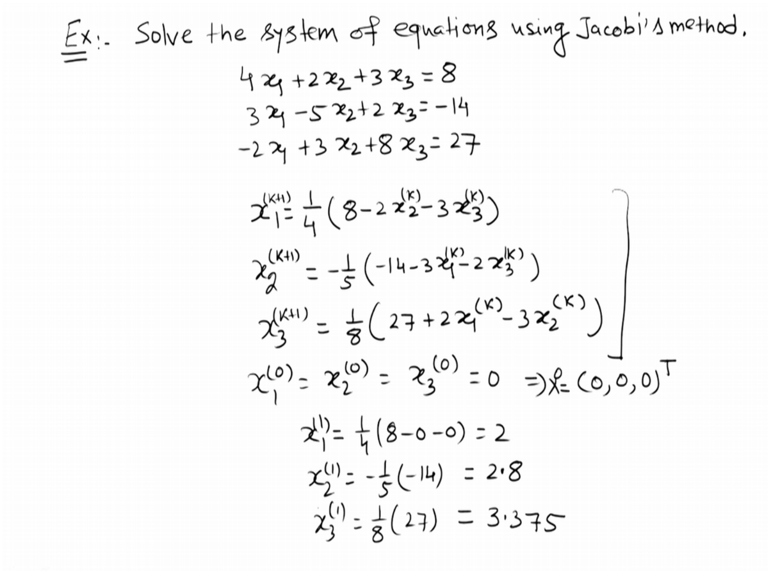

Solve the system of equation 4 x 1 plus 2 x 2 plus 3 x 3 equals to 8 3 x 1 minus 5 x 2 plus 2 x 3 equals to minus 14 and minus 2 x 1 plus 3 x 2 plus 8 x 3 equals to 27 and solve this using Jacobi iterative method. So, here I am having x 1 equals to 1 upon 4 8 minus 2 x 2 minus 3 x 3 ok. So, this is my first equation and it is at k plus 1 iterate it is at k and it is k-th iteration. Similarly x 2 at k plus 1 iteration can be obtained minus 1 upon 5 minus 14 minus 3 x 1 minus 2 x 3 and both are at k-th iteration number k.

Similarly x 3 at k plus 1 iteration is given as 1 upon 8 27 plus 2 of x 1 and this x 1 in k-th iteration minus 3 x 2 at k-th iteration. So, this is the initially scheme if I take the initial solution x 0 x 1 0 equals to x 2 0 equals to x 3 0 equals to 0 means my initial solution is 0 0 0 transpose which is my initial x then what I have.

My x 1 1 will become 1 upon 4 into 8 minus 0 minus 0. So, 2 x 2 1 will become minus 1 upon 5 minus 14. So, this will become 2 point 8 and then x 3 1 will become 1 upon 8 into 27. So, it will be 3 point 3 and then 7 5, so this is the initial value of x 1 x 2 x 3 at first iteration.

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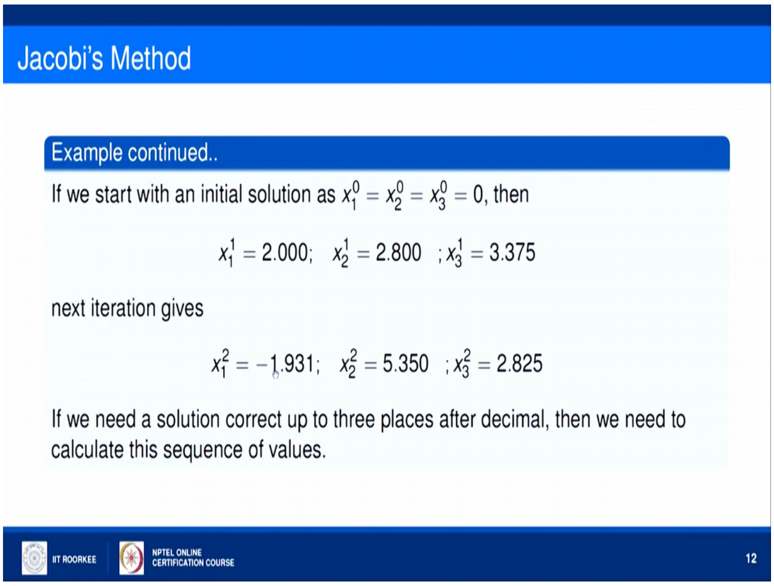

So if I go in the same way in the second iteration I got x 1 as minus 1 point 9 3 1 x 2 as 5 point 3 5 0 and x 3 as 2 point 8 2 5. If we need a solution correct up to 3 decimal places then we need to calculate this sequence of values. And if we go in the same way in the third equation x 1 will be obtained like this x 2 will be this 1.

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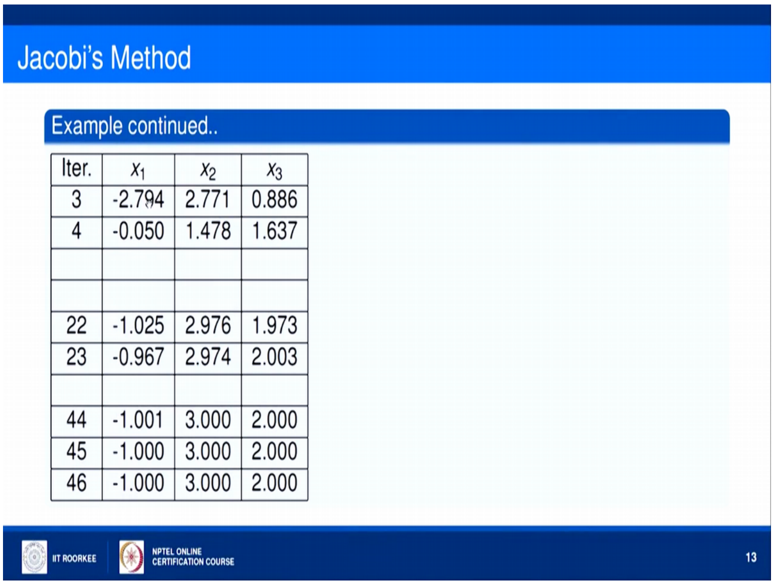

And x 3 will be 0 point 8 8 6 in fourth iteration continuing in the same manner in twenty second iteration value will be like this in twenty third iteration value will be like this still you can see we are having not much flexibility in the value of x 2, but in the value of x 1 and x 2. Moving continuing in the same way, we see in the forty fifth iteration x 1 comes out to be minus 1 x 2 is 3 and x 3 is 2 which remains same in 46 iteration, so it is my exact solution.

So, we have taken 46 iteration for converging to the exact solution from the sequence of approximations and this is the Jacobi method.

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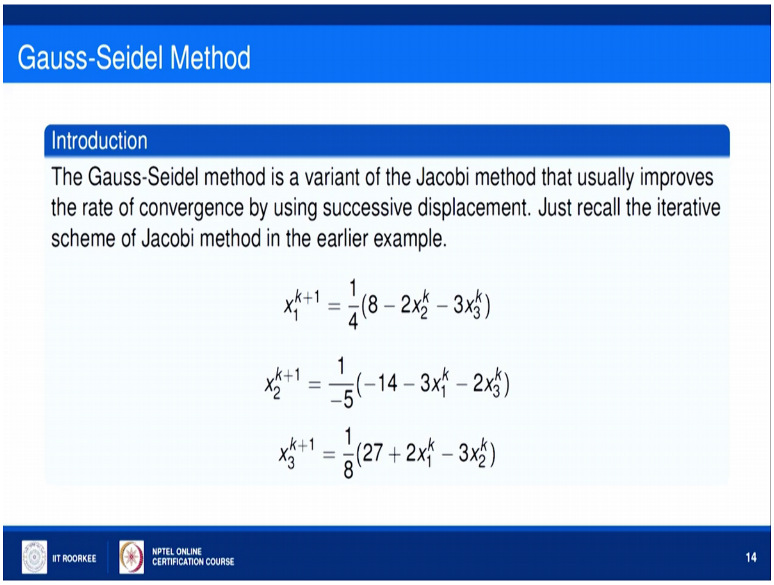

My next method in this category is Gauss-Seidel method which is a bit better when compared to the Jacobi method in terms of convergence. So, the Gauss-Seidel method is a variant of the Jacobi method that usually improves the rate of convergence by using successive displacement in Jacobi we have used the simultaneous displacement. So, here we will use successive displacement, just recall the iterative scheme of the Jacobi method in the earlier example.

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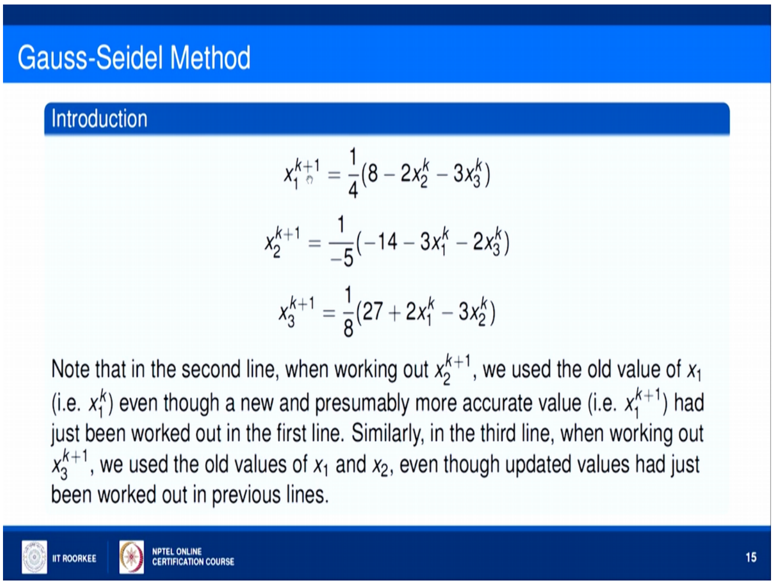

So, this was the iterative scheme of the Jacobi method in the example which we have taken in case of Jacobi method. So, if you observe here I have calculated from the first equation x 1 8 k plus 1 iteration when I am calculating x 2 in k plus 1 iteration using the Jacobi method I am using the approximation which I have obtained for x 1 and x x 3 in k-th iteration.

However if you observe since I have calculated x 1 in k plus 1 iterations, so I am having a more updated value of this x 1 here when I am calculating x 2. Similarly when I am calculating x 3 I am having more updated value of x 1 and x 2 those are from k plus 1 iteration. However, in Jacobi method I am using the values from the k-th iteration.

So what I can do instead of using the old values I can use the more updated values and in that manner I can have a faster convergence. So, in Gauss-Seidel method we use such type of updated values.

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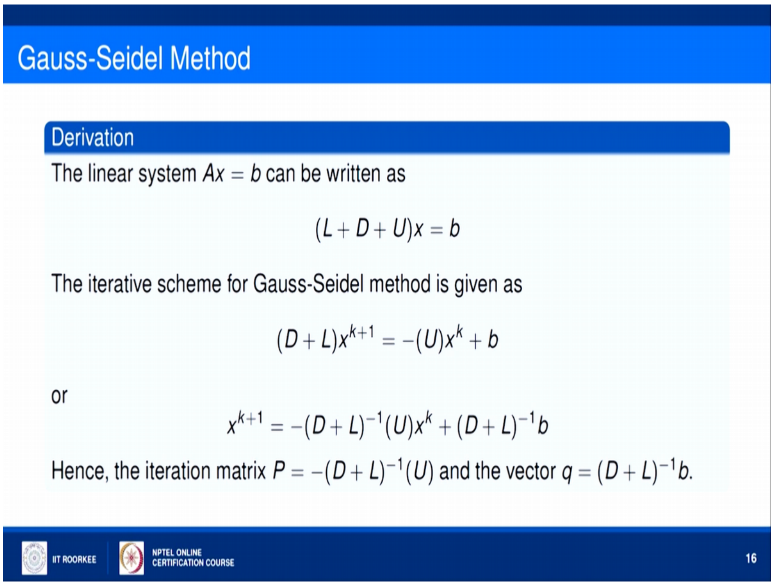

So, if we see the derivation of this method then I am having Gauss-Seidel method.

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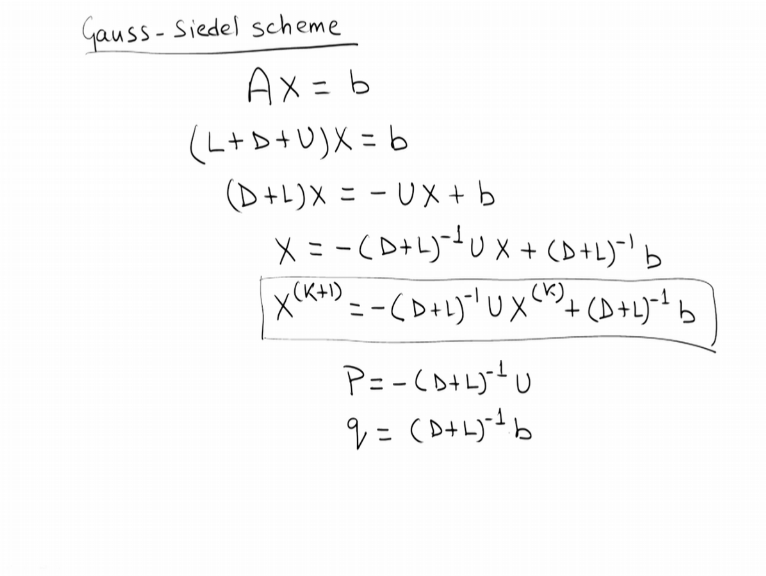

So, again I am having n by n system ax equals to b I will write matrix as the sum of 3 matrices L D and U, where L is lower triangular D is diagonal and U is an upper triangular matrix into x equals to b.

Now, in this method what I will do I will take D plus L here and what I will take I will take minus U into x in the right hand side. So, from here I can write x equals to minus D plus L inverse into U into x plus D plus L inverse into b and the iterative scheme can be set like this x at k plus 1 means in the left hand side I am taking the value in k plus 1 iteration, which equals to minus D plus L inverse U into x at k-th iteration plus D plus L inverse into b.

So, this is the iterative scheme of the Gauss-Seidel method. So, here if you check the iteration, iteration matrix p comes out to be minus D plus L then inverse of this sum U this is the iteration matrix p and the vector q is D plus L inverse into b.

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$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$
$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$$

$$x_1^{(K+1)} = \frac{1}{a_{11}} \left(b_1 - a_{12} x_2^{(K)} - a_{13} x_3^{(K)} \right)$$
$$x_2^{(K+1)} = \frac{1}{a_{22}} \left(b_2 - a_{21} x_1^{(K+1)} - a_{23} x_3^{(K)} \right)$$
$$x_3^{(K+1)} = \frac{1}{a_{33}} \left(b_3 - a_{31} x_1^{(K+1)} - a_{32} x_2^{(K+1)} \right)$$

. So, if you again take a 3 by 3 system like a 1 1 x 1 plus a 1 2 x 2 plus a 1 3 x 3 equals to b 1 a 2 1 x 1 plus a 2 2 x 2 plus a 2 3 x 3 equals b 2 and a 3 1 x 1 plus a 3 2 x 2 plus a 3 3 x 3 equals to b 3. So, if I need solve this system using the Gauss-Seidel method then my iterative scheme will comes like this. So, x 1 at k plus 1 iteration will be 1 upon a 1 1 b 1 minus a 1 2 x 2 at k-th iteration minus a 1 3 x 3 at k-th iteration.

Then I will take x 2 at k plus 1 iteration this will become 1 upon a 2 2 b 2 minus a 2 1 and please note that this is the change from the Jacobi method, earlier in Jacobi method I have taken x 1 at k-th iteration, but here since I am having x 1 in k plus 1 iteration available with me. So, I will have x 1 at k plus 1 iteration minus a 2 3 x 3 at k-th iteration, and then x 3 in k plus 1 iteration this will become 1 upon a 3 3 b 3 minus a 3 1 and I am having x 1 at k plus 1 iteration minus a 3 2 x 2 at k plus 1 iteration. Because now I am having x 1 and x 2 both available at k k plus 1 iteration, so this is the scheme of the Gauss-Seidel method.

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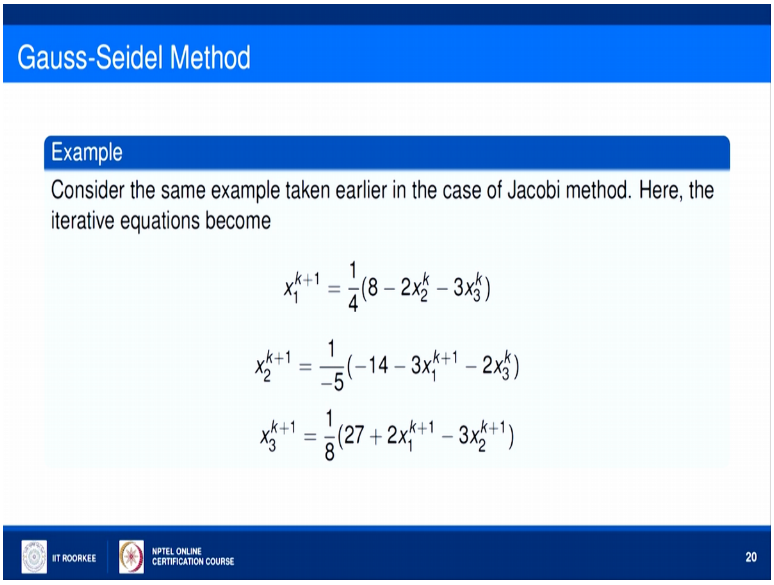

So, consider the same example which we have taken earlier in the case of Jacobi method. So, we will solve the same example using the Gauss-Seidel method. So, according to whatever just I have told I have told you according to that iterative process my iterative equations becomes like this. So, x 1 at k plus 1 iteration will be 1 upon 4 8 minus 2 x 2 k minus 3 x 3 k which is similar as in case of Jacobi method.

Then I am having x 2 at k plus 1 iteration it will become 1 upon minus 5 minus 14 minus 3 x 1 k plus 1. So, please note this 1 this is the change from the Jacobi method minus 2 x 3 at k-th iteration x 3 at k plus 1 iteration will become 1 upon 8 27 plus 2 x 1 k plus 1 minus 3 x 2 k plus 1.

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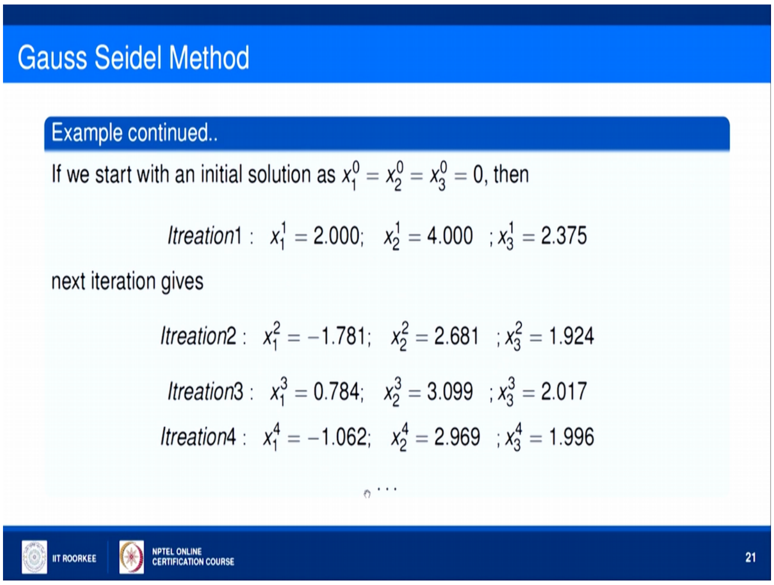

So, in this way if I start with 0 0 0 means initial case for x 1 0 for x 2 0 and for x 3 0 then, in the first iteration my x 1 become becomes 2 x 2 becomes 4 and x 3 becomes 2 point 3 7 5. Then in the second iteration x 1 will become minus 1 point 7 8 1 x 2 becomes 2 point 6 8 1 x 3 becomes 1 point 9 2 4. In third equation x 1 becomes 0 point 7 8 4 x 2 becomes 3 point 0 9 9 and x 3 becomes 2 point 0 1 7, these are the values of x 1 x 2 x 3 in the fourth iteration.

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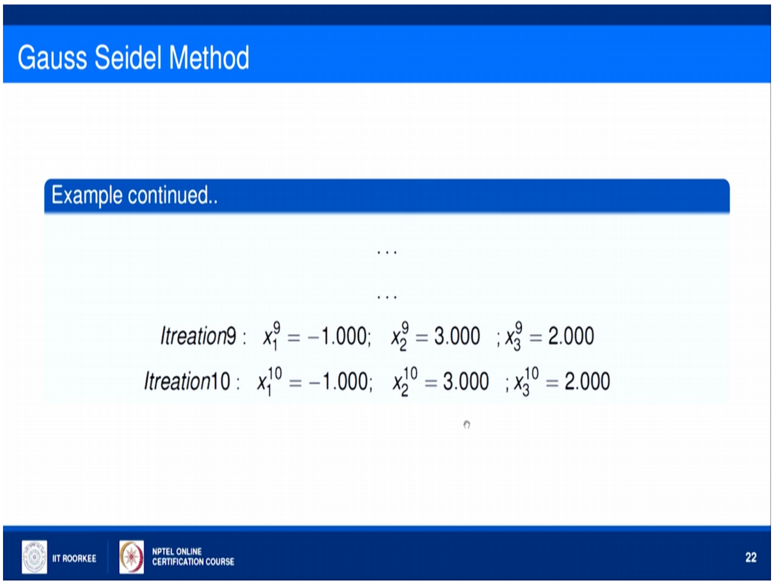

Continuing in the same manner in ninth iteration we observed that x 1 is minus 1 x 2 is 3 x 3 is 2 in tenth iteration x 1 is minus 1 x 2 is 3 x 3 is 2 which is same as we have obtained in the ninth iteration, when we are taking accuracy up to 3 places after the decimal.

So, this is a actually the exact solution also so what we have observed in Jacobi scheme. We have taken 46 iterations for converging to this solution. However, in the Gauss-Seidel method we have obtained the same solution means solution with same accuracy just in 10 iterations. So, in this way we can say the Gauss-Seidel method is a better variant of the Jacobi method, because it makes the usage of successive displacements more updated values.

So, in this lecture we have learned the two iterative schemes those are very basic schemes one is the Jacobi method, and the other one is Gauss-Seidel method. In the next lecture we will learn one more scheme called successive over relaxation, and then we will discuss the convergence criteria and few results about the convergence of these three schemes. And then we will go to non-stationary iterative methods like steepest descent conjugate gradients and other Krylov subspace methods for solving large and sparse linear systems.

Thank you very much.