Matrix Analysis with Applications
Dr. Sanjeev Kumar
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture - 30
Introduction to Ill-Conditioned Systems

Hello friends. So, welcome to the lecture on Introduction to Ill-Conditioned Systems. So, in this lecture we will learn when a system will be called ill-conditions, apart from that we will learn how to measure the ill-conditioning of a given system. So, for introducing the ill-conditioned system let us take one example.

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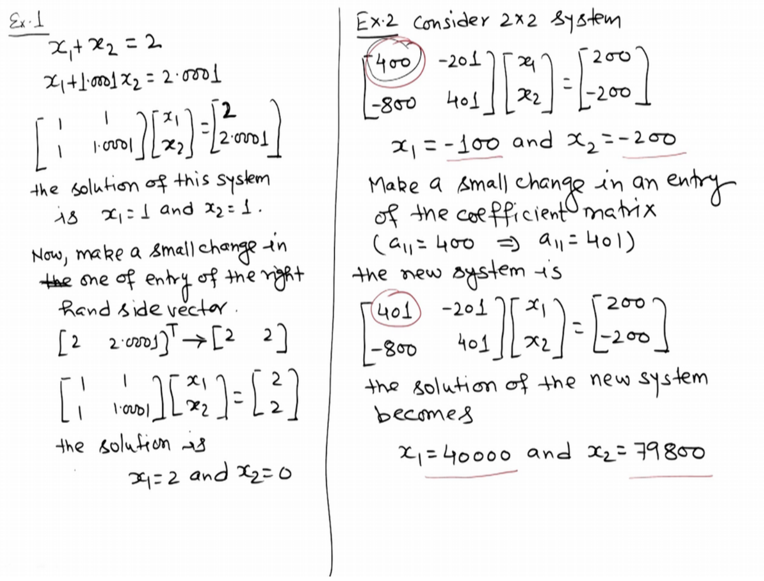

So, consider a 2 by 2 system x 1 plus x 2 equals to 2 and then x 1 plus 1.000 x 2 equals to 2.0001. So, here the coefficient matrix is one 1 1 1.0001 the unknown variable column is x 1 x 2 and right hand side column vector is 2 and 2.0001 now the solution of this system is x 1 equals to 1 and x 2 equals to 1 this you can verify now make a small change in the in one of the entry of the right hand side vector that is you are having earlier the vector 2 2.0001 transpose.

Let us say due to some sensor error or due to some computer what I will say round of sort of thing this 2.0001 will be read as 2. So, now, the new system is 1 1 1 1.0001 x 1 x 2 equals to 2 2 if we see the solution of this system the solution is x 1 equals to 2 and x 2 equals to 0. So, what we have seen if I make a small change in one of the entry of the right hand side vector then my solution is having a large change because earlier solution was x 1 equals to 1 x 2 equals to 1.

But due to this small change now it will become x 1 equals to 2 and x 2 equals to 0 in the same way take one more example. So, here I am having again a 2 by 2 system where the coefficient matrix is 400 minus 2 201minus 800 4001 this is the coefficient matrix the vector is again x 1 x 2 and the right hand side vector is 200 and minus 200.

In the example one we have made a small change in the in entry of the right hand side vector now let us make a small change in one of the entry from the coefficient matrix. So, this is the 2 by 2 system now if I look for the solution of this system the solution is given as x 1 equals to minus 100 and x 2 equals to minus 200 now if I make a small change in an entry of the coefficient matrix because in the earlier example I have made change in the right hand side vector now I am making the change in coefficient matrix.

So, let us say earlier a 1 1 entry was 4 0 0 now make it a 1 1 as 4 0 1. So, the new system is 4 0 1 minus 2 0 1 minus 800 and then 4001 together with x 1 x 2 and right hand side vector is the same 200 and minus 200. So, now, the solution of this new system becomes x 1 equals to 40000 and x 2 equals to 79000and 800.

So, now you just notice I am having only a small change earlier it was 400 now it becomes 4 0 1 how much change I am getting in my solution earlier x 1 was minus 100 now x 1 become 40000 x 2 was minus 200 now x becomes 79000 and 800. So, what we have observed that if you make a small change either in an entry of the right hand side vector or in an entry of the coefficient matrix a very small change tiny change.

We are getting a very large change in our solution so, such type of systems are called ill-conditioned systems.

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If we make small change in my input data (an entry of right hand side vector or coefficient matrix) and get a large change in the solution, then the system is called ill-conditioned system.

Measure of ill-conditioning :-

Conditional number :. Let A be an invertible matrix, then the conditional number of A is denoted by K(A),

$$K(A) = \|A\| \cdot \|A^{-1}\|$$

→ When the conditional number K(A) of ~~a matrix~~ the coefficient matrix becomes large, the system $AX=b$ is regarded as ill-conditioned.

→ If K(A) is near to 1, then the system is called well conditioned.

So, if we make a small change in my input data for the linear system it will become an entry of right hand side vector or coefficient matrix and get a large change in the solution which is output then the system is called ill-conditioned systems. Now, let us how to measure the ill-conditioning of a system for that we need to define conditional number. So, let a be an invertible matrix then the conditional number of a is denoted by ka and it is defined as ka equals to the product of norm of a with norm of a inverse now when the conditional number KA of a matrix or of the coefficient matrix I will write becomes large the system AX equals to b is regarded as ill-conditioned.

So, you find out the conditional number of the coefficient matrix and if it is very large then we will say the system is ill-conditioned if the conditional number of a where A is the coefficient matrix is near to one then the system is called well conditioned. So, hence we can measure the ill-conditioned ill-conditioning of a system by calculating the conditional number and if it is quite large then the system is ill-conditioned.

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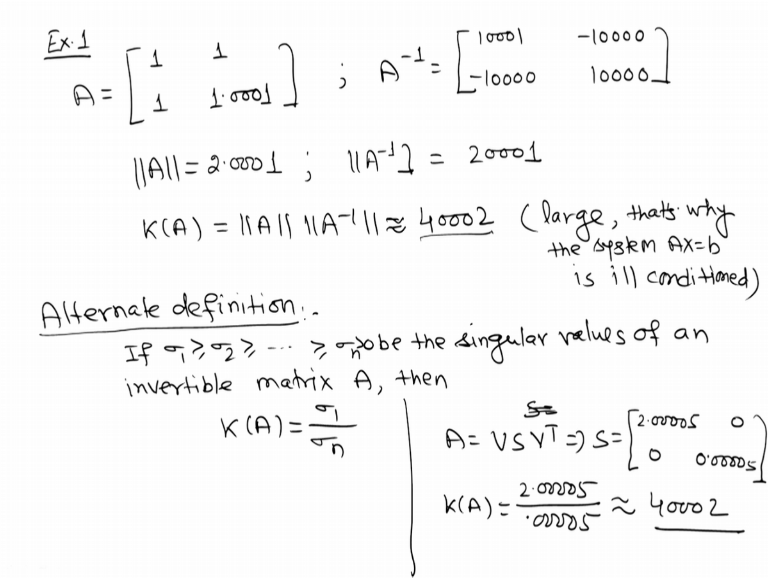

So, if we see those 2 examples. So, in example 1 my coefficient matrix was one 1 1 and 1.0001. So, if I calculate A inverse here A inverse comes out to be 10000 101 and then minus 10000minus 10000 and then finally, 10000. So, here norm of a will become the maximum of sum of the rows of A. So, I am taking the row norm. So, it will be 2.0001 and if I calculate a norm of a inverse this comes out to be 20001.

So, now if I find out the conditional number of a this will be norm of a into norm of A inverse which comes out to be something around 42002 almost of this 1. So, hence point something like this. So, hence it is quite large and that is why the system ax equals to b is ill-conditioned similar kind of analysis we can make for another matrix alternate definition for conditional number is if be the singular values of an invertible matrix a.

So, all will be greater than 0 then conditional number of A is the ratio of the largest singular values upon the smallest singular values. So, if we see in the above example the singular values of a will be. So, if a equals to U S V transpose then S will be 2.000 0 5 0 0 and 0.0005. So, from here conditional number a will be 2.0005 upon 0.00005 and which will be the same 40002 which you have obtained with the definition norm of a into norm of A inverse. So, by this alternate definition you can calculate the conditional number if you know the singular values of A given matrix.

Now, let us make some investigation why we are having this change in what I will say the change in solution a large change in the solution by making a very tiny change in the input data means either an entry of coefficient matrix or an entry of right hand side vector.

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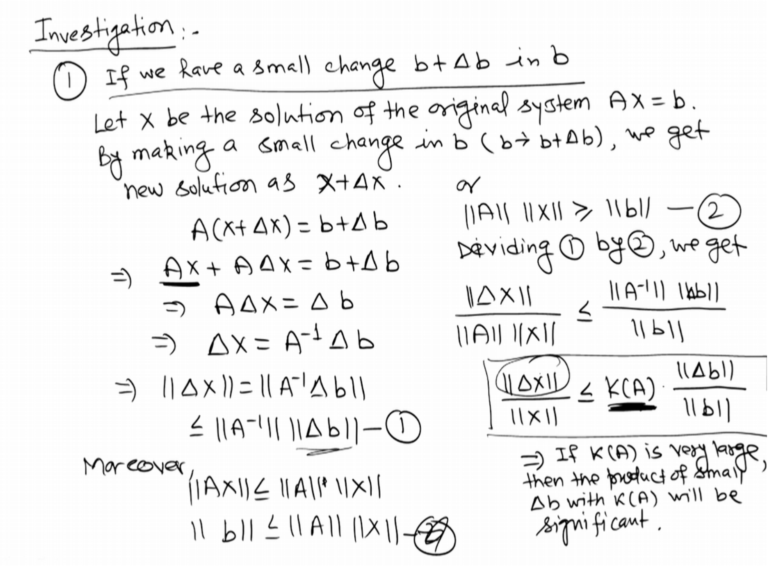

So, first we will check if you we have a small change in b plus delta b in b means in right hand side vector. So, let X be the solution of the original system AX equals to b by making a small change in b that is b becomes b plus delta b we get new solution as X plus delta X. So, what we are having A X plus delta X equals to b plus delta b ok.

So, this I can write ax plus A into delta X equals to b plus delta b, but as you know this ax equals to b. So, this ax I can replace with b. So, from here I am getting A into delta and b will be cancel out from both side A into delta X equals to delta b or from here delta x equals to a inverse into delta b or norm of delta x will be norm of A inverse into delta b this equals less than equals to norm of a inverse, because norm of A into b will be less than equals to norm of A into norm of b into norm of delta b. So, this let us say equation 1 moreover we have norm of AX is less than equals to norm of A into norm of X and X equals to b.

So, norm of b less than equals to norm of A into norm of X let us say 2 or I can write in other way norm of a into norm of X is greater than equals to norm of b. So, let us say this is my equation 2 dividing 1 by 2 we get norm of delta X upon norm of A into norm of X because left hand side of 1 is delta X and left hand side of 2 is norm of a into norm of X and this will be less than equals to norm of a inverse into norm of b norm of a inverse into norm of b delta b, sorry because it is delta b here upon norm of b. If I multiply the this inequality by norm of A I will get norm of delta X upon norm of X in the left hand side which is less than equals to norm of A into norm of A inverse and norm of A into norm of A inverse is my conditional number into delta norm of delta b upon norm of b.

So, from here you can observe if I am if the conditional number is very large if I am having a small delta b I will get a large delta X. This means if K is very large then the product of small delta b with large K A will be significant and that is why I will get a significant delta X means significant change in my final solution. So, this is the investigation when I am making the change in the right hand side vector and from here we have seen that why for the large conditional number a small change in right hand side vector will give a large change in the solution.

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(II) $A \rightarrow A + \Delta A$

$X \rightarrow X + \Delta X$

$(A+\Delta A)(X+\Delta X) = b$

$$\Rightarrow \frac{\|\Delta X\|}{\|X+\Delta X\|} \leq \underline{K(A)} \cdot \frac{\|\Delta A\|}{\|A\|}$$

$\Rightarrow$ ____ $\times$ ____ $\times$ ____ .

Now, take the second case when I am making a small change in my coefficient matrix means if I am having the original coefficient matrix A. And now I am making a small change and it is becoming a plus delta A and if earlier my solution was X now it becomes X plus delta X then what I will be having A plus delta A into X plus delta X equals to b, because there is no change in the right hand side vector by making the same kind of analysis which we have done earlier in the first case I will get delta X upon X plus delta X will be less than equals to conditional number of a into delta norm of delta a upon norm of a.

So, again if conditional number of A is quite large a small delta A multiplied with a large KA will give you a very large value and that is why I am getting a large change in delta X. So, this 2 kind of investigations still such that why conditional number is very important for ill-conditioning and if conditional number is near about 1 then a small change multiplied with an entry which is quite close to 1 will give you the small change delta X.

In this lecture we have learnt what are the ill-conditioning systems what is conditional number what is the relation between conditional number and singular values and we have investigated why the system becomes ill-conditioned when we are having the large conditional number.

In the next lecture we will learn how to solve such type of ill-conditioned system by defining a proper regularization term.

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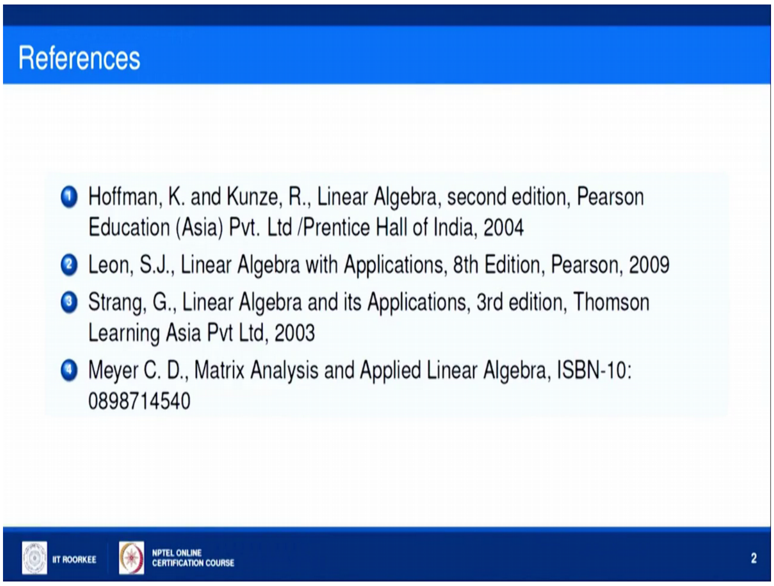

So, these are the references.

Thank you very much.