Matrix Analysis with Applications
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Lecture - 28
Single Value Decomposition

Hello friends. So, welcome to the lecture on Singular Value Decomposition. So, singular value decomposition or SVD in short, is a very powerful tool of a linear algebra and matrix analysis. It is having variety of applications ranging from data analysis then row rank approximation to image and signal processing.

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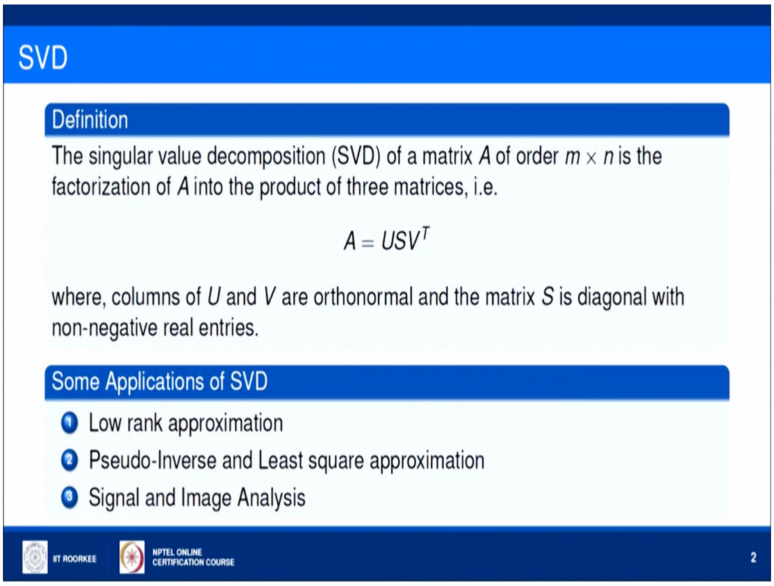

So, the definition of SVD is the singular value decomposition of a matrix A of order m by n is the factorization of a into the product of three matrices that is U S and V such that A equals to U into S into V transpose, where U and V are having orthonormal columns and the matrix S is diagonal with non-negative real entries.

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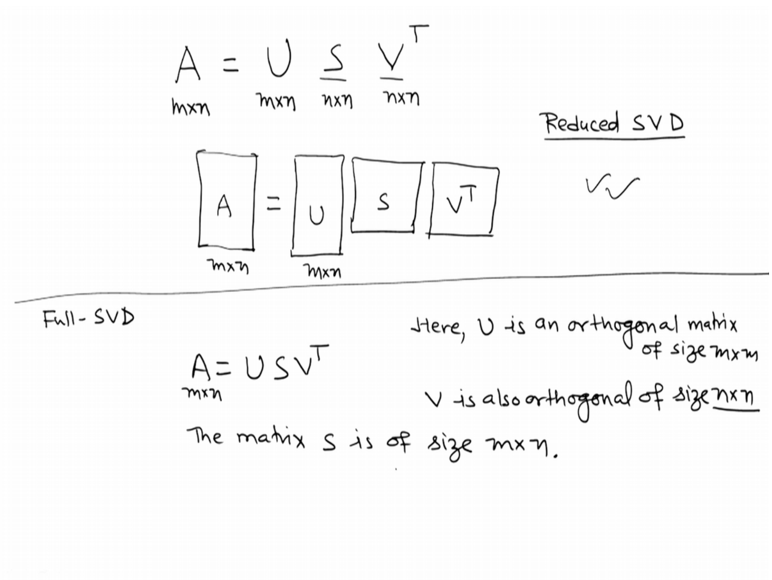

So, what I want to say the matrix A is having size m by n. So, this matrix A I am writing as the product of matrix U S and V transpose. Here, the matrix U is of size m by n and all the n columns of U are orthonormal. The matrix n the matrix S is a n by n diagonal matrix and the matrix V, V transpose is n by n; so the V. So, like this so, if I am having a matrix A of like this so, m by n, so from the size itself you can see this A is having more rows than columns then it will be equals to m by n matrix U and then a matrix n by n matrix S which is diagonal and V transpose. So, this is called reduced singular value decomposition.

Another type of singular value decomposition is called full SVD or full singular value decomposition. So, there again we write A as the product of U, S, and V transpose, here U is an orthogonal matrix of size, if this is m by n then U will be of size m by m, V is also orthogonal of size n by n and the matrix S is of size m by n. So, this is full singular value decomposition and this is reduced singular value decomposition. So, let us see this further.

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$$\underset{m\times n}{A} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}^T$$

$u_i \in \mathbb{R}^m$

$v_i \in \mathbb{R}^n$

Here, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ (non-increasing order)

σ_i's are called singular values of A.

$\Rightarrow$ Singular values are non-negative.

So, here I am saying A is m by n matrix. So, it will become a matrix U which will be having column u 1, u 2, up to u n and each u i will belongs to m dimensional space, means having the m component and then I am having matrix S which is having a matrix of n by n and it is a diagonal matrix having diagonal entries are sigma 1, sigma 2 up to sigma n and then I will be having the matrix V which is V transpose which is again having n by n matrix. So, it will be having n columns and each v i belongs to n dimensional real vector space means each column is a vector of this one. Moreover u i's are orthonormal as well as v i's are orthonormal. So, these are orthonormal vectors.

Here, sigma 1 will be means sigma 1, sigma 2 sigma n will appear in a non increasing order, moreover and all will be non negative these sigma is means sigma 1, sigma 2, sigma 3 are called singular values of A. Hence, I can say singular values are non negative. Furthermore, we will say they are not only non negative they are also real.

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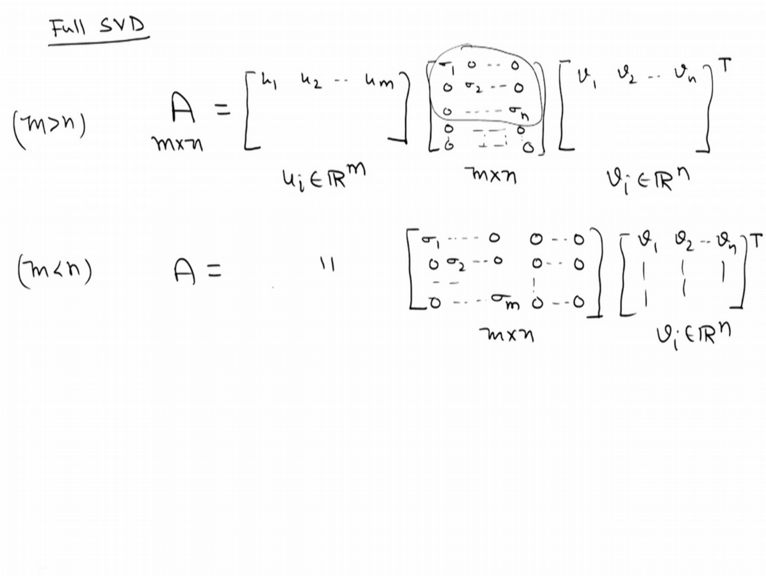

Now, this is the case of reduce SVD if I talk about full SVD then I am having a, which is of size m by n then I am writing matrix U of size m by n. So, here I will be having only m columns and each u i will be having m component here my matrices will be of size m by n.

So, let us take a case first m is greater than n means you are having the more number of rows than columns in matrix A, then S will be written as sigma 1, 0, 0, 0, 0, sigma 2, 0, 0, 0, 0 then you will be having sigma n here. So, a n by n diagonal sub matrix and then m minus n number of 0 rows. So, it is m by n matrix where a m by n sub matrix this one and m minus m 0 rows. Finally, I will be having a n by n orthogonal matrix V transpose. So, it will be like v 1, v 2, v n. So, each v i belongs to R n.

Now, this is the case if m is less than n then A will be this will remain same only S will change here. So, now, the shape of S will be like this sigma 1, 0, 0, sigma 2, 0, 0, 0, sigma m. So, m by m sub matrix which is diagonal and then m minus n number of 0 columns to make it m by n. And then finally, V transpose which is again an orthogonal matrix orthogonal matrix means all v i's are orthonormal all u i's make a orthonormal set.

Now, this is the SVD in both the cases means how what will be the shape of matrix U, what will be the shape of matrix S and what will be the shape of matrix V. Now, how to compute?

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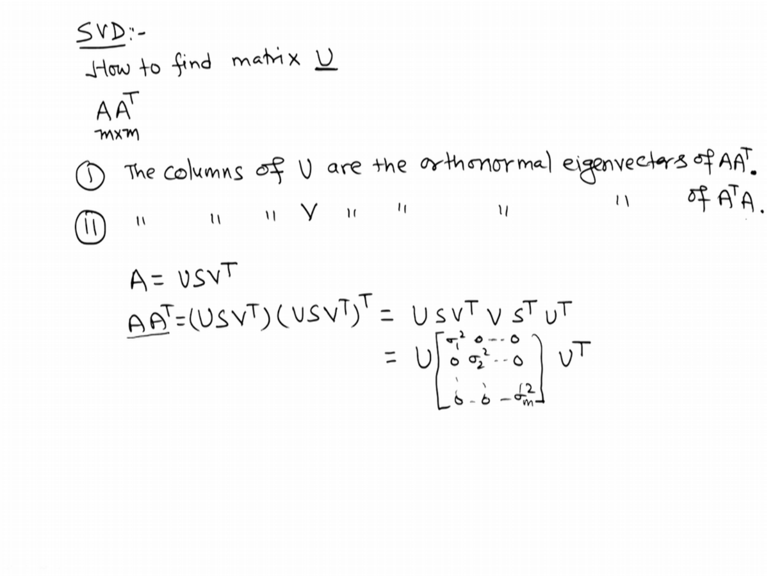

So, how to do SVD? Basically, how to find matrix the three matrix. So, how to find matrix U? So, basically the matrix A is of size of size m by n. So, if I take the matrix A into A transpose the size of this matrix will be of order m by m. So, the size of this matrix is m by m. Now, further more this will be a symmetric matrix.

So, the Eigen values of this matrix means A into A transpose will be real, moreover it will contain an orthonormal set of Eigenvectors since it is symmetric. So, the columns of matrix U will be the orthonormal Eigenvectors of A into A transpose. So, what I want to say the columns of U are the orthonormal Eigenvectors of A into A transpose. So, in this way we will be able to compute matrix U which is orthogonal matrix.

Now, how to compute matrix V which is again orthogonal of size n by n; so basically the columns of V are the orthonormal Eigenvectors of A transpose A and this you can prove quite easily because A is USV transpose. So, A into A transpose will become USV transpose into USV transpose transpose. So, it comes out to be USV transpose into V transpose transpose will become V, S transpose into U transpose.

So, this will become U and it will be having shape like this sigma 1 square, 0, 0, 0, 0, sigma 2 square S into S transpose like this and will depend into U transpose 0, 0, sigma m square. So, here you can see I can write A into A transpose as U this matrix into U transpose and what will be the Eigenvalue of A into A transpose they will be sigma 1, square sigma 2 square and sigma n square.

Hence, it is a diagonalization of A into A transpose, where the Eigenvectors are the columns of matrix U and hence the columns of U are the orthonormal Eigenvectors of A into A transpose. The similar kind of analysis we can do for the columns of V those are the orthonormal Eigenvectors of A transpose into A. So, this is.

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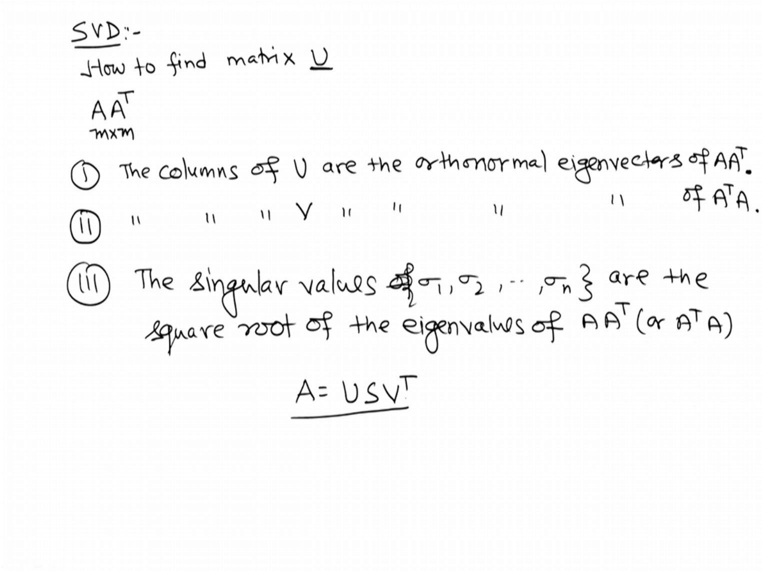

Now, how to find out S? So, S will be containing the singular values of the matrix A and the singular values are the singular values that is sigma 1, sigma 2 up to sigma n are the square root of the Eigenvalues of A into A transpose or A transpose into A because both will be having the same sort of Eigenvalues. If 1 is having the bigger size the rest of the Eigenvalues will be 0 because that is due to rare deficiency. So, in this way we can calculate U, we can calculate V, we can write our matrix S and we can perform the singular value decomposition of A that is USV transpose.

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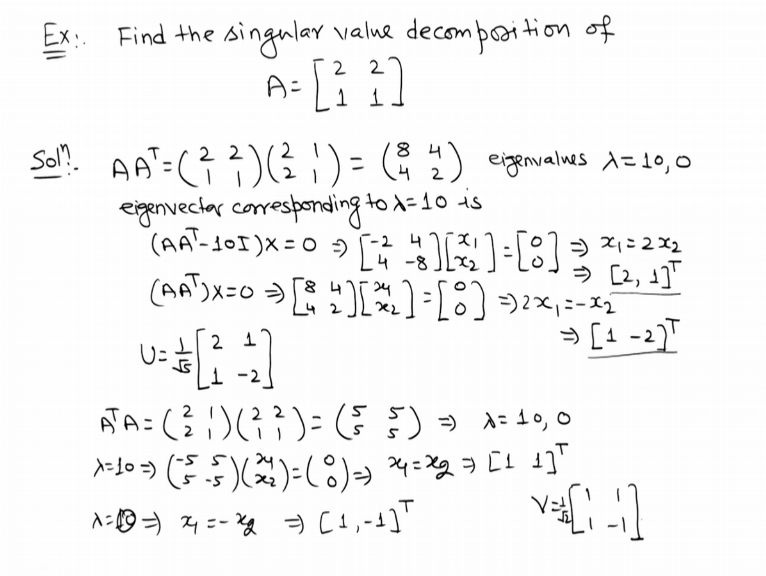

So, let us take an example of it. Find the singular value decomposition of A equals to 2, 2, 1, 1. So, it is a square matrix for simplicity I have taken. I will take an example of rectangular matrix also. So, here A into A transpose will become 2, 2, 1, 1 into 2, 2, 1, 1 which comes out to be 8, 4, 4, 2. So, the Eigenvalue of this A into A transpose will come so, lambda minus 8 into lambda minus 2 minus 16 equals to 0. So, lambda equals to 10 and 0. Here eigenvector corresponding lambda equals to 10 is so, A A transpose minus 10 I into x equals to 0. So, it will be 2 minus 8 2 sorry 2 minus 10 it will become minus 8 minus 10 minus 2, 4, 4 minus 8, x 1, x 2 equals to 0, 0. So, the solution of this will become x 1 equals to 2 x 2.

So, if I take x 2 equals to 1. So, Eigenvector will become. So, x 2 is 1. So, x 1 will become 2, 2, 1 transpose. Similarly, Eigenvector corresponding to lambda equals to 0 will become A into A transpose minus 0 I into x equals to 0. So, this gives me 8, 4, 4, 2, x 1, x 2 equals to 0, 0. So, it means 2x 1 equals to minus x 2. So, from here Eigenvector I can write if I take x minus 1 so, 1 and minus 2 transpose. One more thing is very important here we have to make this Eigenvectors orthonormal Eigenvectors. So, we have to divide this by the norm of these vectors. So, norm of this vector will be root 5, norm of this vector will be root 5. So, my matrix U will become 1 upon root 5 into the first Eigenvector 2 1 is the first column of U and 1 minus 2. So, this is the matrix a U.

Now, I will find the matrix V. So, for that I will take A transpose into A. So, A transpose into A will become 2, 2, 1, 1 is 2, 2, 1, 1. So, this comes out to be 4 plus 1, 5, 5, 5, 5. So, here again Eigenvalues come out to be because both will be having the same Eigenvalue because Eigenvalues of A B and B A are the same. So, A into A transpose A and A transpose into A will be having the same Eigenvalue. Moreover, if I find the Eigenvector corresponding to lambda equals to 10. So, this comes out to be minus 5, 5, 5, minus 5, x 1, x 2 equals to 0, 0 this gives me x 1 equals to x 2. Similarly, if I take lambda equals to 0 I will get so, this gives me an Eigenvector 1 1 transpose and from here I got x 1 equals to minus x 2. So, from here I got another Eigenvector 1, minus 1 transpose.

So, here my matrix V will become Eigenvector corresponding to of A transpose A corresponding to lambda equals to 10. So, it is 1, 1 only thing I have to divide it by the lengths which is root 2 1, 1, 1 minus 1.

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$$U = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} ; \quad V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} ; \quad S = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

So, what I am having U is 1 upon root 5 and then I am having 2, 1 and 1, minus 2 V equals to 1 upon root 2, 1, 1, 1 minus 1. Now, I need to find out matrix S. As I told you my sigma 1 which is the bigger singular value will be the square root of the biggest Eigenvalue of A A transpose or A transpose A, so which is 10. So, it will become root 10, 0, 0 another Eigenvalue is 0. So, root square root of 0 will be 0.

So, here a will be having singular value decomposition U which is 2 upon root 5, 1 upon root 5, 1 upon root 5, minus 2 upon root 5 into S which is root 10, 0, 0, 0 and then V transpose; so 1 by root 2, 1 by root 2, 1 by root 2, minus 1 by root 2. So, this is the singular value decomposition of a square matrix.

Let me take another example in which I am having a rectangular matrix.

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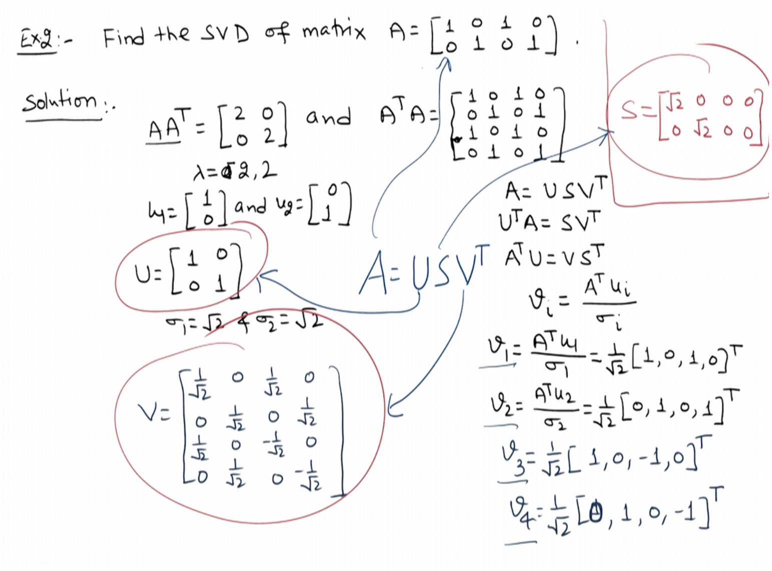

So, example is find the singular value decomposition of matrix A which is given as 1, 0, 0, 1, 1, 0 and 0, 1. So, let us solve this. So, here A into A transpose will become it will be a 2 by 2 matrix, 2, 0, 0, 2 and A transpose A will become 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0 and then 0, 1, 0, 1.

Now, let us first find out U for which I have to use A into A transpose. So, Eigenvalues of A into A transpose will become 2 and 2. So, lambda equals to 2, 2 and Eigenvectors will become U 1 equals to 1, 0 and U 2 equals to 0, 1. So, here my matrix U comes out to be 1, 0, 0, 1. Now, my sigma 1 will become root 2 and sigma 2 also become root 2 because they are the square root of the Eigenvalues of A into A transpose.

Now, I am having a equals to USV transpose, where what I am having where U and V are orthogonal matrices. So, if I premultiply by U transpose I will be having U transpose A equals to S into V transpose. So, by this lesson what I can write so, in a more better way I can have taking the transpose of both side A transpose into U will become V into S transpose. So, from here I can write v i equals to that is the column of V A transpose u i upon sigma i from this lesson because column wise if I want to calculate this one because S is a diagonal matrix. So, that will come in denominator.

So, now if I calculate v 1 from here v 1 will become a transpose into u 1 upon sigma 1. So, A is given to you I can calculate a transpose into u 1 upon sigma 1. So, this comes out to be 1 by root 2, 1, 0, 1, 0 transpose. From here if I calculate v 2 which will become A T u 2 upon sigma 2 which comes out to be 1 by root 2, sigma 2 is root 2 into 0, 1, 0, 1 transpose as you know that V capital V is a 4 by 4 matrix. So, there will be four columns I have just calculated only two columns. So, I need to calculate two more columns.

So, if I am able to find out two more orthonormal vectors that is v 3 and v 4 such that the set v 1, v 2, v 3 and v 4 makes an orthonormal set of Eigenvectors then my job will be done. Please note that here I am not going by the classical process means finding the Eigenvalue of it and then finding the Eigenvectors corresponding to each Eigenvalue, I want to make a shortcut and that I have taken from this particular relation. So, I need to choose two more vectors v 3 and v 4 such that they makes an orthonormal set. So, if I choose v 3 equals to 1 by root 2 1, 0, minus 1, 0 transpose and v 4 is 1 by root 2 so, 0, 1, 0, minus 1 transpose then you can see that v 1, v 2, v 3 and v 4 are orthonormal and this can serve the purpose of columns of the matrix V.

So, here my matrix V comes out to be 1 by root 2. So, this I am writing this vector is the column 0, 1 by root 2, 0, then this will become 0, 1 by root 2, 0, 1 by root 2 then this will be 1 by root 2, 0, minus 1 by root 2, 0 and then this will be 0, 1 by root 2, 0 minus 1 by root 2. So, hence clearly V is an orthogonal matrix. Now, only thing left to write the matrix S. So, as I told you in this case S will be of since my original matrix of size 2 by 4. So, x will be of size 2 by 4 and it will be the Eigenvalue means root 2, 0, 0, root 2 and then 2, 0 columns as I told you in the definition of full SVD.

So, here I am having my U, S and V and here matrix a will be equals to USV T, where U is this one, S is this one and V transpose can be obtain from this V and A is this 1. So, this is another way of doing singular value decomposition without computing the Eigenvectors of a bigger matrix, but here you have to choose these vectors very carefully because this would make a pair a set of orthonormal vectors.

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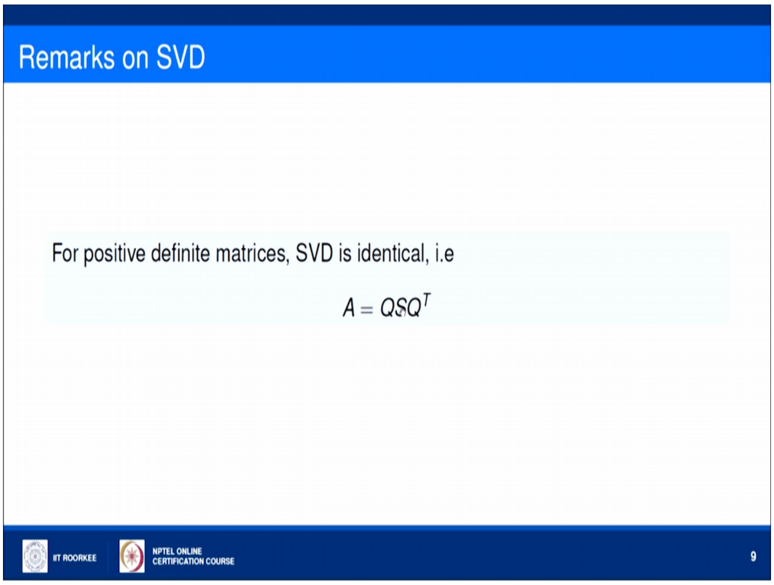

So, there is a remark on singular value decomposition. For the positive definite matrix means if A is a positive define matrix then SVD is identical to QSQ transpose. So, it is identical to some sort of factorization or what I will say diagonalization. Basically it will be diagonalization only

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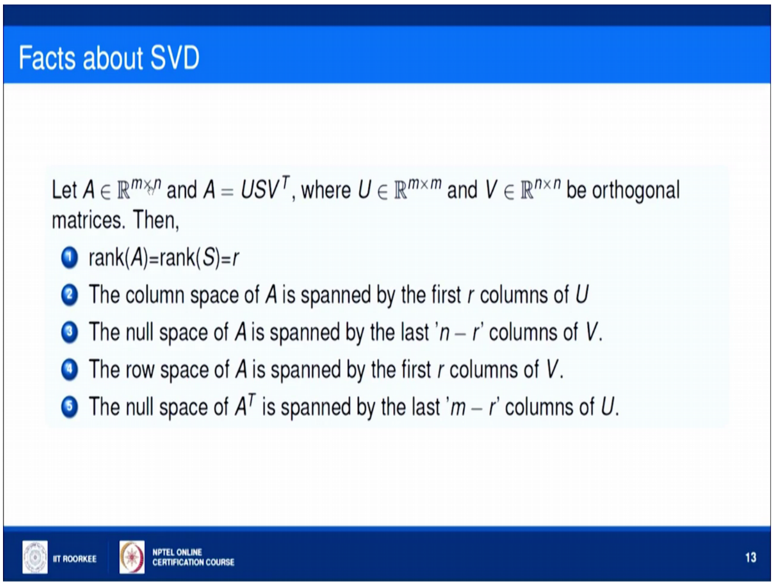

Some facts about singular value decomposition if A is a real matrix of size m by n and the singular value decomposition of a equals to USV transpose, where U is m by m matrix, V is n by n both are orthogonal then rank of matrix A will be number of nonzero singular values of A that is rank of S and that will be r. So, rank of A matrix will be the number of nonzero singular values because singular values will be non negative. So, they may be non nonzero or zero. So, number of nonzero singular values will be the rank of the matrix A.

The column space of A is spanned by the first r columns of U that is the first r columns of U will form a basis for column space of A. The null space of a is spanned by the last n minus r columns of V, means the solution of the system ax equals to 0 that is the homogeneous linear system will be given by the last n minus r columns of V and those will be the columns corresponding to zero singular values the row space of a is spanned by the first r columns of V. And similarly the null space of a transpose is spanned by the last m minus r columns of U.

Hence, if you know the singular value decomposition of a matrix A. Then you can write all the four fundamental subspaces of that particular matrix that is column space, null space, row space and then null space of a transpose.

So, in this lecture we have learnt singular value decomposition. We have taken couple of examples that how to do singular value decomposition of a matrix. In the next lecture, we will find out a relation between least square approximation and singular value decomposition. So, these are the references.

Thank you very much.