Matrix Analysis with Applications
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Lecture- 26
Evaluation of Matrix Functions

Hello friends. So, welcome to the lecture on Matrix Functions. So, in this lecture we will learn how to calculate the functions having matrix as variable. So, first let me define the matrix function.

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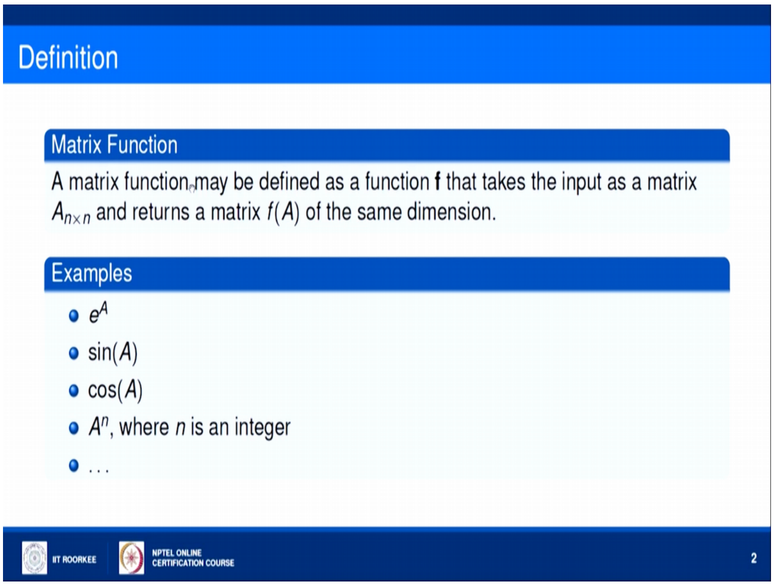

So, a matrix function may be defined as a function f that takes the input as a matrix A of order n by n and returns another matrix f of A of the same dimension. So, we can define various matrix functions for example, such that f of A equals to e raised to power A or f of A let us A raised to power n, where n is any integer.

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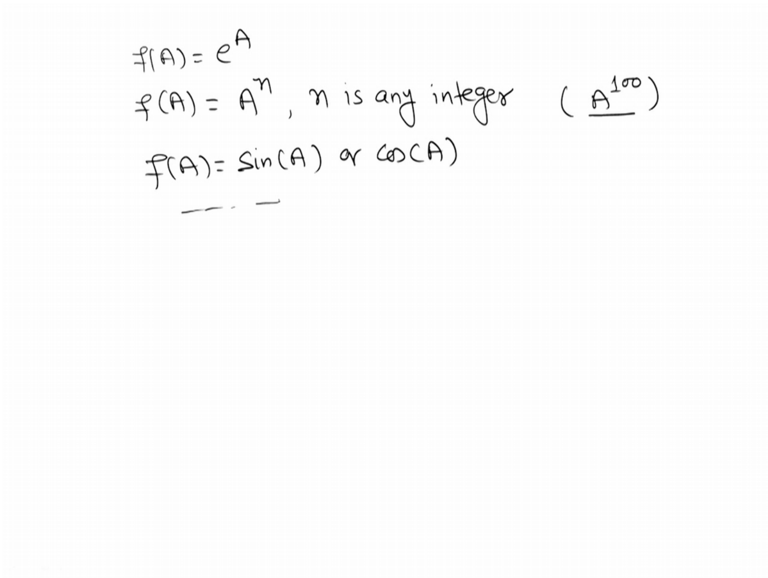

So, for example, if someone ask you to calculate A raised to power 100, which is not easy because you have to multiply the matrix A 100 times. Another may be the trigonometric functions of A for example, sin of A or cos of A or hyperbolic functions and many more, that is the polynomials of A. So, these are the examples of matrix functions.

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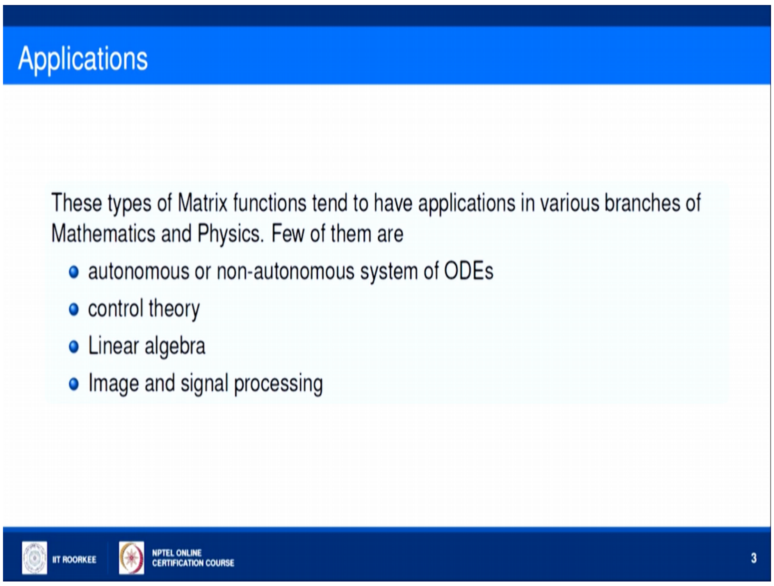

Now, why we calculate or why we evaluate the matrix functions? So, we are having several applications of matrix functions especially in mathematics and physics, few of them I have listed here like autonomous or non-autonomous system of ordinary differential equations, in control theory we are having plenty of applications of the matrix functions, and where we need to evaluate the matrix functions, in linear algebra; obviously, and then we are having applications of these matrix functions in image and signal processing. Because, an image can be consider as a matrix and when you are performing some operation on image you have to evaluate that particular functions means; output of that particular operation will be the value of the functions when image will be the input variable.

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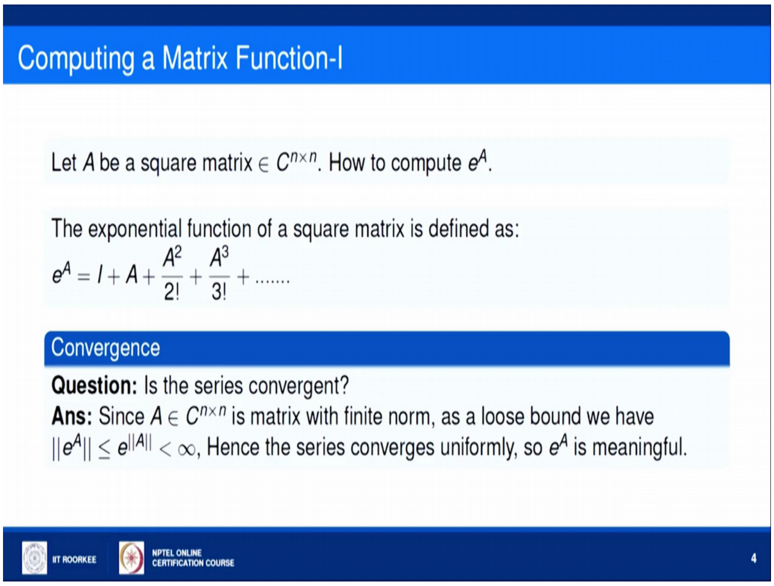

So, let us learn about the computing matrix functions. So, here so, let me take the case 1.

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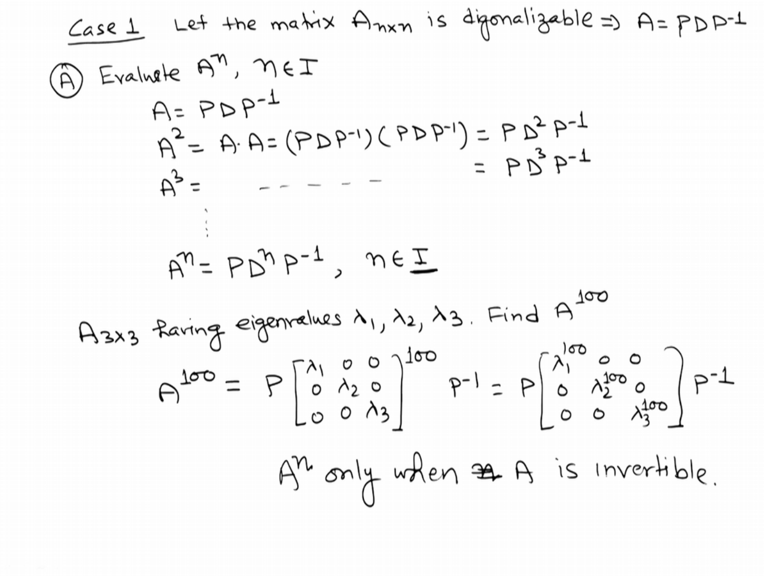

And in case 1 I am taking that let the matrix A which is of size n by n is a is diagonalizable. So, it means I can write the matrix A equals to PDP inverse. Where P is the modal matrix coming from the eigenvectors of A and D is the diagonal matrix having diagonal entries as the eigenvalues of A.

Now, number 1: evaluate A raised to power n where n is any integer. So, what I am having as you know a can be written as P into D into P inverse, if I want to calculate A square, it will become A into A, that is PDP inverse which is A into A PDP inverse. So, this will become P D square P inverse, if I am having A cube. So, in the same way it will become P D cube P inverse.

So, for any given n A raised to power n will become. So, in the similar way A raised to power n equals to P D raised to power n P inverse, where n belongs to I. So, for example, if let us take a is A 3 by 3 matrix having eigenvalue lambda 1, lambda 2, lambda 3, then I can write a raised to power 100.

So, let us say find A raised to power 100. So, A raised to power hundred will become P which is the model matrix for A and then lambda 1 0 0 0 lambda 2 0 0 0 lambda 3 it is D. So, D raised to power 100 into P inverse. So, this will become P lambda 1 raised to power 100 0 0 0 lambda 2 raised to power 100 0 0 0 lambda 3 raised to power 100 into P inverse. So, in this way we can calculate various powers of A given matrix.

So, one more thing need to mention here since I have written A n belongs to I. So, for negative integers we will define A raised to power n only when, n only when matrix A is invertible. So, please take care of it.

Now, let me take the second case.

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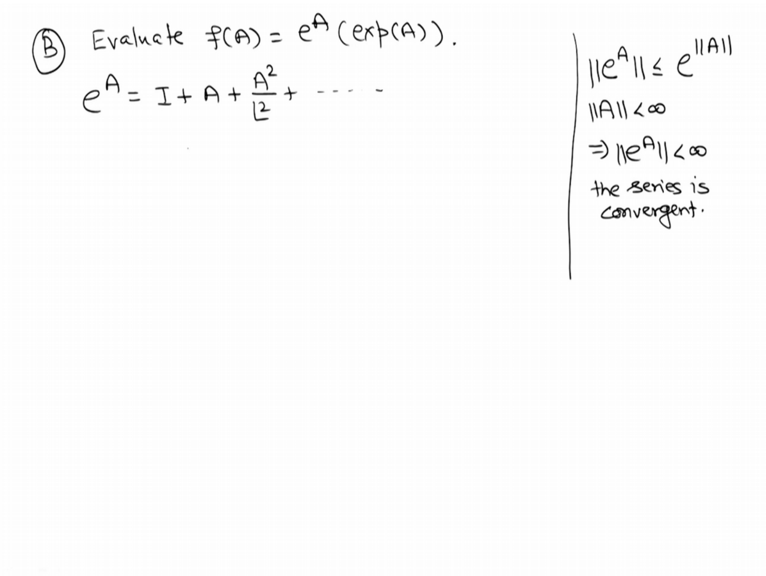

Evaluate f A equals to e raised to power A, where it means exponential of matrix A. So, as we know e raised to power A. So, if I open the series it will become I plus A plus A square by factorial 2 plus and so, on up to infinity. So, I can write this particular function e raised to power A in terms of this infinite series, in terms of this infinite series if the series is convergent. So, there is A question about the convergent of this series.

So, if I talk about convergent I can write e norm of e raised to power A as e raise to power norm of A. And, since A is A matrix of order n by n so, A will be finite, this gives me e raised to power A is less than infinity and hence the series is convergent in fact, uniformly convergent. So, this is about the convergence of this series. So, let us come back to our evaluation.

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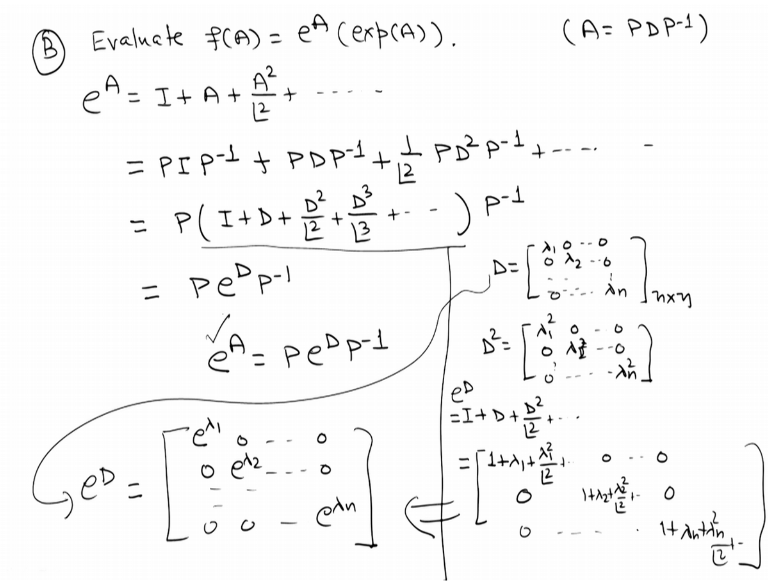

So, I can write this function, I can I write P I P inverse because A is diagonalizable. So, I can write A equals to PDP inverse. So, this A I can write PDP inverse, plus 1 upon factorial 2 and then this will become P A square will become from the previous evaluation P D square P inverse and so on. So, this I can write P I plus D plus D square upon factorial 2 in the same when next term will become D cube upon factorial 3 and so on into P inverse. So, this is P and if you see this round bracket term it is nothing just e raised to power D into P inverse.

So, what I got e raised to power A equals to P e raised to D P inverse. Now, the question is if D is given to you how to evaluate this e raised to power D. So, if let us assume that D equals to lambda 0 0 0 0 or let me write lambda 1 lambda 2 0 lambda n ok. So, it is n by n matrix so, then D square will become lambda 1 square 0 0 0 lambda 1 lambda 2 square 0 and in the same way lambda n square.

So, if I calculate I plus D plus D square by 2 and so on and which is nothing just e raised to power D. So, this comes out to be from I first entry will be 1 plus from here lambda 1 plus lambda 1 square upon factorial 2 from this matrix and so on 0 0, then the 0 the second entry here will become 1 plus lambda 2 plus lambda 2 square by factorial 2 and so on 0 and in this way, 1 plus lambda n plus lambda n square by factorial 2 and so on.

So, this is the matrix e raised to power D and this is nothing just this becomes e raised to power lambda. So, this I can write simply e raised to power D if my D is this one then e raised to power D will become e raised to power lambda 1 0 0 0 0 e raised to power lambda 2 0 0 0 e raised to power lambda n. So, you can calculate e raised to power D quite easily. So, hence you can calculate e raised to power D.

Let us take an example of it.

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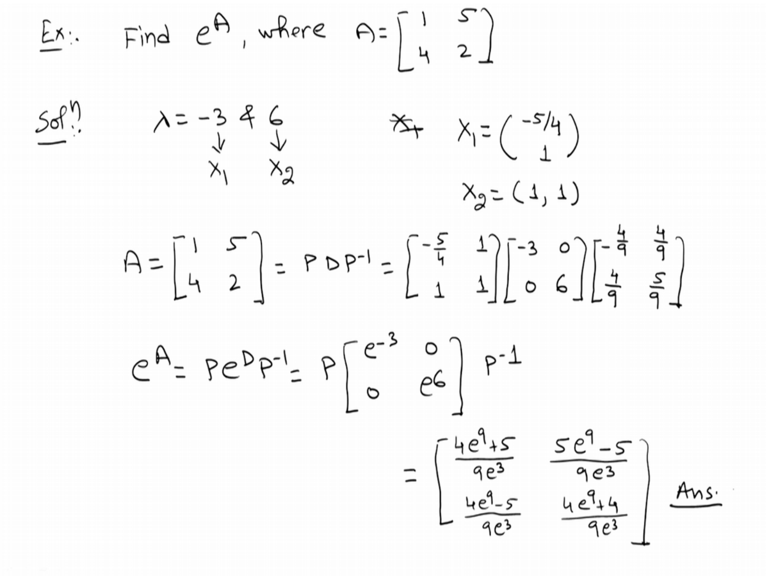

Find e raised to power A where A is given as 1 5 4 2. So, let us see the solution of it. So, here the eigenvalues of A is lambda equals to minus 3 and 6, the eigenvector X 1 corresponding to lambda equals to minus 3. So, the eigenvector corresponding to lambda equals to minus 3 is X 1 and let us say the eigenvector corresponding to lambda equals to 6 is X 2, since both the eigenvalues are different. So, they will be having linearly independent eigenvectors and hence matrix is diagonalizable.

So, if we calculate X 1, that is the eigenvector corresponding to lambda equals to minus 3. So, it comes out to be minus 5 by 4 and 1. Similarly, if I calculate X 2, which is the eigenvector corresponding to lambda equals to 6 it comes out to be 1 and 1.

So, from here I can write the matrix A. So, I can write A equals to P that is my matrix A is 1 5 4 2. So, this is equals to PDP inverse. So, matrix P is minus 5 by 4 1 1 1 means second column here D is lambda 1, which is minus 3 0 0 lambda 2 which is 6 and then P inverse. So, P inverse comes here 4 minus 4 upon 9 4 upon 9 4 upon 9 and 5 upon 9.

Now, e raised to power A will become P e raised to power D P inverse. So, this will become P e raised to power D will become e raised to power minus 3 0 0 e raised to power 6 into P inverse. And, this will become if I evaluate this multiplication or product of these 3 matrices 4 e raised to power 9 plus 5 upon 9 e raised to power 3 will be the first entry 5, e raised to power 9 minus 5 upon 9 e raised to power 3 will be the second entry, 4 e raised to power 9 minus 5 upon 9 e raised to power 3 will be the third entry means, the first entry of the second row and then 4 e raised to power 9 plus 4 upon 9 e raised to power 3 will be the last entry. So, this is my answer that is e raised to power A.

Now, consider the second case. In the first case we have consider that A is A diagonalizable matrix, but suppose A is not diagonalizable then we can use the Jordan canonical form of A for finding the function of A.

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Case II: When A is not digonalizable, then $A = SJS^{-1}$

(A) $A^n = SJ^nS^{-1}$

$[\lambda_0]^n = [\lambda_0^n]$

$$J = \begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix} \Rightarrow J^2 = \begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix}\begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix} = \begin{bmatrix} \lambda_0^2 & 2\lambda_0 \\ 0 & \lambda_0^2 \end{bmatrix}$$

$$J^3 = J^2 J = \begin{bmatrix} \lambda_0^3 & 3\lambda_0^2 \\ 0 & \lambda_0^3 \end{bmatrix}$$

For example, if I want to evaluate case 2 when A is not diagonalizable, then by the Jordan canonical transformation I can write A equals to S J S inverse and how to calculate it you have learn in the previous lecture?. So, now, if I want to evaluate let us say A raised to power n.

So, here A raised to power n will become S into J raised to power n into s inverse and now how to calculate J raised to power n?. So, I will evaluate it with the Jordan blocks of various size. So, let me take a Jordan block of size 1. So, if lambda 0 is there. So, this raised to power n will become simply lambda 0 raised to power n.

So, now, problem with Jordan block of size 1 now if you take Jordan block of size 2 so, lambda 0 1 0 lambda 0. So, let us assume this is my J. So, now, if I calculate J square it will become lambda 0 1 0 lambda 0 into lambda 0 1 0 and lambda 0. So, if I multiply by these 2 matrices it comes out to be lambda 0 square 2 lambda 0 0 and then it will become lambda 0 square, if I calculate J raised to power 3. So, it will be J square into J. So, it means I have to multiply again with this matrix here into this matrix.

So, this becomes lambda 0 cube 3 lambda 0 square 0 and lambda 0 raised to power 3. So, this is about 2 by 2 matrices. Similarly, we can obtain it for 3 by 3 Jordan blocks or Jordan blocks of any size.

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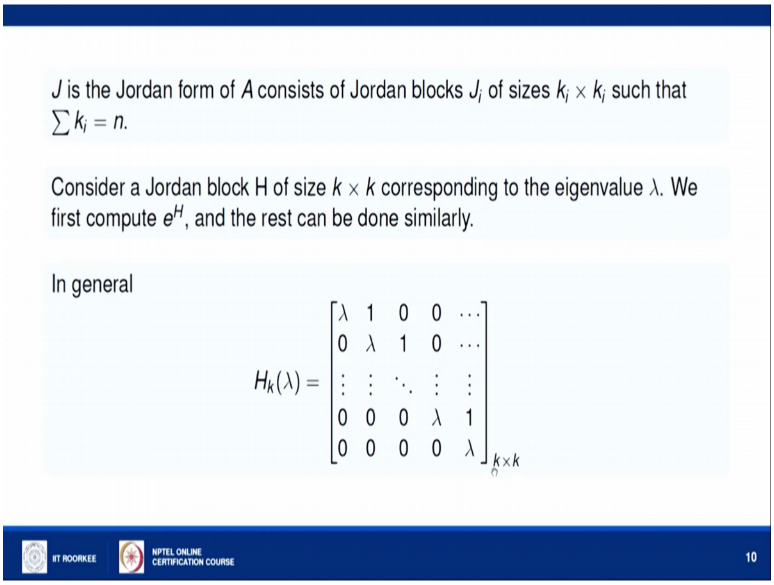

So, in general if I am having a Jordan block of size k by k having diagonal entries at lambda, then the power of n of this Jordan block given is given by, lambda n is the first diagonal entry then nc 1 lambda raised to power n minus 1, nc 2 lambda raised to power n minus 2 and so on.

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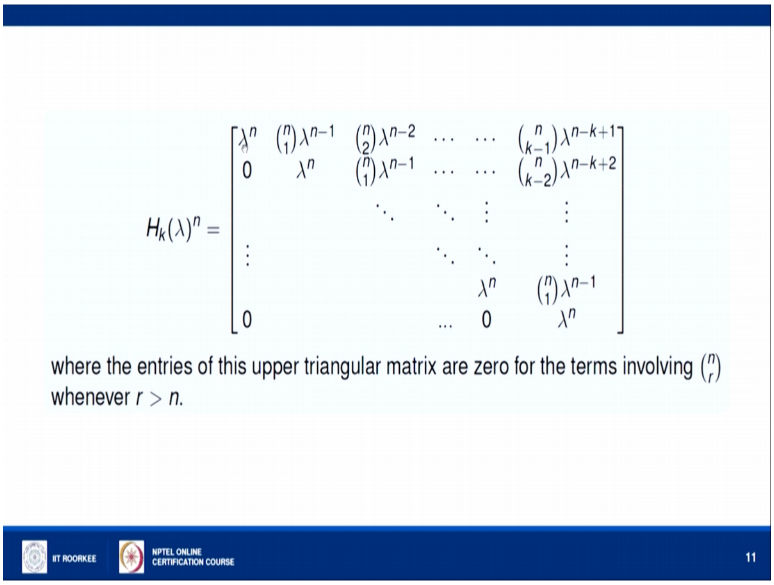

Then 0 lambda raised to power n nc 1 and so on and in this way ok. One more thing you have to note here that the entries of this upper triangular matrix are 0 for term involving ncr whenever r is greater than n.

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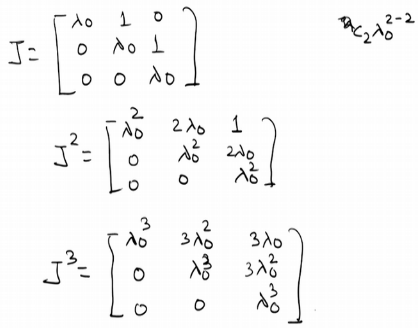

So, if I use this result for calculating if J is let us say A 3 by 3 blocks so, lambda 0 1 0 0 lambda 0 1 0 0 lambda 0. So, using this general formula here J square will become lambda 0 square then nc 1 means 2 c 1 lambda 0 raised to power n minus 1 that is 2 minus 1 1. So, it is become 2 lambda 0, then next term will become nc 2 lambda 0 raised to power n minus 2. So, as you know nc 2 will here n is 2. So, 2 c 2 will become 1.

So, it will become 1 0 lambda 0 square 2 lambda 0 0 0 lambda 0 square if someone ask J cube. So, in the same way now n will become 3. So, lambda 0 cube 3 c 1 will become 3 lambda 0 3 minus 1 will become powers will become 2, then next will become 3 c 2. So, 3 c 2 will again will become 3 and then it will become lambda 0 0 lambda 0 raised to power 3 lambda 3, lambda 0 square 0 0 lambda 0 raised to power 3. So, this particular matrix gives us the general power of a Jordan block.

So, in this way you can calculate A raised to power n is P s into A raised to power n equals to S into J raised to power n into S inverse and where J raised to power n can be calculated block wise using this particular formula, please note that. The Jordan canonical form will be having different blocks. So, you have to apply this formula block wise.

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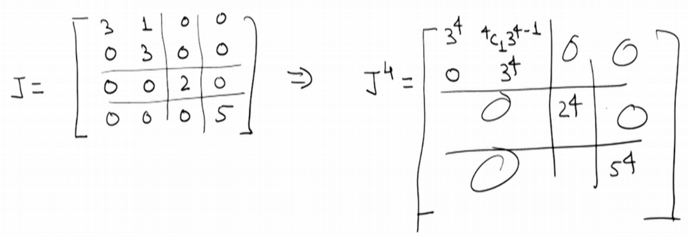

So, for example if I am having m Jordan block matrix like this 3 1 0 0 0 3 0 0 0 0 2 2 0 and 0 0 0 5 like this is my Jordan block matrix. So, here I am having 3 Jordan blocks 1 of size 2 corresponding 2 lambda equals to 3 and rest 2 are of size 1 corresponding to lambda equals to 2 and lambda equals to 5.

So, if someone ask me right J raised to power 4. So, now, I will use I will calculate it block wise. So, here n is 4 k is 2 here. So, first entry will become 3 raised to power 4, second entry will become 4 c 1 into 3 raised to power 4 minus 1, then 0 here and 3 raised to power 4.

So, this is for the first block. Now, second block it will become 2 raised to power 4 and for third block it will be 5 raised to power 4. So, this will become the corresponding J raised to power 4 matrix.

So, you have to calculate it block wise. Now, and it will be quite easy to calculate using the given formula.

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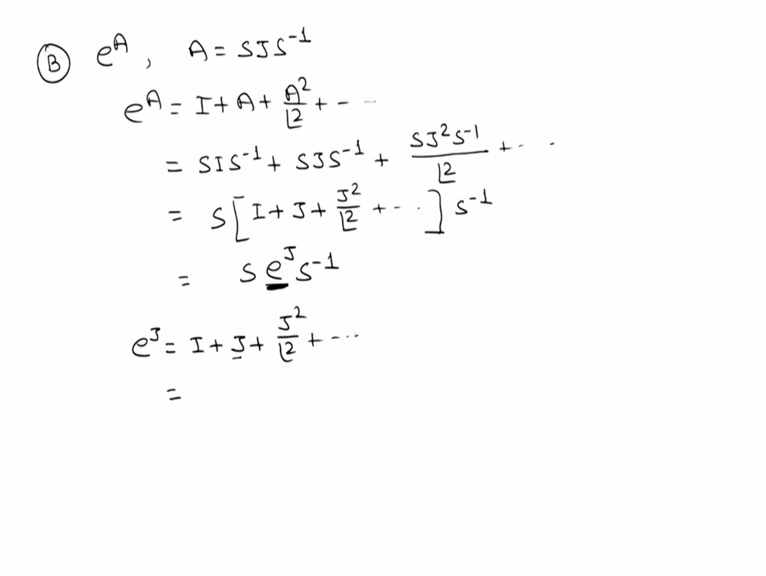

Now, case B evaluate e raised to power A A is given as SJS inverse. So, again e raised to power A will become I plus A plus A square upon factorial 2 and so on. So, this will become SI into s inverse plus SJ into s inverse plus SJ square S inverse upon factorial 2 and so on. So, this I can write S I plus J plus J square upon factorial 2 plus and so on infinite series into S inverse and this will become S into e raised to power J S inverse.

Now, how to calculate this e raised to power J. Again, you know that e raised to power J can be written as I plus J plus J square upon factorial 2 and so on and you know ho[w]- if J is given how to calculate J square and various powers of J. So, if I see this.

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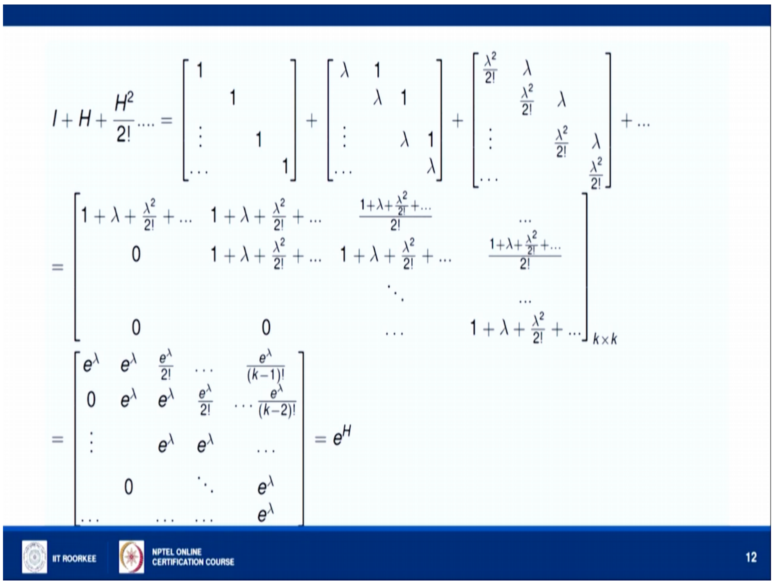

So, this will be if I take A Jordan block of size k. So, I plus means I am writing J, but it is H in slide. So, you can consider it J Jordan canonical. So, I plus H plus Jordan block. So, h is A Jordan block here.

So, I plus H plus H square H square by factorial 2 up to infinite terms. So, I is this h is this one H square is given by this one. So, if I add all these. So, final e raised to power x comes out to be e raised to power lambda e raised to power lambda e raised to power lambda upon factorial 2 and it will go up to e raised to power lambda upon factorial k minus 1.

The second row will be 0 e raised to power lambda e raised to power lambda and it will go up to e raised to power lambda upon factorial k minus 2 and so on. So, in this way you can evaluate e raised to power H.

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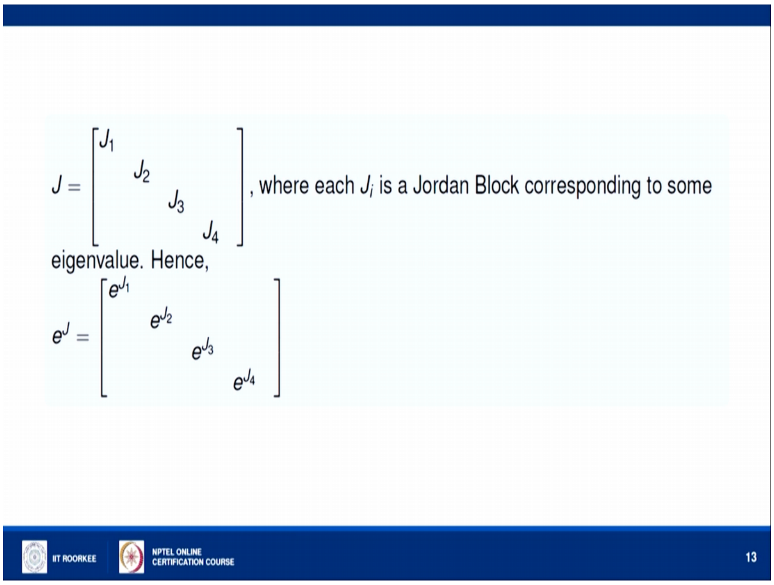

And, since J is a Jordan having various Jordan blocks. So, this formula will be have to apply each Jordan blocks separately best on the size at given eigenvalue. So, let me take one more example of this.

So, let me take a interesting example.

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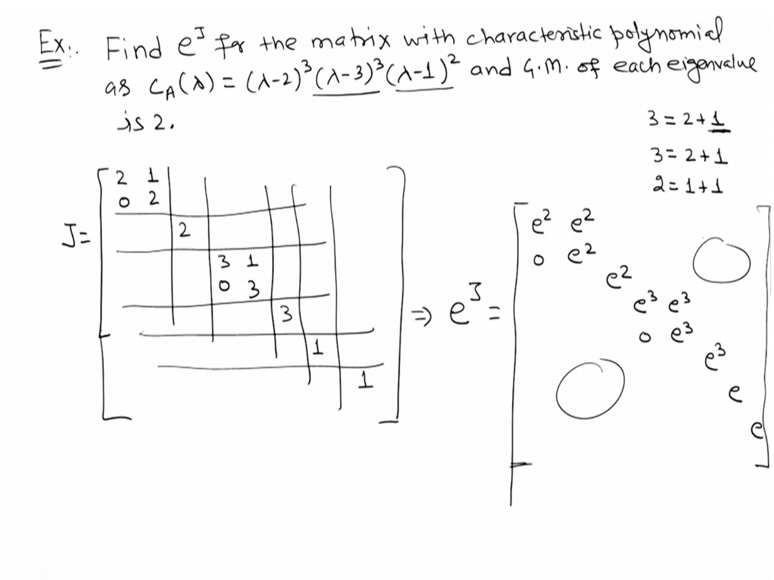

Find e raised to power J for the matrix with characteristic polynomial as C of A lambda equals to lambda minus 2 raised to power 3, lambda minus 3 raised to power 3, lambda minus 1 raised to power 2 and geometric multiplicity of each eigenvalue is 2 ok.

So, first of all we have to write the Jordan canonical form of such A matrix. So, if I write the Jordan canonical form. So, total size of the matrix becomes 3 plus 3 plus 2 8 by 8 matrix. So, here lambda equals to 2 will occur 3 times of in the main diagonal and geometric multiplicity is 2. So, there will be 2 Jordan blocks corresponding to lambda equals to 3. So, if I make the factors of 3 as 2 integers. So, it will become 2 plus 1. So, here again I am iterating that I do not consider the order 2 plus 1 or 1 plus 2 I am taking the same ; however, if you talk about the complete Jordan canonical form they will be different matrices.

So, here 2 1 0 2 so, this will be the first Jordan block corresponding to lamda equals to 2 another 1 will come from here. So, it will be simply A Jordan block of size 1. Now, come to this factor here again 3 geometric multiplicity is 2. So, 2 blocks. So, 3 factored in 2 integers as 2 plus 1. So, I will be having 3 1 0 3 and again 3 here and finally, come lamda minus 1 square. So, here algebraic multiplicity is 2 of lamda equals to 1 geometric multiplicity is 2 so, there will be 2 factor.

So, 2 integers with sum is 2 is 1 plus 1 positive integers. So, then I will be having 1 and 1. So, this is my matrix J now what I need to calculate e raised to power J. So, first I will calculate for this. So, it will become e square e square, because e raised to power lambda e raised to power lambda then e raised to lamda upon factorial 2 0 e square. Then I will come for this one. So, it will become e square, then I will come to this Jordan block e raised to power 3 e raised to power 3 0 e raised to power 3. Then I will come here e raised to power 3 e and e and rest will be 0.

So, this is the corresponding e raised to power J for such a given matrix. So, in this way you can write e raised to power J and e raised to power A will become s into e raised to power J into S inverse. So, in this lecture we have learn how to evaluate various functions especially e raised to power n and e raised to power A, for a given matrix A.

When a diagonalizable and A is not diagonalizable. If, you are having trigonometric function you can always write trigonometric function in terms of exponential function. So, you can calculate exponential using this process and then by performing the corresponding operations you can evaluate the given trigonometric function. So, with this I will end this lecture.

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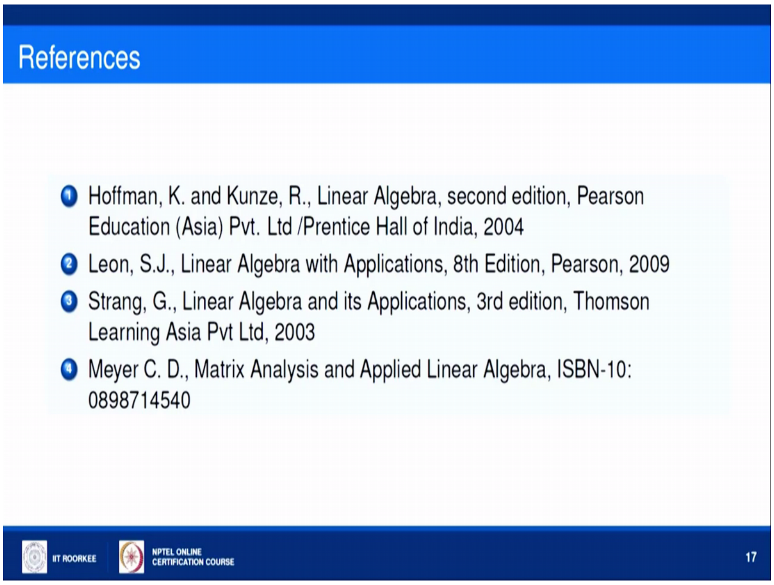

These are the references.

Thank you very much.