Matrix Analysis with Applications
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Lecture 25
Generalized Eigenvectors and Jordan Canonical Form

Hello friends, so, welcome to the lecture on Generalized Eigenvectors and Jordan Canonical Form. So, as we know that, if a matrix of order n by n having n linearly independent Eigen vectors, then the matrix can be written as P into D into P inverse or in other way other words we can say, the matrix is diagonalizable or the matrix is similar to a diagonal matrix, where the diagonal entries are the Eigen value of the matrix A.

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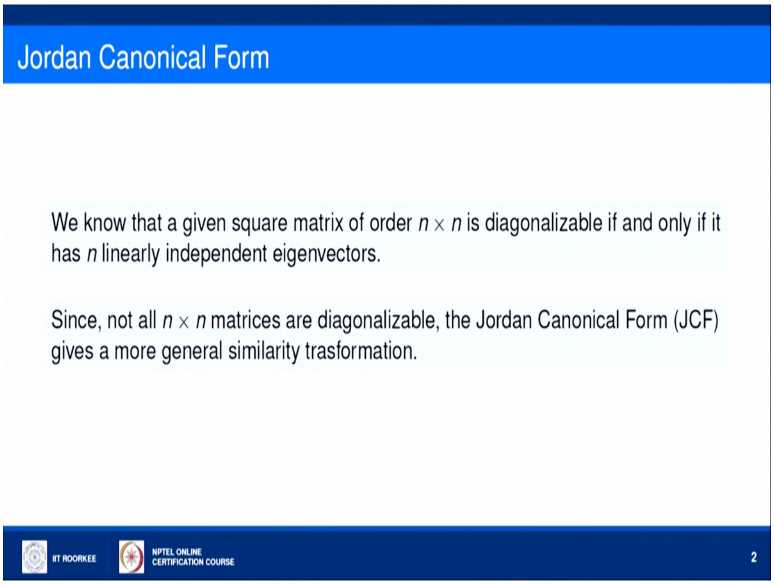

So, if the matrix is not diagonalizable means the matrix does not have n linearly independent eigenvectors. Then, how to find out a similar type of transformation so that, at least we can write this matrix as P J into P inverse where J is a block diagonal matrix. So, I want to say that, if matrix is diagonalizable then, I can write P into D into P inverse where D is a diagonal matrix. And if matrix is not diagonalizable, then write it P into J into P inverse, where J is a block diagonal matrix. So, it is a generalized similar transformation where we are reducing a given matrix into block diagonal matrix.

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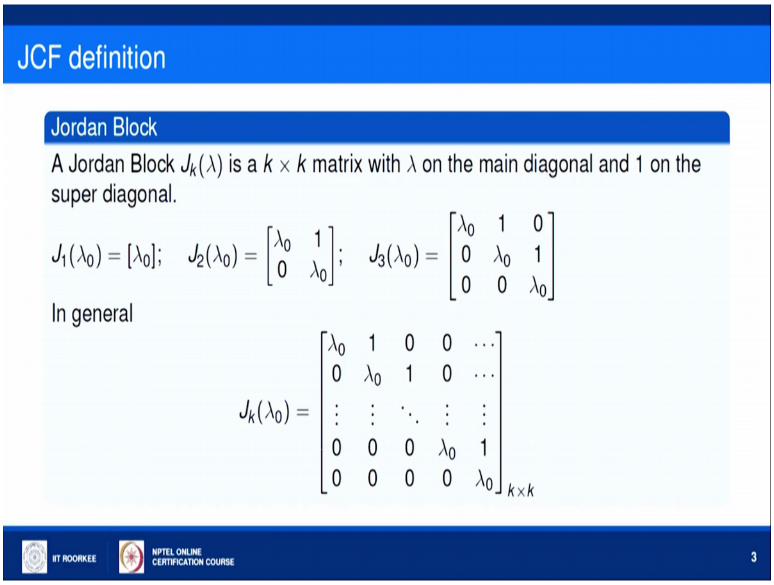

So, for doing this, we have to talk about Jordan blocks. So, the definition of Jordan block is a Jordan block corresponding to a given eigenvalue lambda is a k by k matrix with lambda on the main diagonal and one on the super diagonal.

So, for example, if I want to write a Jordan block of size 1 with respect to eigenvalue lambda equals to lambda 0. So, j 1 lambda 0 is given as by this matrix. If I want to write a Jordan block of size 2 corresponding to eigenvalue lambda equals to lambda 0, then it will be lambda 0 lambda 0 in the main diagonal. So, 1 in the super diagonal and 0 will be below.

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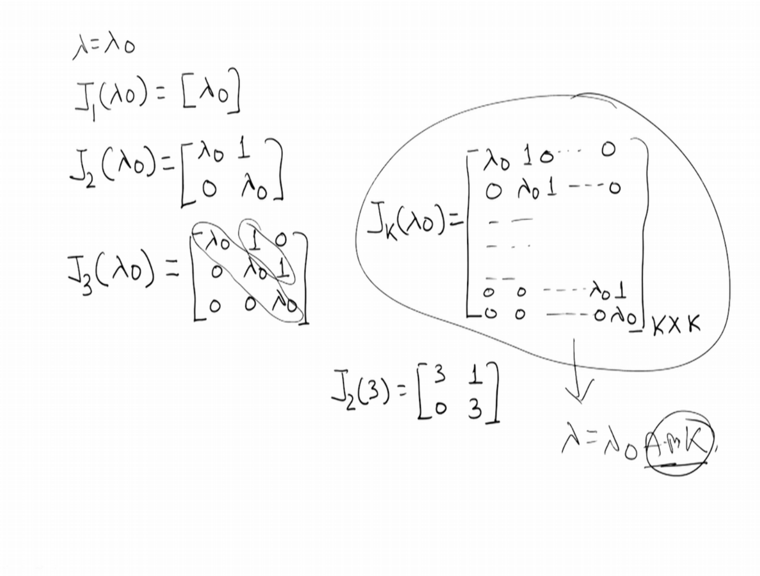

Similarly, a Jordan block of size 3 corresponding to eigenvalue lambda equals to lambda 0 can be written in the same way. So, lambda 0 1 0 0 lambda 0 1and 0 0 lambda 0 here, you can notice that lambda 0 are in the main diagonal and 1 is in the super diagonal. So, in the similar way, a Jordan block of size k corresponding to eigenvalue lambda equals to lambda 0 can be written as lambda 0 1 0 0 lambda 0 1 0 0 0 lambda 0 1 and finally, 0 0 0 lambda 0. So, it is a k by k matrix. So, this is the definition of Jordan block of different size.

So, if someone ask you write a Jordan block of size 2 corresponding to eigenvalue lambda equals to 3, so, it can be written as size 2 eigenvalue is 3 so, it is given as 3 1 0 3. Now, we will see some properties of Jordan blocks; so, a Jordan block has only 1 eigenvalue lambda equals to lambda 0. So, for example, if you are having this particular matrix, it is a Jordan block of size k. So, the eigenvalue of this will be lambda equals to lambda 0 with algebraic multiplicity k.

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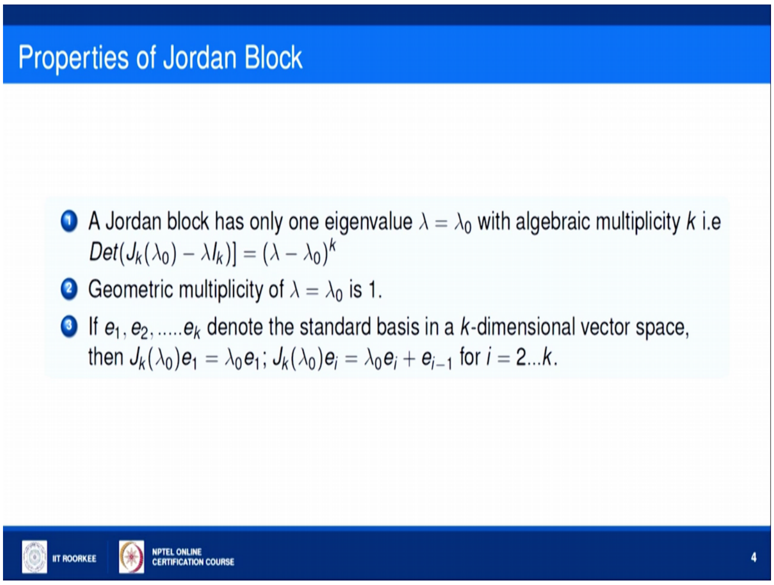

So, algebraic multiplicity of this will be k and the determinant of a Jordan block of size k corresponding to lambda equals to lambda 0 will be given as lambda minus lambda 0 raised to power k. The geometric multiplicity of lambda equals to lambda 0 of a Jordan block of size k will be 1 means, there will be only 1 linearly independent eigenvector corresponding to a given Jordan block whatever be the size.

And third important property is, if e 1 e 2 e k denotes the standard basis in a k dimensional vector space, then j k lambda 0 e 1 is given as lambda 0 e 1 and j k lambda 0 e i is given as lambda 0 e i plus e i minus 1 where i is varying from 2 to k. So, suppose I want to find out j k lambda 0 e 2. So, it will become lambda 0 e 2 plus e 1.

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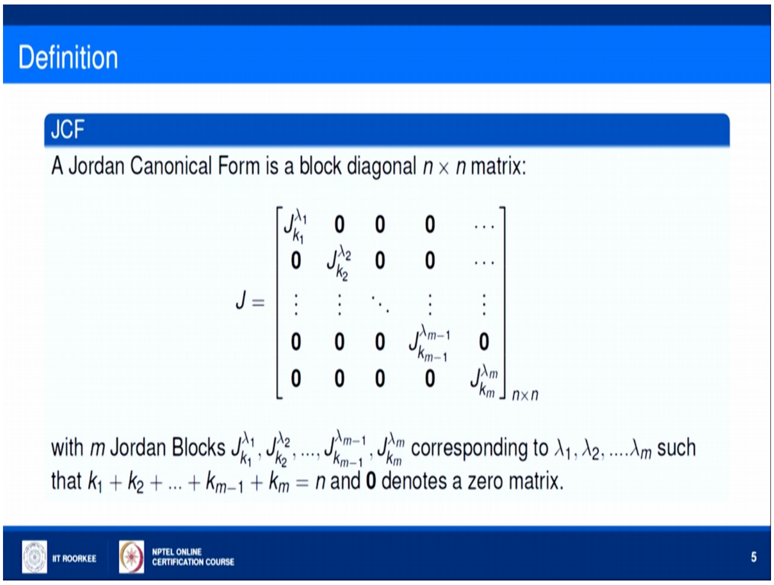

Now, after learning about Jordan blocks, let us divide the define the Jordan canonical form. So, a Jordan canonical form is a block diagonal n by n matrix given like this. So, here, basically what we are having? We are having these m Jordan blocks corresponding to eigenvalues lambda 1 lambda 2 up to lambda n and respective size are k 1 for the Jordan block corresponding to eigenvalue lambda 1 k 2 is the size of the Jordan block corresponding to eigenvalue lambda 2 and so on.

So, in this way, if a the matrix is n by n matrix, then k 1 plus k 2 plus up to k m should be equals to n. Please note it is a block diagonal matrix and all these are 0 blocks. So, block diagonal and we are having 0 blocks above the block diagonal and below the block diagonals. So, here we are having m Jordan blocks as I told you and this complete matrix is called Jordan canonical form. So, and if I know the algebraic multiplicity and geometric multiplicity of different eigenvalues for a given matrix, then I can write the Jordan canonical form of that matrix that we will take some example of that.

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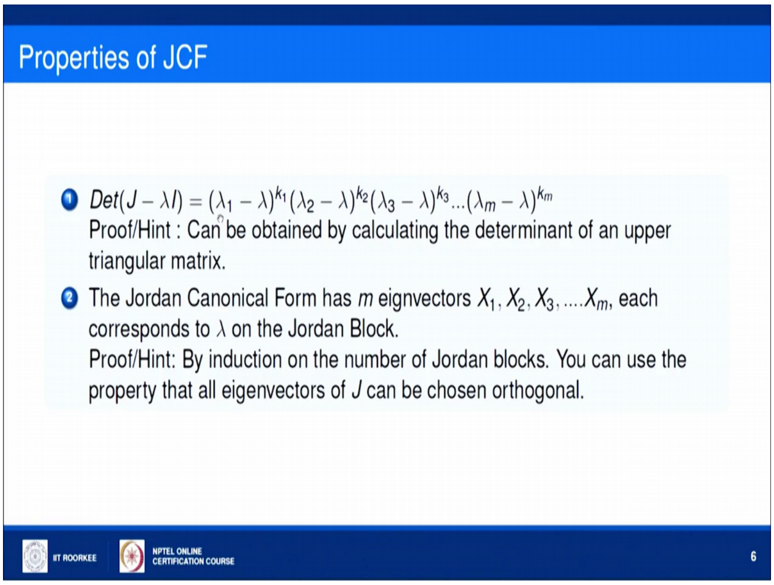

Now, as you can see, the determinant of this Jordan met canonical form is given as lambda 1 minus lambda raised to power k 1 lambda 2 minus lambda raised to power k 2 and so on up to lambda m minus lambda raised to power k m and this can be easily obtained by using the concept of finding the determinant of the block diagonal matrix where the determinant of the complete matrix will be the product of determinants of different block matrices.

Similarly, corresponding to each block as I told you, when I was defining the Jordan blocks, that corresponding to each Jordan blocks there will be only one linearly independent eigenvector. So, in that way, I will be having this particular Jordan canonical form will be having m eigenvectors given as X 1 X 2 X 3 up to X m each corresponding to lambda on the Jordan blocks and this can be prove by the method of induction.

Now, we are coming to a very important theorem and this tells us about this is called Jordan canonical form.

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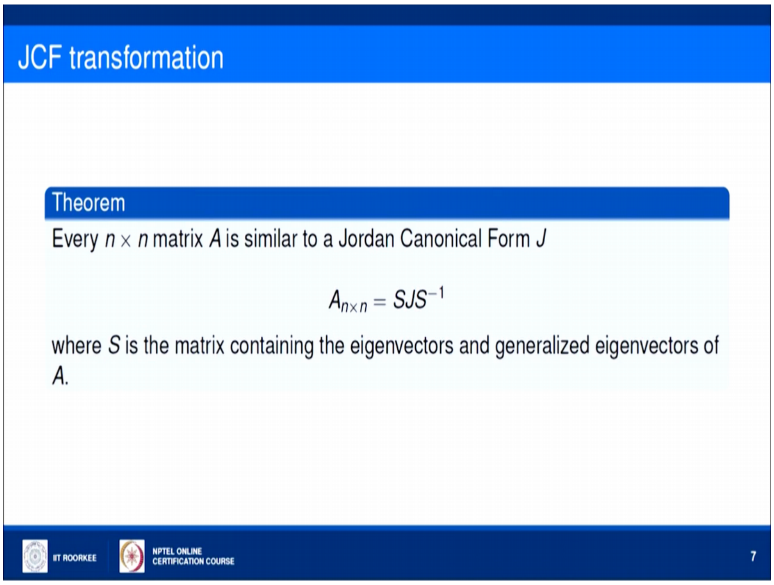

Similar to a transformation so, this particular transformation tells us that every square matrix of order n is similar to a Jordan canonical form j of the similar size that is if A is a matrix of order n by n then a is similar to a Jordan canonical form J such that A can be written as S J S inverse. So, where S is the matrix containing the eigenvectors and generalized eigenvectors of A.

Now, please note it here; if A is a diagonalizable matrix, in that case, A will become P J P inverse where P comes from the eigenvectors of A because, then if A is diagonalizable, A will contain n linearly independent eigenvectors and if I write those eigenvectors S columns of P, then I will get the model matrix P and J will be D where diagonal entries are the eigenvalues of A and P inverse. So, if A is diagonalizable. So, this Jordan canonical transformation will become diagonalization. So, we can consider Jordan canonical transformation as the generalization of classic diagonalization transformation.

So, in this way, now question arise when A is not diagonalizable, means A does not have n linearly independent eigenvectors then, how to write this matrix S? Because, if A is a n by n matrix and I am able to find only let us say some m linearly independent eigenvectors corresponding to different eigenvalues of A, then I will be able to write only m columns of S. So, from where I will write raised n minus m columns?

So, those columns I will write by finding the generalized eigenvectors of the matrix A. So, for writing this particular matrix S, we need to learn how to find out the generalized eigenvectors of a matrix. So, let me define generalized eigenvector of a square matrix.

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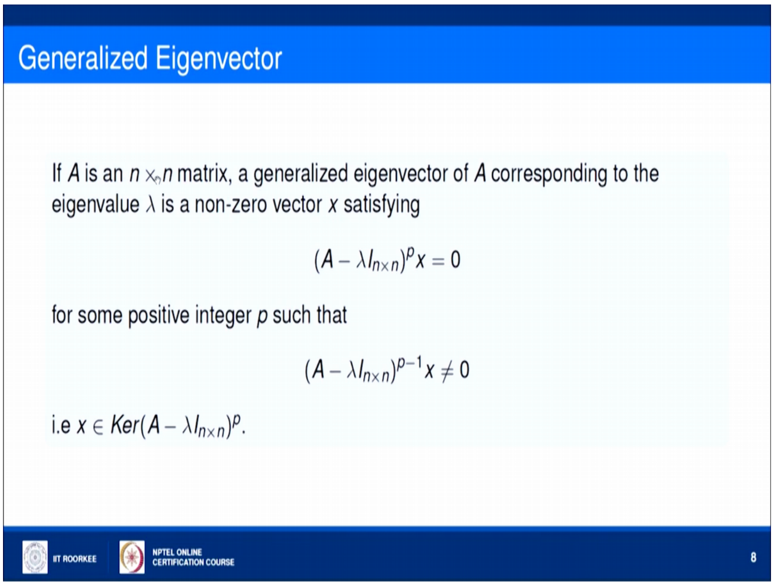

So, if A is a square matrix of order n, a generalized eigenvector of a corresponding to the eigenvalue lambda is a non 0 vector X satisfying a minus lambda i raised to power p into X equals to 0.

For some positive integer p such that a minus lambda I raised to power p minus 1 X is not equals to 0. So, a minus lambda I raised to power p into X equals to 0 where X is a non 0 vector, but a minus lambda I raised to power p minus 1 into X is not 0. So, we can say that, a generalized eigenvector is a member of null space of a minus lambda I raised to power p. Let us take an example to find out the generalized eigenvector.

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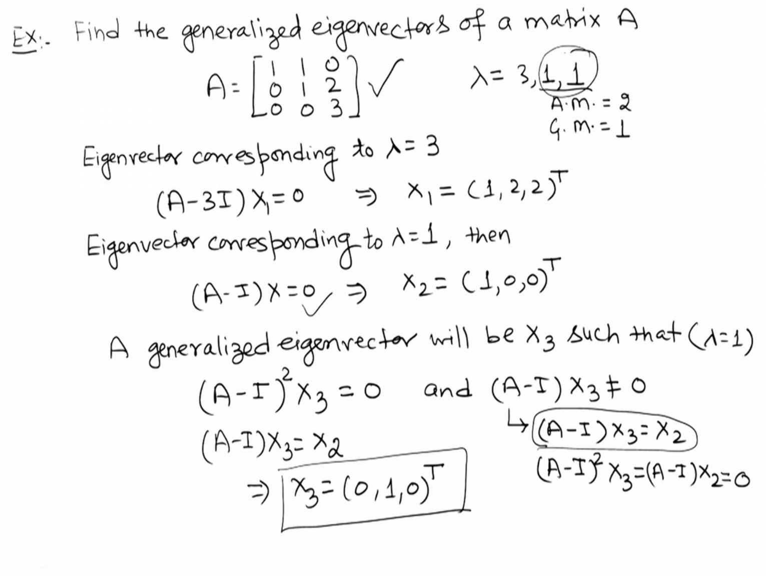

So, find the generalized eigenvectors of a matrix a where a is given as 1 1 0 0 1 2 and 0 0 3. Now, if I see here, the A is a upper triangular matrix. So, eigenvalues of A will become 3 1 1. If I calculate the eigenvector corresponding to lambda equals to 3, then A minus 3 I into X will become 0 and let us say it is X 1. So, from here, I get an eigenvector X 1 equals to 1 2 2 transpose. Now, similarly if I calculate the eigenvector corresponding to lambda equals to 1, then A minus I into X equals to 0 and from here I get only 1 linearly independent eigenvector that comes out to be 1 0 0.

So, here what we can say that the algebraic multiplicity of A is 2 while geometric multiplicity of lambda equals to 1 is only 1. So, hence A is not a diagonalizable matrix. So, if A is not a diagonalizable matrix and as I told you, we can write A as S into J into S inverse by the Jordan canonical transformation. So, for writing the matrix S, I need to find out 1 generalized eigenvector corresponding to lambda equals to 1. So, it means a generalized eigenvector will be X 3 such that A minus I and that is, I am talking about corresponding to lambda equals to 1. So, A minus I X 3 and A minus I square X 3 equals to 0.

And A minus I X 3. So, not be equals to 0. So, since I want to take A minus I X 3. So, not be equals to 0. So, if from here, I take A minus I X 3 equals to X 2 because, as I told you X 2 is an eigenvector. So, it is a non 0 eigenvector.

So, this particular equation satisfy this condition of generalized eigenvector if I multiply both side by A minus I, then it will become A minus I square into X 3 and it comes out to be A minus I X 2 and from here, A minus I X 2 is 0 because X 2 is an eigenvector.

So, hence I need to find out n eigenvector or generalized eigenvector let me say X 3 which is satisfying this particular condition, so that, I can calculate by using or by solving this non homogeneous system of equations. So, if I solve it, here it will become A minus I X 3 equals to X 2. So, this gives me X 3 equals to 0 1 0 transpose. So, here X 3 is a generalized eigenvector of the matrix a corresponding to eigenvalue lambda equals to 1. So, in this way, we can calculate the generalized eigenvectors.

Once you find out the raised m minus n generalized eigenvectors, then you are having m linearly independent eigenvectors corresponding the square matrix A of size n and then what you have done you have calculated n minus m generalized eigenvectors corresponding to different Eigen values. So, what you can do? You will be having n total eigenvectors and generalized eigenvectors and those n vectors. You can write as the columns of A matrix and that matrix will become matrix S.

So, if X 1 X 2 X 3 X n is the set of all linearly independent eigenvectors and generalized eigenvectors of the matrix A, then S will be the matrix having columns as these eigenvectors and generalized eigenvectors. So, let us take an example to write the Jordan canonical transformation of a given matrix.

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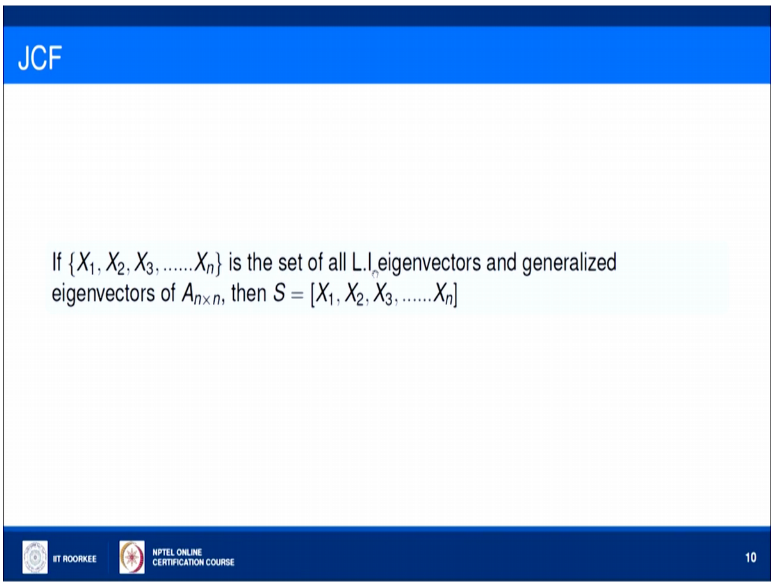

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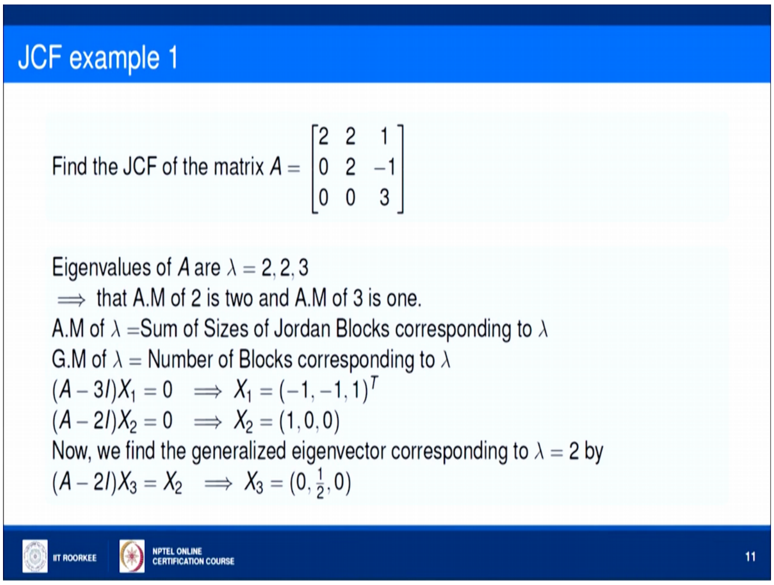

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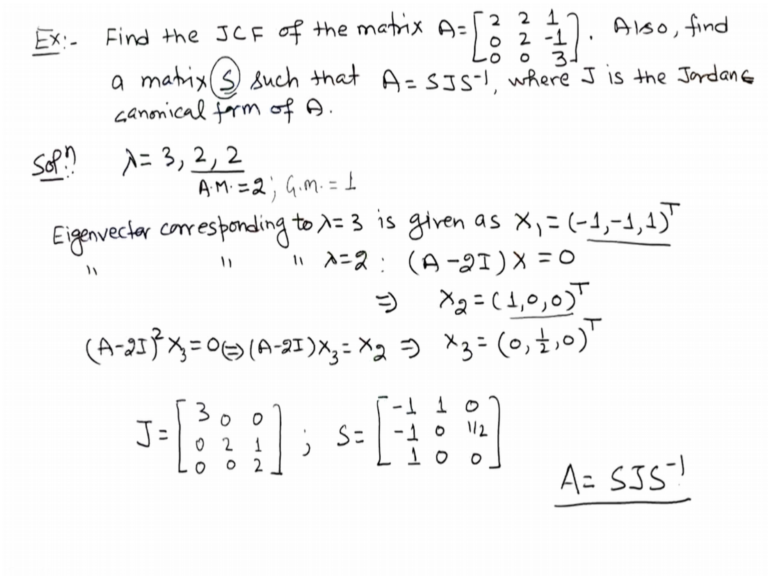

So, example is find the Jordan canonical form of the matrix A equals to 2 2 1 0 2 minus 1 and 0 0 3 also find a matrix S such that A equals to S J S inverse where J is the Jordan canonical form of A. So, let me solve this particular example. So, first of all, I need to find out eigenvalues of A and again you can see A is an upper triangular matrix. Eigenvalue will be given by the diagonal elements.

So, here eigenvalues are lambda equals to 3 2 2. The algebraic multiplicity of lambda equals to 2 is 2. Now, we will see what is the geometric multiplicity of this.

If the geometric multiplicity of the eigenvalue lambda equals to 2 is 2, then the matrix is diagonalizable if it is 1, then it will be we have to find out 1 generalized eigenvector to write the matrix S. So, the eigenvector corresponding to corresponding to lambda equals to 3 is given as let us say X 1 and this comes out to be minus 1 minus 1 and 1 transpose.

Now, eigenvector corresponding to lambda equals to 2. So, it means a minus 2 I into X equals to 0. So, from here what I got I got only one linearly independent eigenvector which is let me write as X 2. So, X 2 becomes 1 0 0 because, when I will write A minus 2 I, the first equation will become 0 2 X 2 plus X 3 equals to 0.

Second equation will give us that X 3 0. So, from there, I will get X 1 is also 0 and third equation will give me again X 3 equals to 0. So, here, I will get X 2 equals to 0 equals to X 3 and X one is arbitrary. So, I have chosen X 2 X 1 as 1 means X 1 X 2 X 3 are different components of X 2.

Now, I need to find out one generalized eigenvector corresponding to lambda equals to 2. So, if I solve for that A minus 2 I square X equals to 0 which is equivalent to solving A minus 2 I, let me write this X 3 equals to X 2. So, if I do it, I will get X 3 as the generalized eigenvector and this comes out to be 0 1 by 2 0.

So, after doing this, now I need to write matrix S and the matrix J. So, here my matrix J as I told you, I can write with only the information about algebraic multiplicity and geometric multiplicities of the different eigenvalues of A. So, here 3 is having algebraic multiplicity 1 geometric multiplicity 1. So, there will be A 1 by 1 block of 3. Now, the algebraic multiplicity of the eigenvalue is 2. So, algebraic multiplicity of a given eigenvalue tells us that how many what will be the total size of the sum of various blocks corresponding to this eigenvalue.

So, here it is saying these that algebraic multiplicity is 2. So, it will be a 2 by 2 blocks corresponding to eigenvalue lambda equals to 2 the geometric multiplicity of lambda equals to 2 is one. So, geometric multiplicity tells us the total number of blocks corresponding to that eigenvalue. So, algebraic multiplicity tells size total size geometric multiplicity total blocks.

So, algebraic multiplicity is 2 geometric multiplicity is 1. So, only 1 block of size 2. So, I will be having a block of size 2 corresponding to eigenvalue lambda equals to 2. So, in this way this particular matrix becomes the Jordan canonical form of A.

Now, I can write the matrix S as the minus 1 minus 1 1 1 0 0. So, this vector as column this vector as column and third column will come from here. So, these are the matrices J and S such that A equals to S J S inverse and you can verify it later on.

So, this is the overall process for find using the Jordan canonical transformation for finding the matrix S and a Jordan canonical form J of a given matrix.

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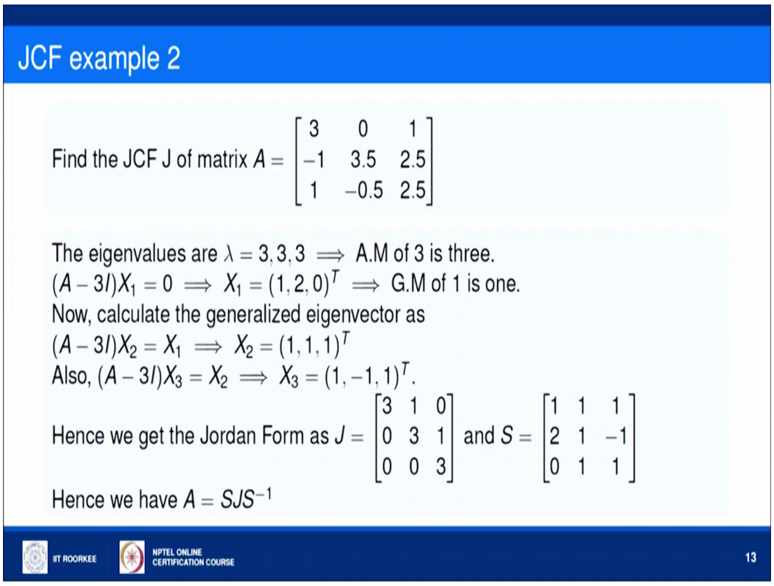

Similarly, if I take this example so, find a Jordan canonical form J of this matrix.

So, if I solve it here, the eigenvalues coming out to be lambda equals to 3 3 3. So, algebraic multiplicity of lambda equals to 3 is 3.

If I calculate the eigenvector corresponding to eigenvalue lambda equals to 3 then A minus 3 I X 1 equals to 0 this gives me X 1 equals to 1 2 0. So, hence the geometric multiplicity of lambda equals to 3 is 1.

Now, calculate the generalized eigenvector. So, A minus 3 I X 2 equals to X 1. So, from that I got X 2 equals to 1 1 1 and the another because I need to calculate 2 generalized eigenvectors for writing the matrix S.

So, another generalized eigenvector can be written as A minus 3 I cube X 3 equals to 0 which I can have A minus 3 I X 3 equals to X 2 from this relation. So, from there I got X 3 equals to 1 minus 1 1 transpose.

So, hence J will means total block is 1 total block is 1 size is 3. So, a Jordan block of size 3. So, 3 1 0 0 3 1 0 0 3 and S will become 1 2 0 is the first column 1 1 1 is the second column and 1 minus 1 is the third column.

Hence we have A equals to S J S inverse. So, I have taken a couple of examples for finding the Jordan canonical transformation for a given matrix. If matrix is diagonalizable, then Jordan canonical form will be equal to the diagonal matrix having eigenvalues as the main diagonal entries.

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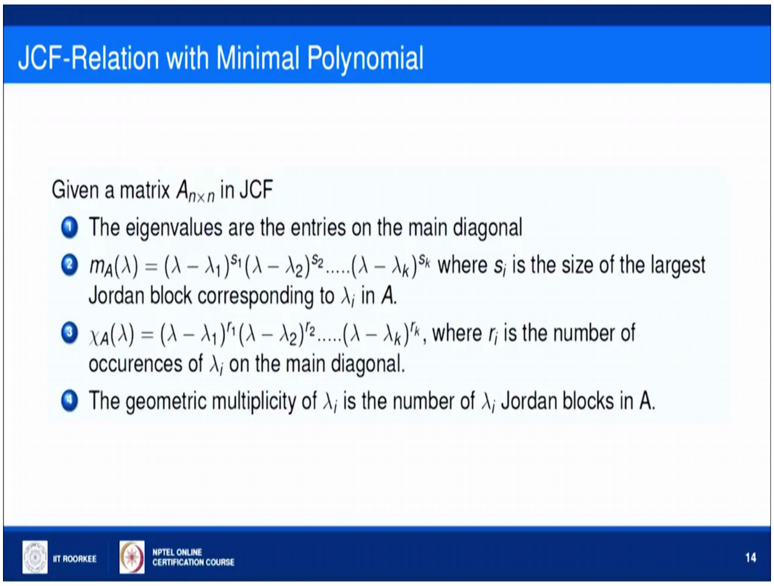

So, let me explain the relation of Jordan canonical form of a matrix with minimal polynomial it given matrix A of order n, then the J C of F of A the eigenvalues are the entries on the main diagonal.

So, if the minimal polynomial of A is m lambda and it is given as lambda minus lambda 1 raised to power s 1 lambda minus lambda 2 raised to power s 2 and upto lambda minus lambda k raised to power s k where s I is the size of the largest Jordan block corresponding to lambda I in A.

So, powers in the minimal polynomial corresponding to different terms, different factors will give you the size of largest block corresponding to that particular eigenvalue. And if lambda minus lambda 1 raised to power r 1 into lambda minus into lambda minus lambda 2 raised to power r 2 up to lambda minus k raised to power r k is the characteristic polynomial, then r I is the number of occurrence of lambda I on the main diagonal which is obvious. So, the geometric multiplicity of lambda I is the number of lambda I Jordan blocks in A because each Jordan block will give you only 1 linearly independent eigenvector.

So, let us take an example corresponding to this particular relation. So, consider a 6 by 6 matrix A having characteristic polynomial lambda minus 3 raised to power 4 into lambda minus 2 raised to is raised to power 2 and minimal polynomial is lambda minus 3 raised to power 3 lambda minus 2 raised to power 2.

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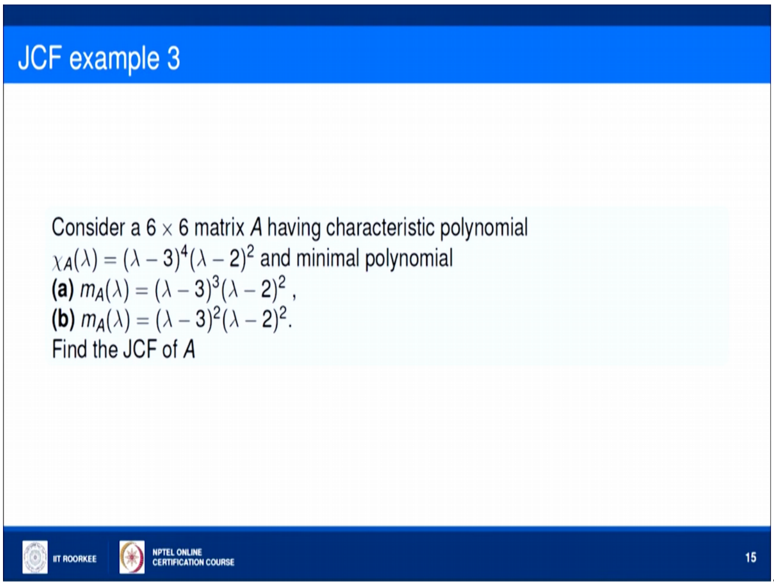

So, here what I am having I am having.

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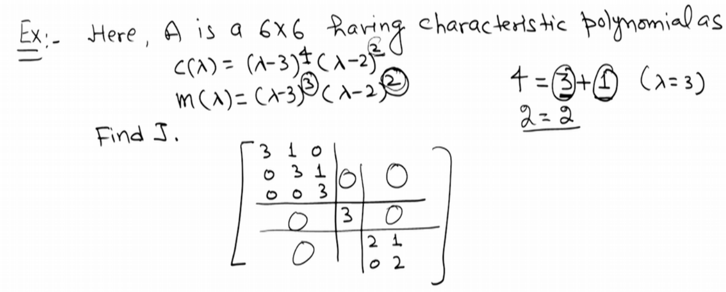

An example here a is a 6 by 6 matrix having characteristic polynomial as C of lambda equals to lambda minus 3 raised to power 4 into lambda minus 2 raised to power 2 and minimal polynomial is m lambda equals to lambda minus 3 raised to the power 3 and lambda minus 2 raised to the power 2.

So, find J mean Jordan canonical form of A. So, here as I told you, that these powers will give you the size of maximum biggest Jordan block corresponding to these eigenvalues and these are the number of occurrence of these eigenvalues on the main diagonal. So, here I am having 4. So, the lambda equals to 3 will occur 4 times on the main diagonal out of which the biggest Jordan block will be having size 3.

So, 4 equals to 3 which is the biggest Jordan block, then what is rest one? So, from here, I can get an information that, thus there will be 2 Jordan blocks corresponding to lambda equals to 3; one of size 3 another one of size 1. So, hence I can have 3 1 0 0 3 1 0 0 3. This is the block of size 3 another one of size 1. So, 3.

And then, here I am having 2-time occurrence of lambda equals to 2 and the maximum Jordan block will be having size 2. So, 2 equals to 2. So, there will be only one Jordan block of size 2. So, 2 1 0 2 and rest are 0 blocks. So, in this way, this is the Jordan canonical form of A if characteristic polynomial and minimal polynomials are given in this way.

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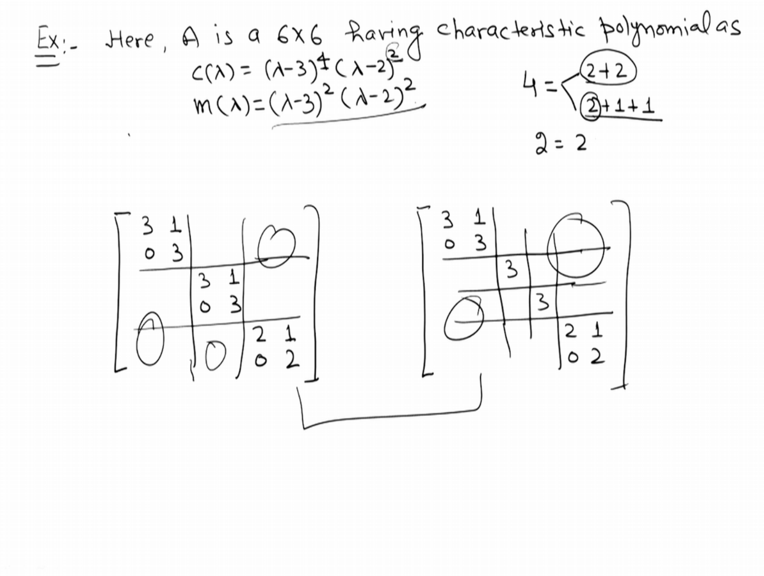

If minimal polynomial is lambda minus 3 raised to the power 2 and another one lambda minus 2 raised to power 2. So, now, what is happening? I am having total 4 size corresponding to lambda equals to 3 and the size of the biggest block is 2.

So, there will be 2 ways of writing 4 having the biggest entry is 2; one is 2 plus 2 another one is 2 plus 1 plus 1 and the other one is 2 which I have to have biggest one 2. So, 2 like this.

So, the possible Jordan block form will be if I take this particular thing, so, there will be 2 blocks corresponding to lambda equals to 2 each of size 2. So, 3 1 0 3, then 3 1 0 3, then what I am having here? I am having 2 1 0 2. So, this is one of the possible Jordan canonical form. Obviously, you can interchange the Jordan blocks.

So, here I am not taking consideration of reordering of the Jordan blocks. I am taking them as the symmetrics. The other possibilities if you use this combination , so, in this combination, what I am saying one of the Jordan blocks corresponding to lambda equals to 3 of size 2 and two are of size 1 1. So, it I am having 3 1 0 3 and 2 are of size 1 1 and then I am having 2 1 0 2. So, this is m and these are 0s. So, this is the 2 possibilities for this the this these are the 2 possibilities of the Jordan canonical form J of this matrix A having this characteristic polynomial and this minimal polynomial.

So, hence I can say that, the information you can write the Jordan canonical form of A matrix if you know either the algebraic and geometric multiplicity of each eigenvalues or you can get n information, if you know the minimal polynomial as well as characteristic polynomial of that particular matrix.

So, in this lecture we have learn about Jordan canonical transformation and how to write Jordan canonical form of a given matrix.

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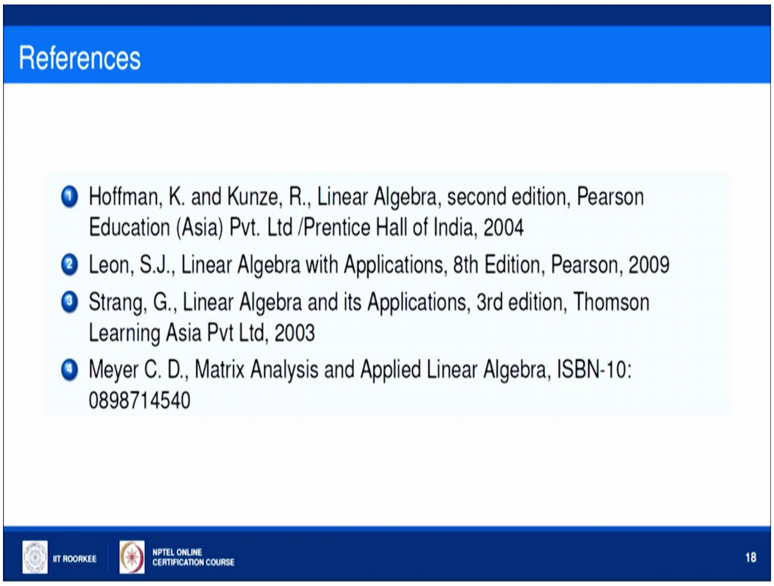

These are the references in the next lecture we will learn evaluation of matrix functions.

Thank you very much.