Matrix Analysis with Applications
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Lecture – 24
Gram Matrix and Minimization of Quadratic Forms

Hello friends. So, welcome to the 24th lecture of this course. So, in this lecture we will learn about gram matrix and we will learn how to find out a suitable minimizer for the given quadratic form.

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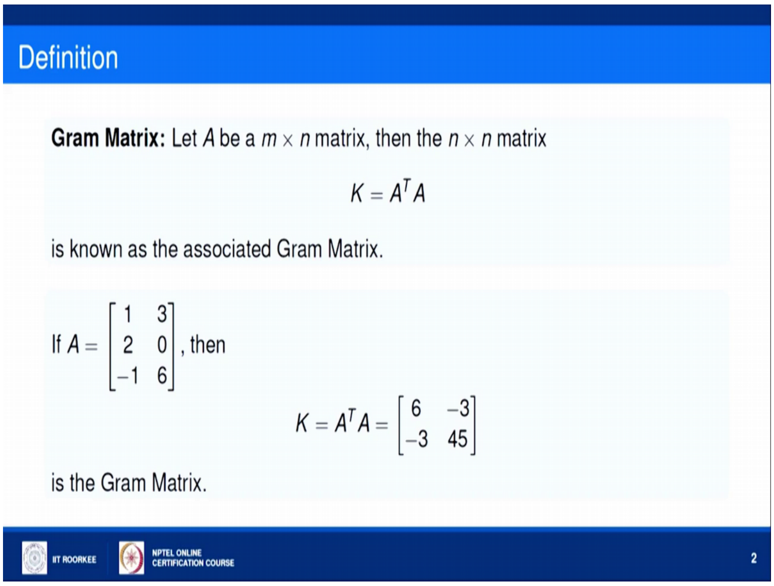

So, first definition in this lecture comes as a gram matrix. So, a gram matrix can be defined in this way let A be a m by n matrix. So, A can be a rectangular matrix also, then the n by n matrix a transpose into a equals to K is known as the associated gram matrix.

So from here, you can see since, it is a since. It is a product of the transpose of a matrix with that matrix itself. So, gram matrix will be always a symmetric matrix. If I take an example that, A is a 3 by 2 matrix. Where, first row of row of A is 1 3 2 0 and minus 1 6 then a transpose a comes out to be 6 minus 3 minus 3 and 45.

So here a transpose, A will become 2 by 2 matrix and this is the gram matrix of A.

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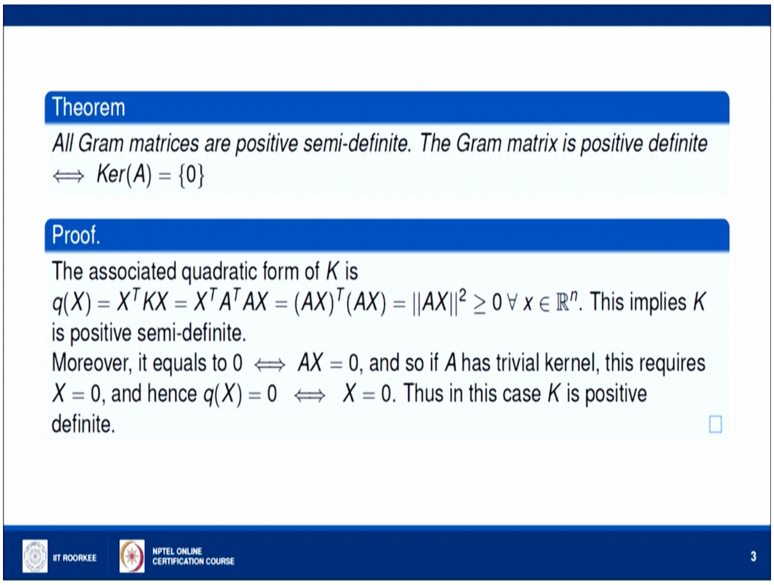

Now, we are having a very important result related to gram matrices. So, all gram matrices are positive semi definite, moreover the gram matrix is positive definite if and only if the kernel of A equals to 0, means the kernel is having only 0 vector.

So, let us see the proof of this.

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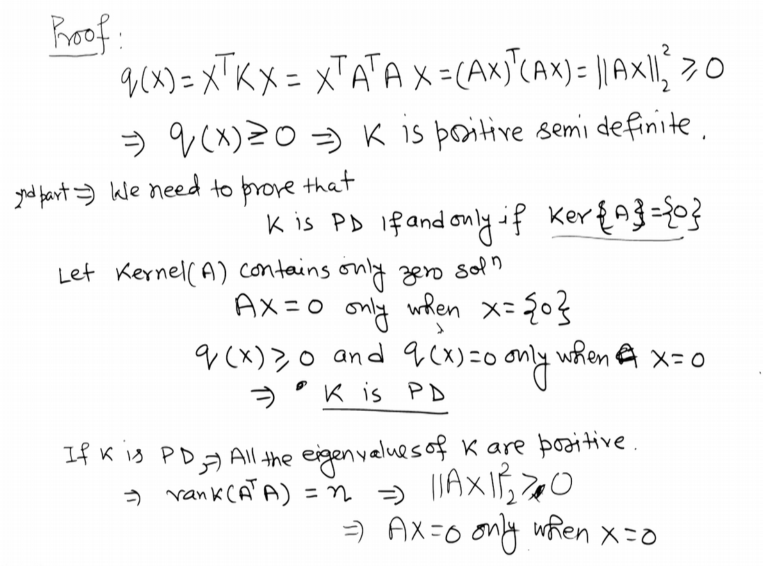

So, first we have to prove that all gram matrices are positive definite. So, if I take A So, if I take the quadratic form of a gram matrix K, it can be written as X transpose K into X. Where, K is the gram matrix. So, this can be defined as a transpose into A into X this can be written as A X transpose into a X. Which is the norm of which will be always greater than equals to 0.

So, since the quadratic form of a gram matrix will be always greater than equals to 0. So hence, K is positive semi definite; this is the first part of the theorem. Now, in the second part we need to prove that, the matrix K is positive definite if and only if kernel of the matrix A contains only 0 vector.

So, let kernel of A contains only 0 solution. In this case, the system AX equals to 0, implies that X equals to 0. Since, we are having that q X greater than equals to 0 then and q X equals to 0. Only when, X equals to 0 because, kernel is having only 0 vector. So, hence K is positive definite.

So, this we have done that, if kernel of A equals to 0, then K is positive definite. So, let us do the same result from the other way that is, if we assume that, if K is positive definite then we need to see. So, that the kernel of A is having only 0 vector. So, if K is positive definite means all the eigenvalues of K are positive. It means the rank of the matrix a transpose A is n. If K is A n by n matrix means it is a full rank matrix.

Norm of a AX square is greater than equals to 0 or. In fact, it will be strictly greater than 0. So, hence AX will be 0 only when X equals to 0. This means the kernel of a contains only the 0 vector. So, this is the proof of other side.

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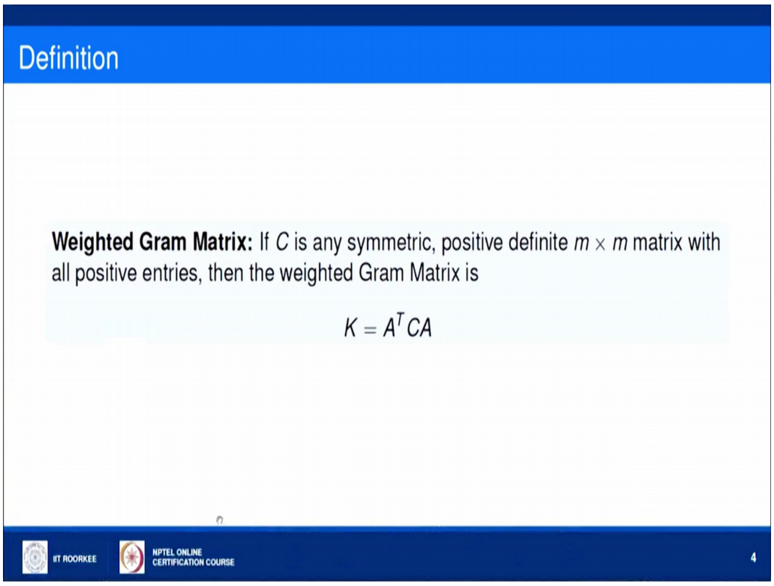

In the same way, we can define weighted gram matrix. So, if C is any symmetric positive definite m by m matrix with all positive entries. Then the weighted gram matrix is K equals to a transpose CA.

Weighted gram matrix is K equals to a transpose CA.

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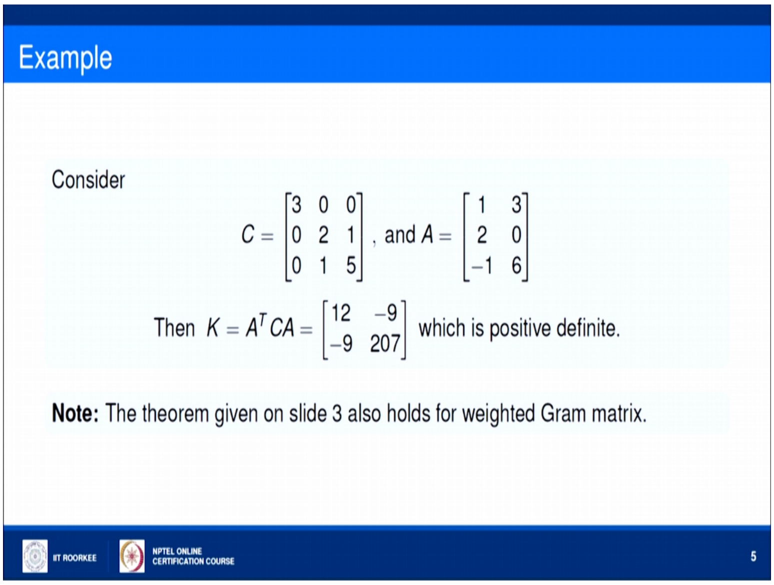

So, for example, if I take C equals to this one, it is symmetric as well as positive definite matrix then and A is the same as we have taken in earlier case. Then, the associated gram matrix is A transpose G CA which is given as 12 minus 9 minus 9 and 2 0 7 which is again positive definite.

So, all the result which we have discuss in the earlier theorem holds for weighted gram matrix also. Now, we will look for the minimization of quadratic forms.

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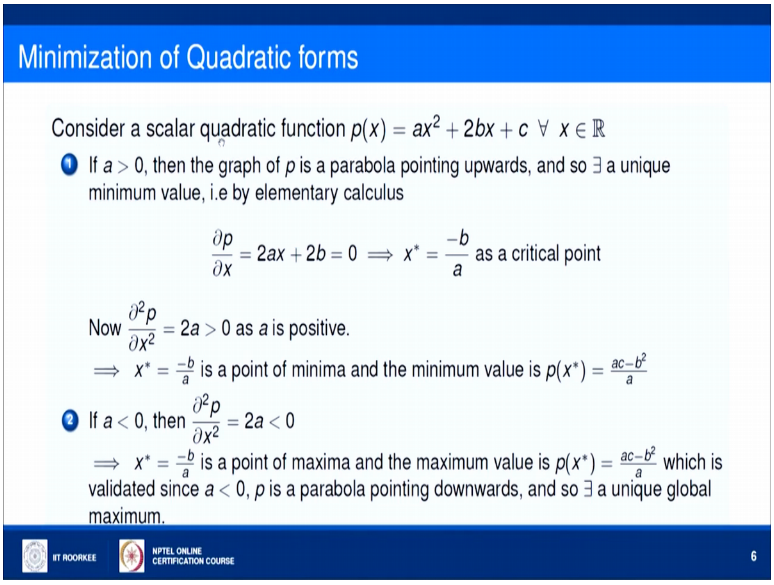

So, consider a scalar quadratic form px equals to ax square plus 2 bx plus c for all x belongs to R. So, here what we are having we are having a.

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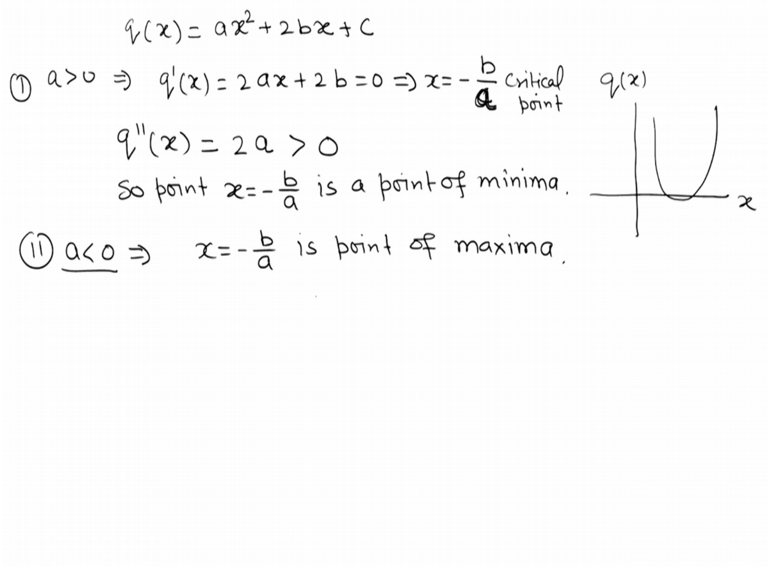

Scalar quadratic form q x equals to ax square plus 2 b x plus c. Now, consider the case, when a is greater than 0. So, when a is greater than 0 it will be a parabola pointing upward and if I see the minimization of these.

So, it will be having an unique global minima. So, if I see this so, q dash x will become 2x plus 2b equals to 0 which gives me 0 equal x equals to minus b upon a is the critical point. Now, if I check the sufficient condition of minimization on this, then the double derivative of q with respect to x can be given as 2 twice of a and if a is positive then twice of a is also positive. So, point x equals to minus b upon a is a point of minima. Similarly,, if I take a is less than- 0. In this case, the point x equals to minus b by a comes out to be a point of maxima.

So, generally these are the necessary and sufficient conditions we can use for looking the maxima and minima of a scalar quadratic form, but if I am having a vector quadratic form, where x is not a scalar, but it is a vector form n. Dimensionally space then, how to find out the minima of the quadratic form? So, we will learn this application now.

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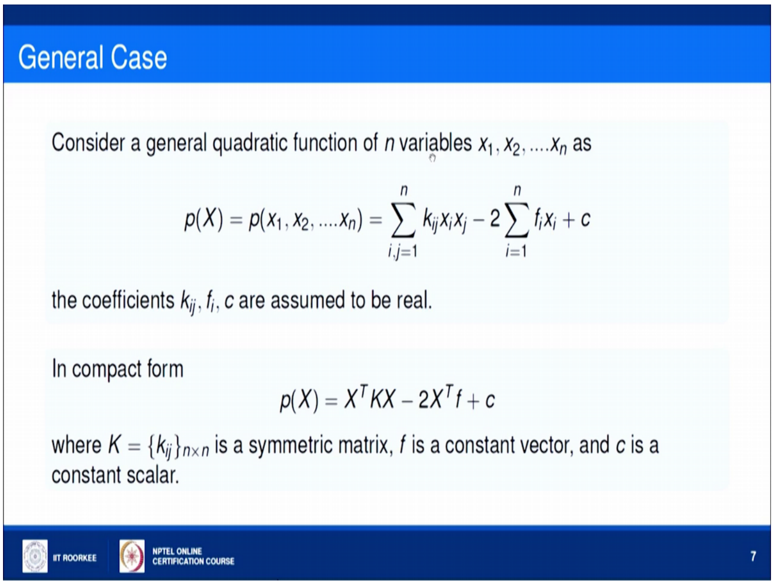

So, consider a general quadratic function of n variables x1 x2 xn as pX where X is n dimensional vector having component x1 x2 xn and this can be written as sigma ij equals to 1 to n kij xi xj minus twice of summation i equals to 1 to n fi xi plus c.

Where, the coefficient kij fic are assumed to be real. So in compact form, this particular general quadratic function can be written in by the matrix notations. So, here it can be written as X transpose KX minus 2 times X transpose f plus C.

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$$p(x_1, x_2, x_3) = 2x_1^2 + 3x_2^2 - 5x_3^2 - 2x_1x_2 + 2x_2 + 3$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - 2 \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + 3$$

$$K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix} \text{ and } f = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}; \quad c = 3$$

$$p(X) = X^T K X - 2X^T f + c$$

Where K is a symmetric matrix f is a constant vector and C is a constant scalar. For example, if you are having a general quadratic form of 3 variables that is let us say 2x1 square plus 3 x 2 square minus 5 x 3 square minus 2x 1 x 2 plus 2 times x 2 plus 3.

Then in matrix form I can write it as x1 x 2 x 3 multiplied with the symmetric matrix 2 3 minus 5 and then minus 1 minus 1 because the coefficient of x1 x2 is minus 2 the coefficient of x 2 x 3 as well as x one x 3 are 0. So, all these entries will be 0 into x1 x2 x3.

Then what I am having minus 2 times minus 2x T f. So, x T will become again x 1 x 2 x 3 and f will be here a column vector 0 minus 1 0 plus 3. So, if I see here, K comes out to be 2 minus 1 0 minus 1 3 0 and 0 0 minus 5 and the vector f is 0 minus 1 0 the scalar C is 3. So, this form p X can be written as XTKX minus 2 times XT f plus C. So, in this way we can write any quadratic function of n variables into a compact form by using this concept.

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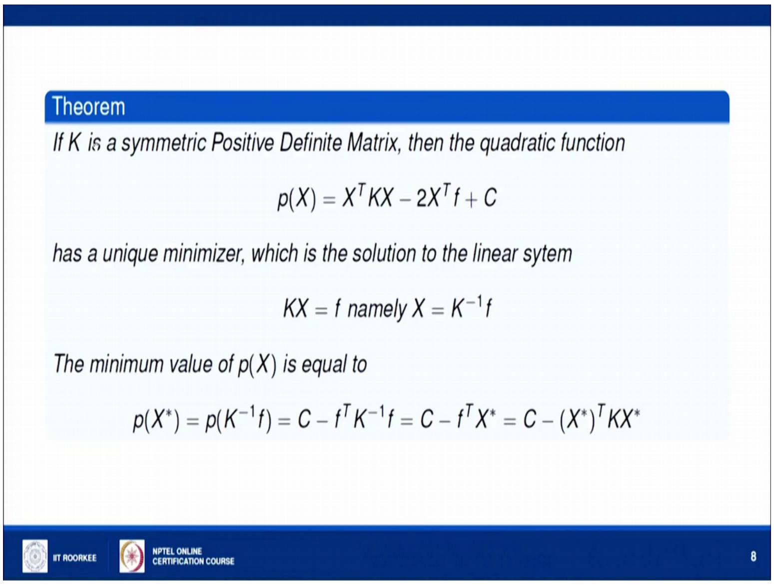

Now, this is the theorem which tells us about the minimization of a general quadratic form. So, if K is a symmetric positive definite matrix then the quadratic function pX equals to X transpose KX minus twice of XT f plus C has a unique minimize.

So, please note down if the unique minimization or global minima of this particular quadratic function depends on the property of K. So, if K is symmetric and positive definite then this particular quadratic function will certainly have a global minima.

And this global minima is the solution of the linear system KX equals to f namely as X equals to K inverse f. The minimum value of p X is equals to in any one of this form. So, p X star equals to p of K inverse f which is C minus f transpose K inverse f. So, this K inverse f I can replace with the X star which is the K solution of this system. So, C minus f transpose X star or this also I can write C minus X star TKXX star. So, let us see the proof of this.

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Proof: Since K is a PD matrix $\Rightarrow$ K is non-singular and hence $X^* = K^{-1}f$ exists.

Now for any $X \in \mathbb{R}^n$, we can write

$$p(X) = X^T K X - 2X^T f + C$$
$$= X^T K X - 2X^T K X^* + C$$
$$= (X - X^*)^T K (X - X^*) + [C - (X^*)^T K X^*]$$

$$X^T K X^* = (X^*)^T K X$$

So, since K is a positive definite matrix. This implies K is nonsingular and hence X equals to K inverse f exists, it means K is invertible. Now, for any X belongs to n dimensional real vector space. We can write p of X equals to X transpose KX minus twice XT f plus C. This equals to XTKX minus 2XT and from here, I can write f equals to K into X star plus C.   .

Or this particular expression, I can write also X minus X star transpose KX minus X star plus C minus X star transpose KX star only writing this I have use the symmetry of K because K is a symmetric matrix. That is K equals to K transpose to identify X transpose KX star equals to X star transpose K into X. So, here I have use the symmetry of K.

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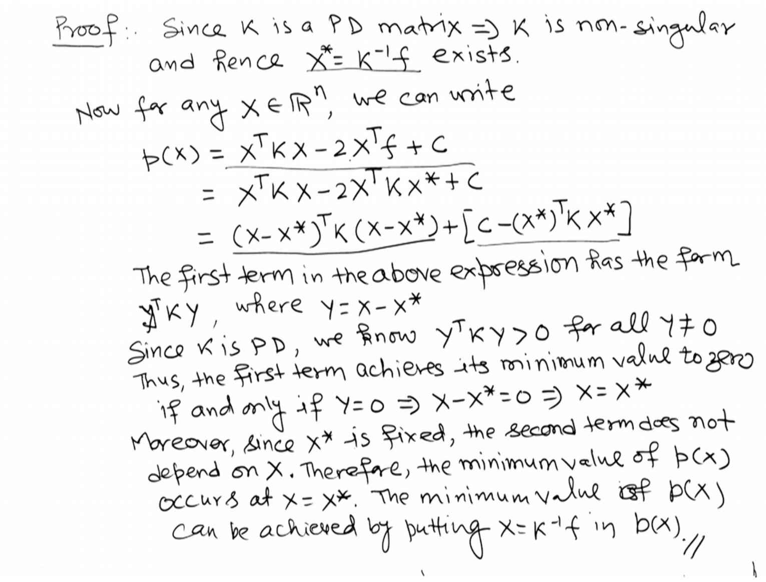

The first term means this term. In the above expression, has the form YTKY where Y equals to X minus X star. Since, K is positive definite. We know YTKY will be greater than will be a strictly greater than 0. For all Y those are not equals to 0. Thus, the first term means the underlined term achieves. It is minimum value because, it is a quadratic term to 0 if and only if, Y equals to 0 because, then only this term will become 0 and this implies Y equals to X minus X star. So, X minus X star equals to 0 or X equals to X star.

Moreover, since X star is fixed the second term does not depend on X because X star is fixed here and this term does not contain any expression in X. Therefore, the minimum value of pX occurs at X equals to X star ok. And where X star equals to K inverse f which is the proof of first part.

And the minimum value is or can be obtained just by substituting X equals to K inverse f in this expression and from that we can get any form. So, the minimum value of pX can be achieved by putting X equals to k inverse f in p X means, in this expression, which will give you the second part of the second statement of the theorem proof of the second statement of the theorem.

So, this is the proof of this theorem. Now, let us see an example of this theorem. So, example is like this.

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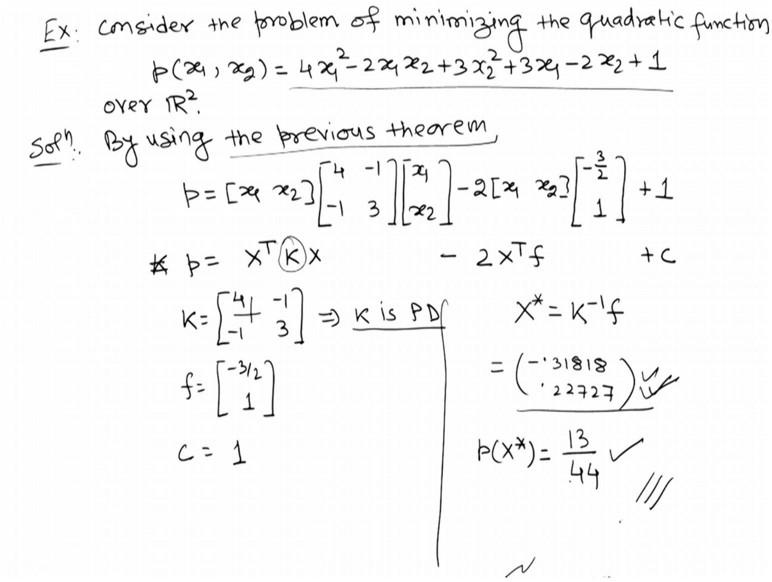

Consider the problem of minimizing the quadratic expression or quadratic function px1 x2 equals to 4 x1 square minus 2x1 x2 plus 3x2 square plus 3x1 minus 2x2 plus 1 over r 2 means all real values of x1 and x2.

So, by using the. So, how to solve this particular thing? So, by using the previous theorem which we have stated just now if the associated matrix K will be positive and symmetric positive definite and symmetric. Then, it will be having a unique minima. So, let us write this p in the matrix form. So, it will become x 1 x 2 and then the matrix 4 minus 1 minus 1 and 3 and then x 1 x 2 then I am having minus 2 times x t into f. So, x t is x 1 x 2 and what will be f here. So, if we see the coefficient of x one it is plus 3. So, it will become minus 3 by 2 and if we see the coefficient of x2 it is minus 2. So, it comes out to be1 plus 1.

So, here what I am having, if I compare with X transpose KX minus 2 times KT f plus C. So, K is 4 minus 1 minus 13 f is minus 3 by 2 and 1 and C is 1. If we see the matrix K, first principal minor of K is positive and the determinant K is 13 which is again positive. So, here K is positive definite. So, since K is positive definite, so according to the previous theorem the quadratic function p will be having a unique minima.

Now, this particular function will be having a unique minima at X star X star equals to K inverse into f. So now, if I calculate the K inverse, by the previous theorem X star equals to K inverse f and this comes out to be minus 0.318 18 and 0.22727.

So, this is the point on which the minima of this function will exist and the minimum value of this comes out to can be calculated using Any of the expression of this theorem means, either C minus f TK inverse f or C minus f transpose X star or C minus X star TK X star.

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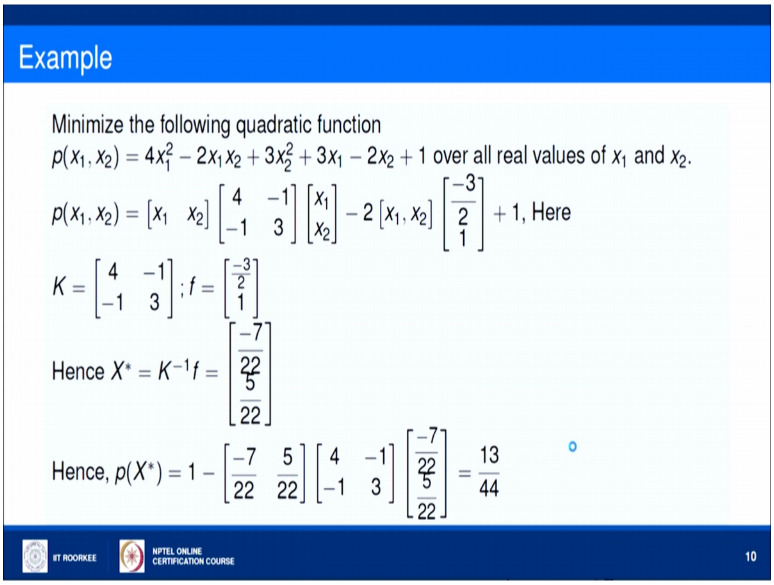

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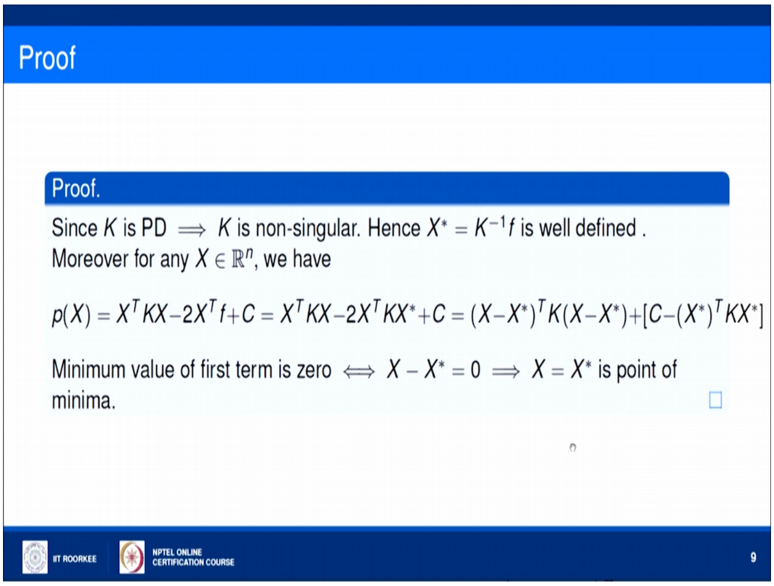

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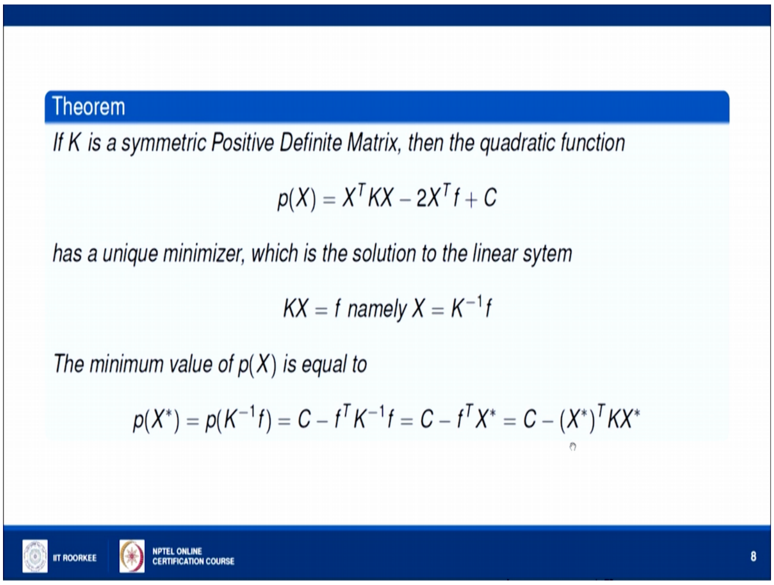

Where, X star is given by this vector and this minimum value is 13 by 44. So, this is the point of minima and this is the minimum value of p at this point of minima. So, in this way by using the previous theorem, we can find out the unique minimizer of a given quadratic function; however, the condition as I stated earlier the matrix K should be positive definite.

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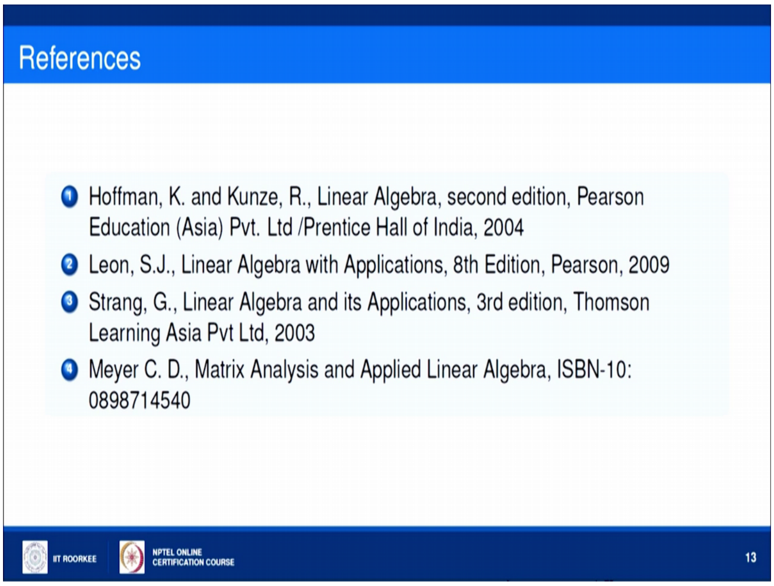

So, with these in this lecture we have learnt gram matrix and how to minimize a given quadratic function of n variables.

In the next lecture, we will learn another canonical transformation that is called Jordan canonical form. Which exist when a matrix is not diagonalizable? So, with this I will end this particular lecture. These are the references for this lecture.