Matrix Analysis with Applications
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Lecture - 23
Positive Definite and Quadratic Forms

Hello friends. So, welcome to the 23rd lecture of this course. So, as you know in the last lecture we have discussed the definition of positive definite matrices. And then we have discussed few methods that how to check whether a given matrix is positive definite or not. We have also learn the properties of eigenvalues of positive definite matrices. In this lecture we will extend the same concept and we will see few applications of positive definite matrices in particular when you are having a positive definite quadratic form.

As you know that in the last lecture, I have defined a positive definite quadratic form as q of X equals to X transpose A X, where A is a symmetric matrix of order n. And X is a vector from the n dimensional vector space.

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$q(X) = X^T \underline{A} X$

$A \in \mathbb{R}^{n \times n}$

$X \in \mathbb{R}^n$

$q(X) = X^T D X$

$\Rightarrow$ diagonal representation of the quadratic form.

Now, if this matrix A becomes D. D means we are having a diagonal representation of A then the corresponding quadratic form that is X transpose D X is called diagonal representation of the quadratic form.

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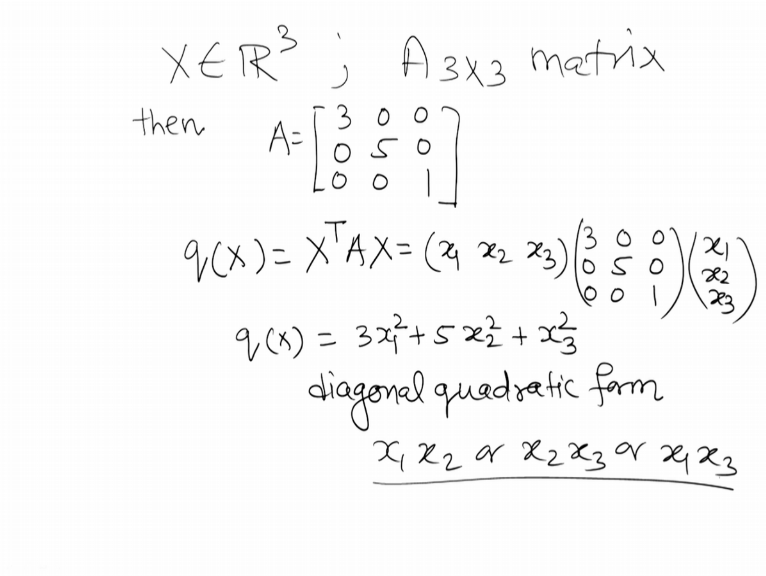

For example, if you consider X belongs to R 3 and A be a 3 by 3 matrix. Then if you take A equals to 3 0 0 0 5 0 0 0 1. Then the quadratic form q X which is given as X transpose into A into X which will be x 1 x 2 x 3 multiplied with a 3 0 0 0 5 0 0 0 1 into x vector. Then this comes out to be 3 x 1 square plus 5 x 2 square plus x 3 square. So, this particular quadratic form is a diagonal quadratic form. So, I can say in other word that a quadratic form is said to be diagonal quadratic form if there is no cross term or cross product term.

So, here my meaning of cross product term means a term something like x 1 x 2 or x 2 x 3 or x 1 x 3. So, all these are having 0 coefficient.

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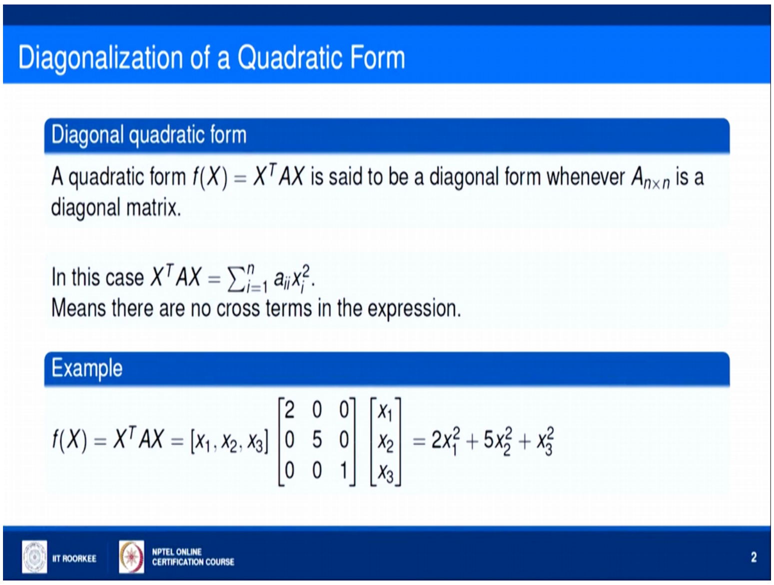

So, I can represent this quadratic form basically in this way. So, X T A X so, here A is a diagonal matrix. So, i equals to 1 to n a ii into x i square, where x i are the components of the vector X.

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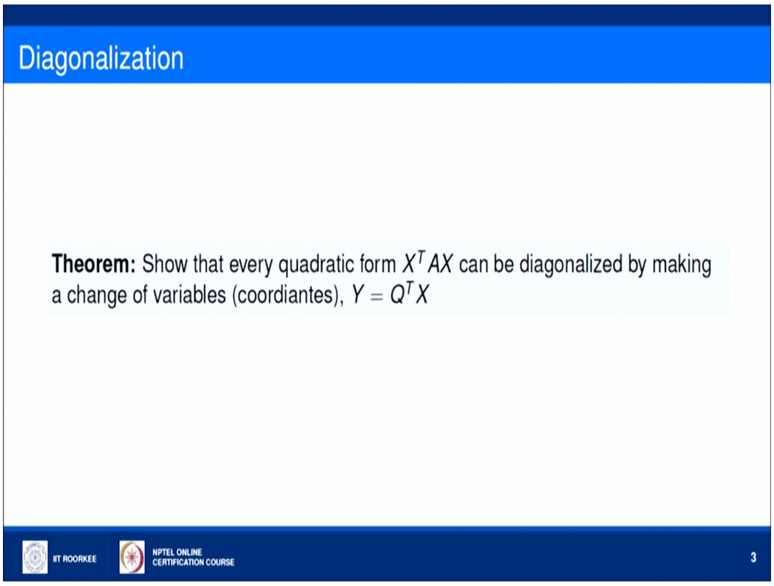

This is another example of the quadratic form. Now, we are having a very important resultant and we will see the application of this theorem later on. So, that every quadratic form X transpose AX can be diagonalized by making a change of variable or coordinates, let us say Y equals to Q transpose X. So, let us see the proof of this.

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Proof: Let $q(X) = X^T A X$ be a quadratic form, where A is a symmetric matrix of order n and $X \in \mathbb{R}^n$.

Now, as we know that A is symmetric, so there exists a diagonal matrix D and an orthogonal matrix Q such that

$$A = QDQ^T$$

Now $q(X) = X^T Q D Q^T X$

Now define a transformation

$$Y = Q^T X \; ; \quad Y^T = X^T Q$$

$$\rightarrow q(X) = Y^T D Y = \sum_{i=1}^{n} d_{ii} y_i^2$$

$\Rightarrow$ Quadratic form in diagonal representation. //

So, proof let q X equals to X transpose AX be a quadratic form, where a is a symmetric matrix of order n and X is a vector from the n dimensional really vector space. Now, as we know that A is symmetric. So, there exists a diagonal matrix D and an orthogonal matrix Q such that, A can be written as Q D Q transpose; means as we discussed earlier also if a is a symmetric matrix it is always diagonalizable, and here the diagonal matrix D will be having the eigenvalues of A as the main diagonal entries. And Q will come from the eigenvectors of A and those you can write as an orthogonal set of eigenvectors.

Now, so if I substitute this form of A in my quadratic form. So, I can write X transpose into Q into D into Q transpose into X. Now define a transformation a vector Y, which is having the same dimension as of X as Q transpose X. Then Y transpose will become X transpose into q and this particular quadratic form can be written as Y transpose into D into Y which is can be written as sigma equals to 1 to n dii into y i square which is a quadratic form, but a, but in diagonal representation. So, this is the result which we have to prove and this is the proof of this.

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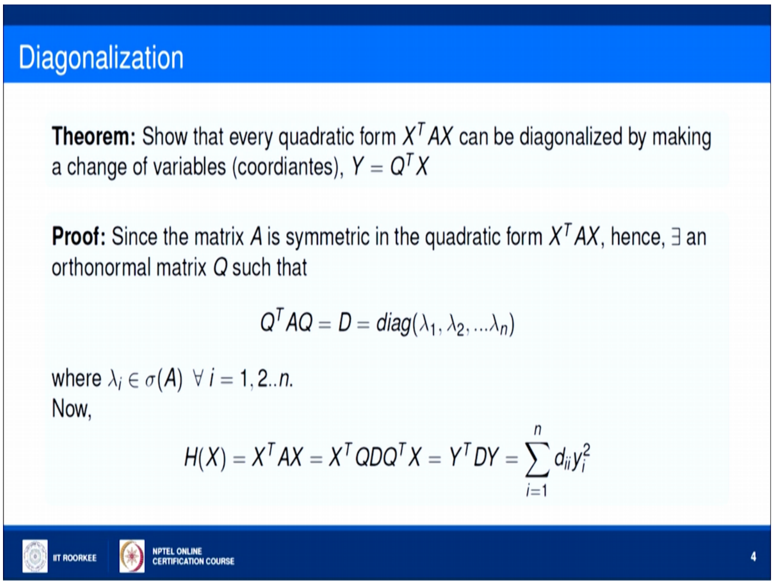

So, this is the proof which I have done just now.

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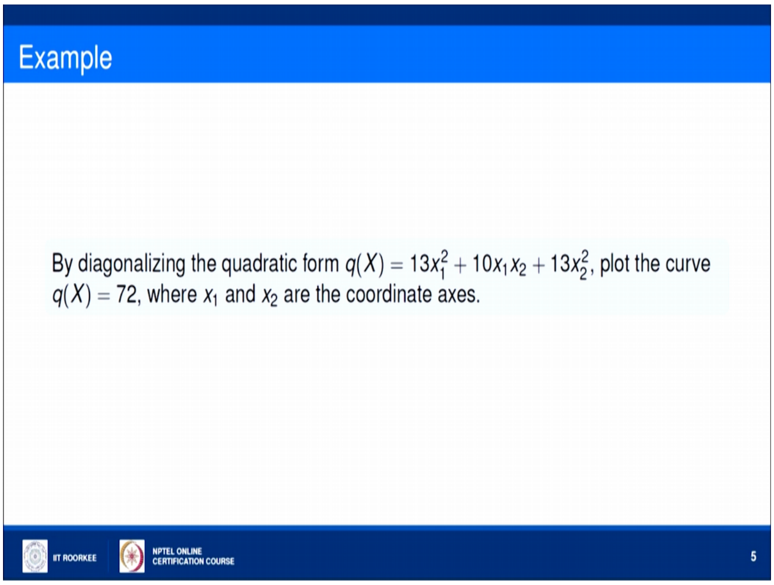

Now, let us take an application of this theorem. So, by diagonalizing the quadratic form q X equals to 13 x 1 square plus 10 x 1 x 2 plus 13 x 2 square plot the curve q X equals to 72, where x 1 and x 2 are the coordinate axis.

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$$q(X) = 13x_1^2 + 10x_1x_2 + 13x_2^2 = 72$$

Now $q(X) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 13 & 5 \\ 5 & 13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 72$

Here the matrix $A = \begin{bmatrix} 13 & 5 \\ 5 & 13 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 13-\lambda & 5 \\ 5 & 13-\lambda \end{vmatrix} = 0 \Rightarrow (13-\lambda)^2 - 25 = 0$$

$$\lambda = 8, 18$$

Eigenvector corresponding to eigenvalue $\lambda = 8 \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$(A - 8I)X = 0 \Rightarrow \left.\begin{matrix} 5x_1 + 5x_2 = 0 \\ 5x_1 + 5x_2 = 0 \end{matrix}\right] \Rightarrow x_1 = -x_2$$

Eigenvector corresponding to $\lambda = 18 \longrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\left.\begin{matrix} -5x_1 + 5x_2 = 0 \\ 5x_1 - 5x_2 = 0 \end{matrix}\right] \Rightarrow x_1 = x_2$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 18 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

So, what curve we need to plot that is q X equals to 13 x 1 square plus 10 x 1 x 2 plus 13 x 2 square and this is equals to 72. So, we need to plot this particular curve. Now, q X can be written as x 1 x 2 13 5 5 and 13 into x 1 x 2 and this equals to 72. So, here the matrix A equals to 5 equals to 13 5 5 and 13, which is associated with the quadratic form q X. Now let us perform the diagonalization of the matrix A. So, here if I calculate the eigenvalue of the matrix A it comes out to be 13 minus lambda 5 5 and 13 minus lambda and this equals to 0.

So, if I solve it, it will be 13 minus lambda whole square minus 25 equals to 0. This gives me eigenvalue as lambda equals to 8 and 18. Now, if I calculate the eigenvector corresponding to eigenvalue, lambda equals to 8, then A minus 8 I into x equals to 0 gives me 5 x 1 plus 5 x 2 equals to 0 the equation from the first row from the matrix a minus 8 I and the equation from the second row will be the same.

That is 5 x 1 plus x 2 equals to 0 5 x 1 plus 5 x 2 equals to 0. Now from these 2 equations I can write x 1 equals to minus x 2. So, the eigenvector corresponding to lambda equals to 8 can be written as if I take x 1 as 1. So, x 2 will become minus 1 and if I want to write it as a unit vector so, it will become 1 by root 2 into 1 n minus 1. Similarly, if I calculate the eigenvector corresponding to eigenvalue lambda equals to 18, then it comes out to be 5 so 18 minus 13 minus 18 minus will become minus 5 x 1 plus 5 x 2 equals to 0 and the second equation will become 5 x 1 minus 5 x 2 equals to 0.

So, from here I am getting x 1 equals to x 2 and the corresponding eigenvector will become unit eigenvector 1 by root 2 and 1 by root 2. So, in this way I can write the matrix A as p or q, where q is 1 by root 2 minus 1 by root 2 1 by root 2 1 by root 2. And the matrix D. So, d will be 8 0 0 18 and then again q transpose. So, it will become 1 by root 2 minus 1 by root 2 1 by root 2 and 1 by root 2.

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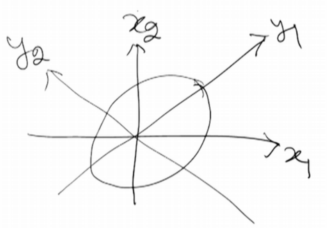
$$q(x) = (x_1 \;\; x_2)\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 0 & 18 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{Now} \quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$13x_1^2 + 10x_1x_2 + 13x_2^2 = 72$$

$$\Rightarrow \quad 8y_1^2 + 18y_2^2 = 72$$

$$\Rightarrow \quad \boxed{\frac{y_1^2}{9} + \frac{y_2^2}{4} = 1}$$



Now, the associated quadratic form can be written as q X equals to x 1 x 2 and then matrix 1 by root 2 minus 1 by root 2 1 by root 2 and 1 by root 2, which is my matrix q. Then I am having matrix d 8 0 0 18 and then I am having q transpose which is 1 by root 2 minus 1 by root 2 1 by root 2 and 1 by root 2 into x 1 x 2. So now assume another vector Y which is y 1 y 2 this equals to 1 by root 2 minus 1 by root 2 1 by root 2 and 1 by root 2 that is Q transpose into X. So, x is here x 1 x 2.

So, basically my y 1 is 1 by root 2 x 1 minus 1 by root 2 x 2 and y 2 is 1 by root 2 x 1 plus 1 by root 2 x 2. So, in these way I can write the quadratic form 13 x 1 square plus 10 x 1 x 2 plus 13 x 2 square 72, in an new variables y 1 and y 2 and it will become basically 8 y 1 square plus 18 y 2 square these equals to 72 or this I can write y 1 square upon 9 plus y 2 square upon 4 equals to 1, which is an standard equation of the ellipse.

Now, if I want to plot this particular curve. So, if these are the axis x 1 and x 2. So, then the representation of y 1 will be x 1 minus x 1 minus x 2 equals to 0 or basically x 1 plus x 1 equals to x 2.

Similarly, the representation of y 2 is x 1 minus x 2, by using these now along. So, this is my y one this is my y 2. So, my curve will become like this. So, in these way the given curve is an ellipse having this orientation respect to original coordinate x is x 1 and x 2. So, in this way we can plot any quadratic form by applying the suitable diagonalization and changing the variables. Now let us see another application of quadratic forms.

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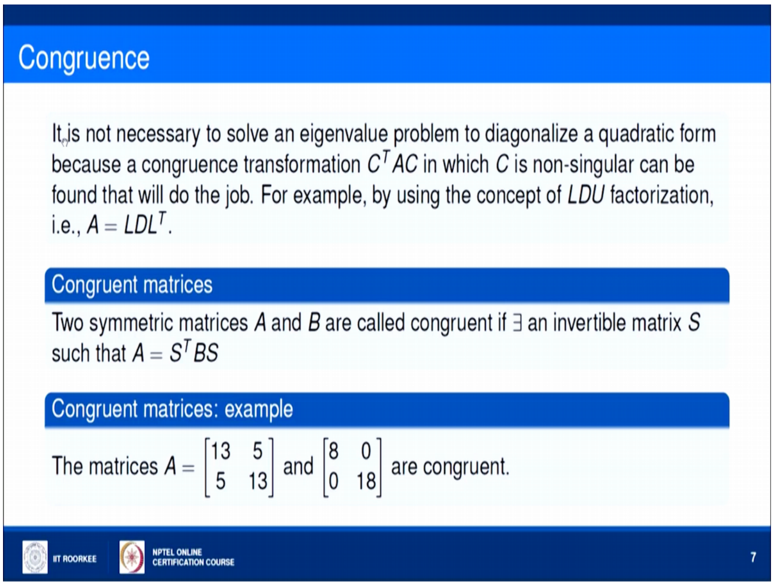

So, it is not necessary to solve an eigenvalue problem to diagonalize a quadratic form because when you are performing the diagonalization of the associated quadratic form and the corresponding matrix you have need to calculate eigenvalues and then eigenvectors. So, it will take lot of computation. So, alternative is a congruence transformation C transpose AC, in which C is a non-singular can be found that will to the same job. For example, by using the LDU factorization and it since here in the quadratic form as a symmetric matrix; so, always we can write this matrix U as L transpose.

So, I can write A equals to LD into L transpose which is a congruent transformation C transpose AC, where l equals to C. Now, the two symmetric matrices A and B are called congruent, if they are exist an invertible matrix S such that; A equals to S transpose BS.

For example, here the matrices A 13 5 5 5 13 and 8 0 0 18 are congruent because just now we have seen that there exist a matrix Q which is 1 by root 2 minus 1 by root 2 1 by root 2 and 1 by root 2 which is a equals to Q D Q transpose where 8 0 0 18 and a is 13 5 5 and 13. Now, my next definition in the same category is inertia.

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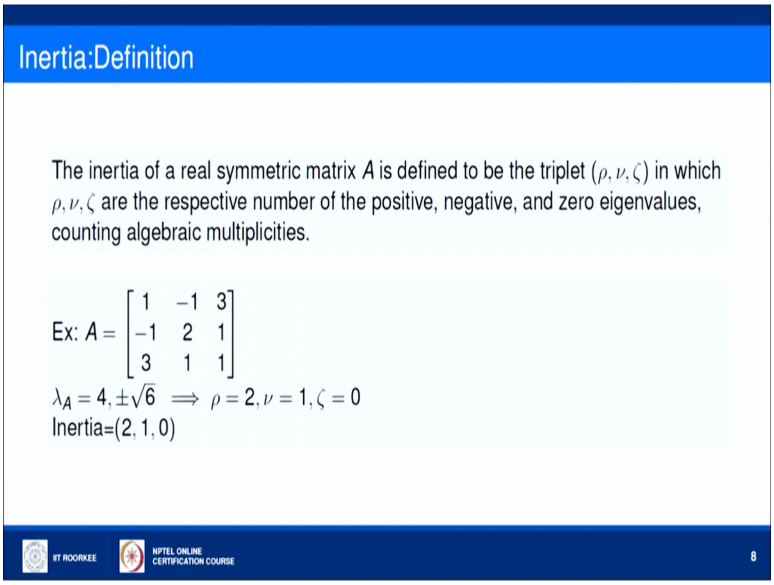

So, how to define an inertia of a real symmetric matrix? So, the inertia of a real symmetric matrix A is defined to be the triplet rho nu zeta in which rho nu and zeta are the respective number of the positive, negative and 0 and 0 eigenvalues. So, inertia of a matrix is a triplet in which first component is the number of positive eigenvalues, the second component is the number of negative eigenvalues, and the third component is the number of 0 eigenvalues counting the algebraic multiplicities. So, if I take this example.

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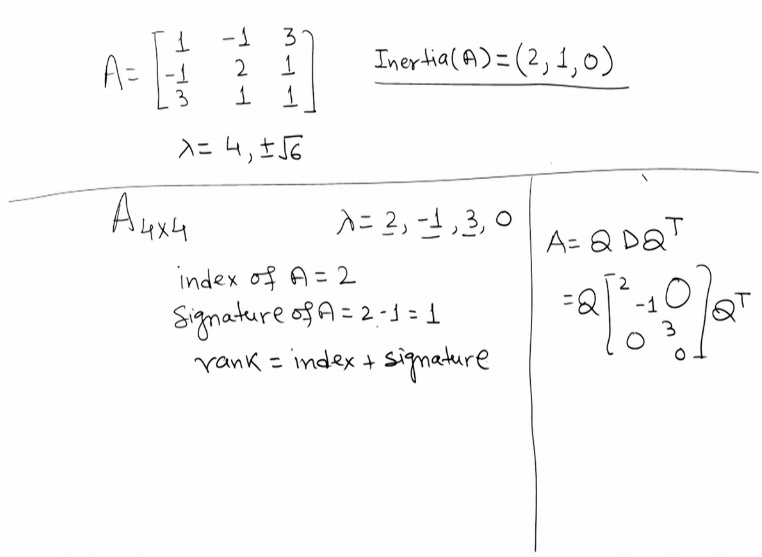

So, my example is A equals to 1 minus 1 3 minus 1 21 3 1 1. So, what is the inertia of this matrix? So, inertia of A can be found just from the eigenvalues of A. So, if I calculate the eigenvalues of A these comes out to be 4 plus minus root 6. So, if I see here I am having 2 positive eigenvalues of A, one is 4 another one is square root 6.

So, here row is 2. So, row is 2 number of negative eigenvalue is 1 and number of 0 eigenvalue is 0. So, this triplet is the inertia of matrix A. Now, we will see a very important result that is called Sylvester law of inertia.

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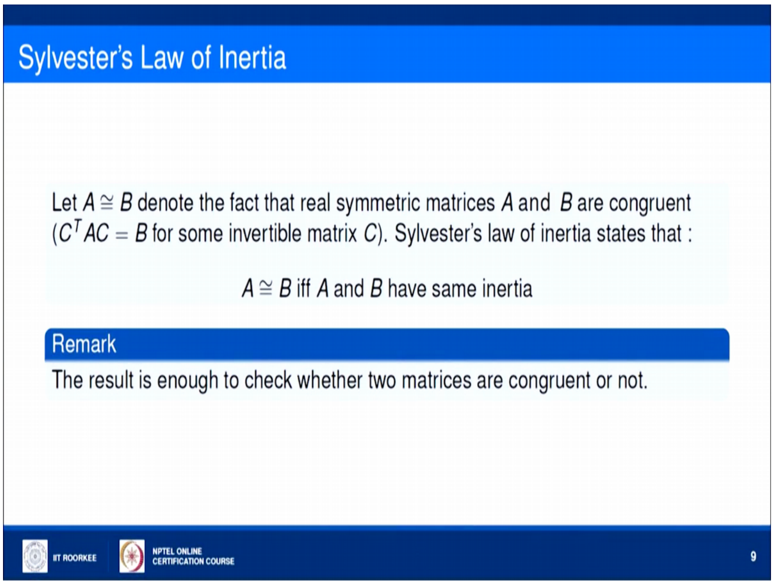

So, this particular theorem tells us that that two matrices A and B both are Symetric real symmetric matrices the same result hold for the Hermitian matrices also. So, A and B are congruent if and only if A and B have the same inertia; means A and B are congruent to each other means you can always find a non-singular matrix C such that; C transpose AC equals to B, if and only if the number of positive eigenvalues, number of negative eigenvalues and number of 0 eigenvalues for A and B are same.

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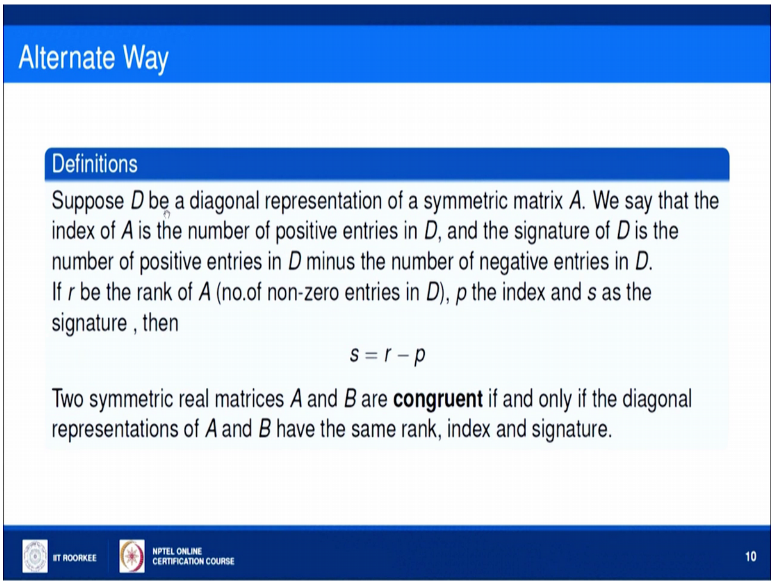

So, in this way we can say the result is enough to check whether two matrices are congruent or not. The alternate statement of this theorem is suppose D be a diagonal representation of a symmetric matrix A. We say that the index of A is the number of positive entries in D, and the signature of D is the number of positive entries in D minus number of negative entries in D, if r be the rank of A. So, r will be the number of non-zero entries. In the diagonal representation of a because number of non-zero eigenvalues, then p the index and s as the signature then s equals to r minus p.

So, for example, if you are having a matrix A which is a 4 by 4 real symmetric matrix and eigenvalues of a let us say you are having 2 minus 1 3 and 0. Then we will define index of A equals to number of positive eigenvalues or number of positive entries in the diagonal representation of a since A is symmetric. So, I can always write a equals to Q D Q transpose. So, q 2 minus 1 3 0 into Q transpose.

So, here index of A is the number of positive entries. So, it is 2 signature is number of positive entries minus number of negative entries. So, here I am having two positive entries and one negative entry. So, 2 minus 1 comes out to be 1. And as I told you rank of a is index plus signature and which is the number of non 0 entries in the diagonal representation of the matrix A.

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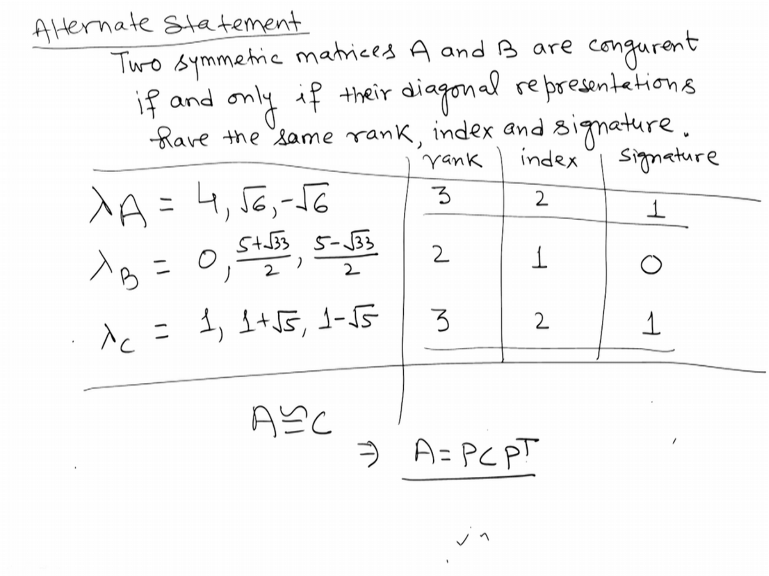

So, again the Sylvester law of inertia can be defined like this the two matrices two symmetric matrices A and B are congruent if and only if their diagonal representations have the same rank index and signature. So, this is the alternate definition or alternate statement I will write. So, let us consider an example of this.

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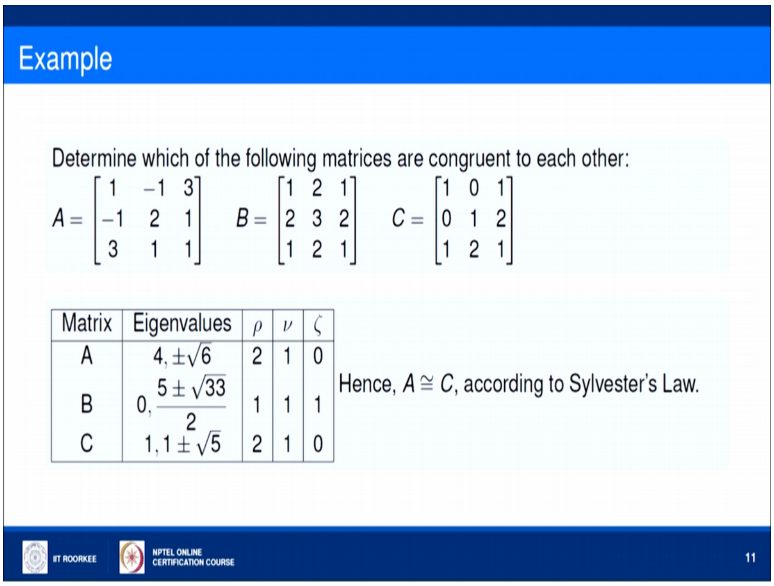

So, determine which of the following matrices are congruent to each other. So, all the 3 matrices are given where the matrix A is 1 minus 1 3 minus 1 2 1 3 1 1. The matrix B is 1 2 1 2 3 2 and 1 2 1 and C is having first row as 1 0 1 0 1 2 and 1 2 1. So, if I calculate the eigenvalues of matrix A. So, eigenvalues of matrix A comes out to be 4 root 6 and minus root 6. The eigenvalue of matrix B is 0 5 plus square root 33 by 2 and 5 minus square root 33 by 2. And the eigenvalues of the matrix C are 1 1 plus root 5 and 1 minus root 5.

Now, if I talk about the rank index and signature of these 3 matrices. So, the rank of matrix A is the number of non 0 entries in the diagonal representation of A and which will be 3 because the diagonal representation will contain the eigenvalues the index will be the number of positive eigenvalues or positive entries in D. And the signature is number of positive entries minus number of negative entries so 2 minus 1 1.

For this matrix the rank is 2 because one of the entry 0, index is 1 because only 5 plus root 33 by 2 is the positive entry and signature is 1 minus 1 0. For this matrix again I am having rank 3 index is 2 and signature is 1. So, if you see here the matrix A and matrix C are having the same rank index and signature. So, A and C are congruent A and C are congruent. A equals to PC P transpose by using either LDU factorization. So, if you go while the earlier statement of this Sylvester law of inertia.

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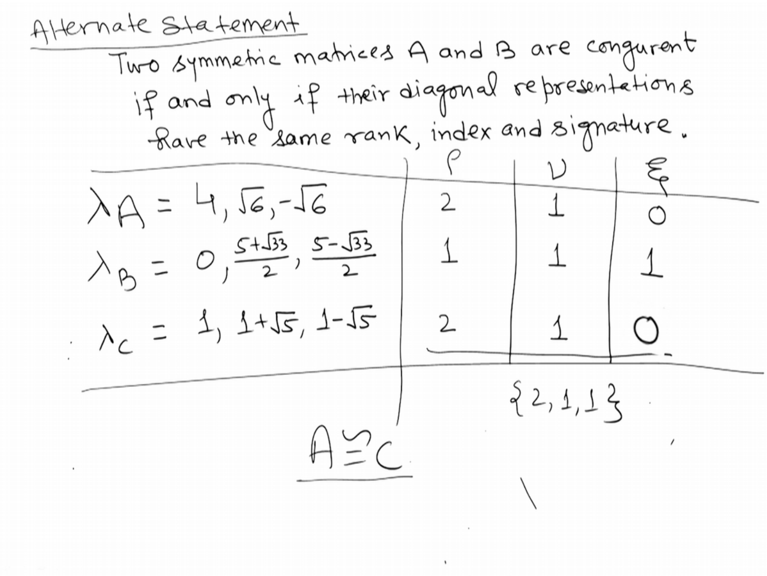

So, there I need to calculate rho nu and zeta. So, rho is defined as the number of positive eigenvalues. So, here it is 2 nu is the number of negative eigenvalues which is 1, and zeta is the number of 0 eigenvalues. So, 0 here it will be 1 1 and 1 and the third one will become 2 1. So, again A and C are having the same inertia because for A this is 2 1 1 which is same for C. So, again A and C are congruent to each other in the same way we can have a diagonal form or quadratic form.

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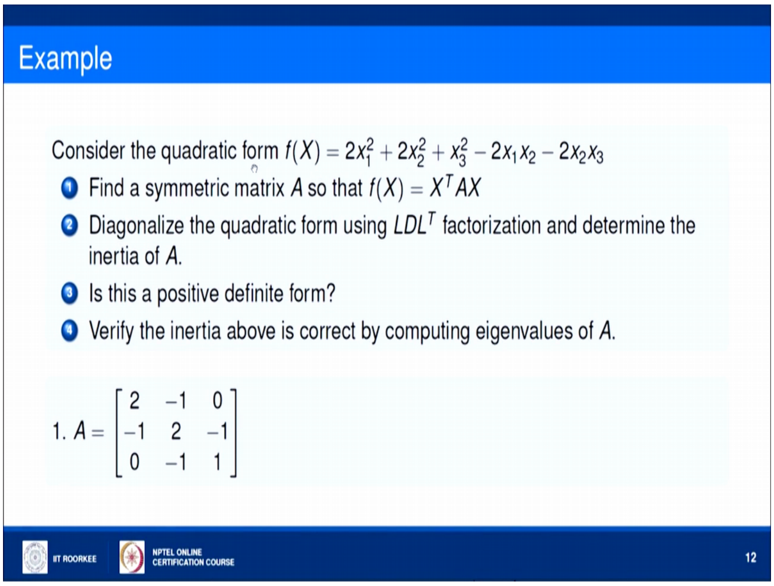

So, consider the quadratic form f X equals to 2 x 1 square plus 2 x 2 square plus x 3 square minus 2 x 1 x 2 minus 2 x 2 x 3. So, find a symmetric matrix A. So, that f X equals to X transpose AX.

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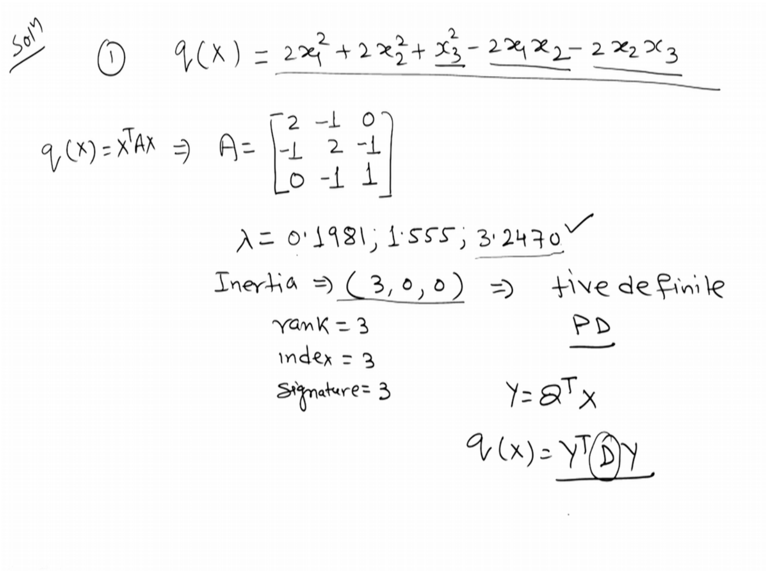

So, here q X given as so, solution q X given as 2 x 1 square plus 2 x 2 square plus x 3 square minus 2 x 1 x 2 minus 2 x 2 x 3. So now, I can the associated matrix is q X equals to X T AX. Then A equals to so here will come the coefficient of x 1 square 2 the second diagonal mean the diagonal entry in second row will be the coefficient of x 2 square and in the same way the diagonal entry in the third row will be the coefficient of x 3 square minus 2 x 1 x 2. So, I will take minus 1 here and minus 1 here minus 2 x 2 x 3. So, I will take minus 1 here and minus 1 here it 2 3 or 3 2 entries and these are 0.

So, this is the associated form. Now if I see the eigenvalues of this matrix the eigenvalues of this comes out to be 0.1981 1.555 and 3.2470. So, here if I calculate the inertia of this inertia will be 3 0 0 here rank is 3, index is 3 and signature is again 3. So, in this way since all the eigenvalues are positive. So, I can say and if rank equals to index equals to signature, then i will say the matrix is positive definite PD and hence the associated quadratic form is also positive definite.

I will ask you to work out on this and find out a suitable transformation Y equals to Q transpose X such that; q X you can write as YT D Y means; the corresponding diagonal representation of the quadratic form. And from you will see the entries of D will be 0.9811 0.555 and 3.2470.

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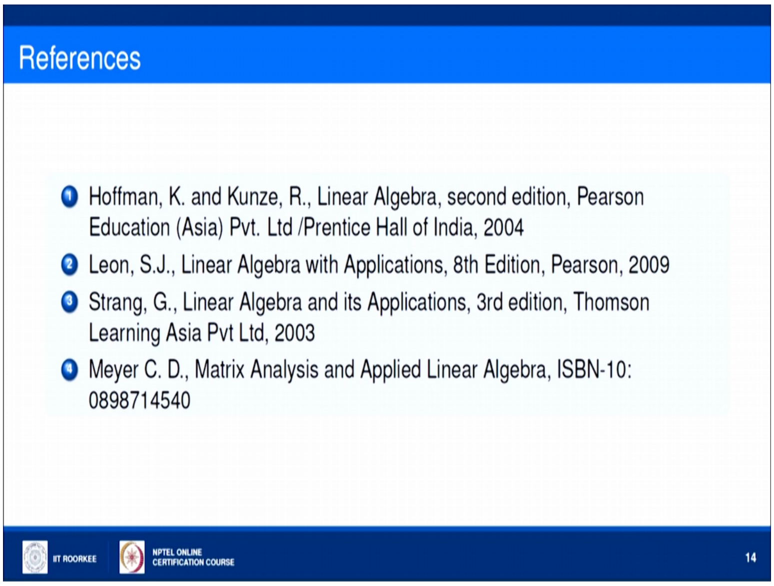

So, with this I will end this lecture. So, in this lecture we have seen a couple of applications of the quadratic form these are the references.

Thank you very much.