Matrix Analysis with Applications
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Lecture – 22
Positive Definite Matrices

Hello friends, so, welcome to the 22nd lecture of this course. So, in the last lecture, we have learnt few results about normal matrices. In this lecture, we will discuss another important class of matrices those are called Positive Definite Matrices. So, as we know that if we are having a symmetric matrix, then all the eigenvalues are real, that is true, and we have seen the proof of this in the previous unit.

However, what is additional we will learn in this lecture that the eigenvalues are not only the real, but positive also or just non negative.

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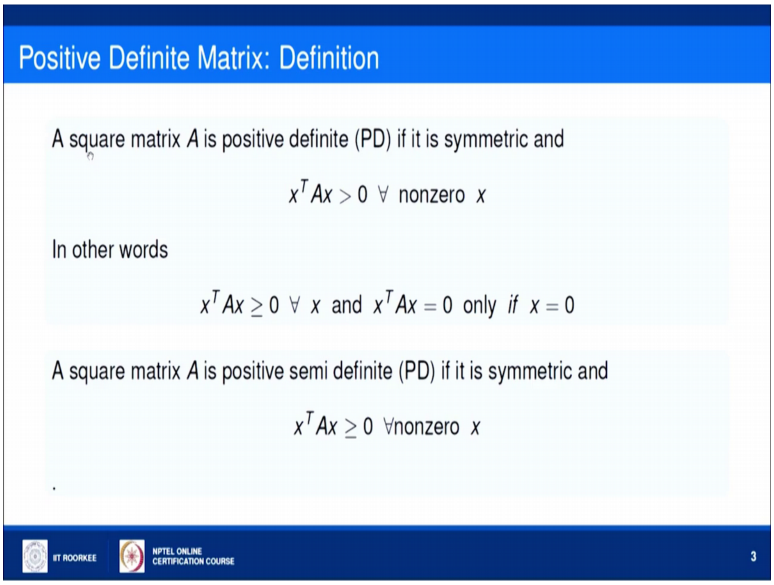

So, a square matrix A is called positive definite if it is symmetric, and for a non 0 vector x, x transpose into A into x, which is a scalar and x would be positive. So, whatever non 0 vector you take x, x transpose A into x would come out to be positive. If this is true, then we will say the square matrix A is a positive definite matrix.

In short I am saying it PD P for positive and D for definite. The same definition in other words can be written like this; x transpose into Ax greater than equals to 0 for all x, and this particular expression x transpose A into x will be 0, only if x equals to 0. So, this is the definition of positive definite matrix. A square matrix A is positive semi definite or PSD in short, if it is symmetric, and x transpose Ax is greater than equals to 0 for all non 0 x. So, the only difference is positive definite and positive semi definite is, here it will be strictly greater than 0, for non 0 x here it may be 0 also. The example or I will say that trivial example of a positive definite matrix is identity matrix.

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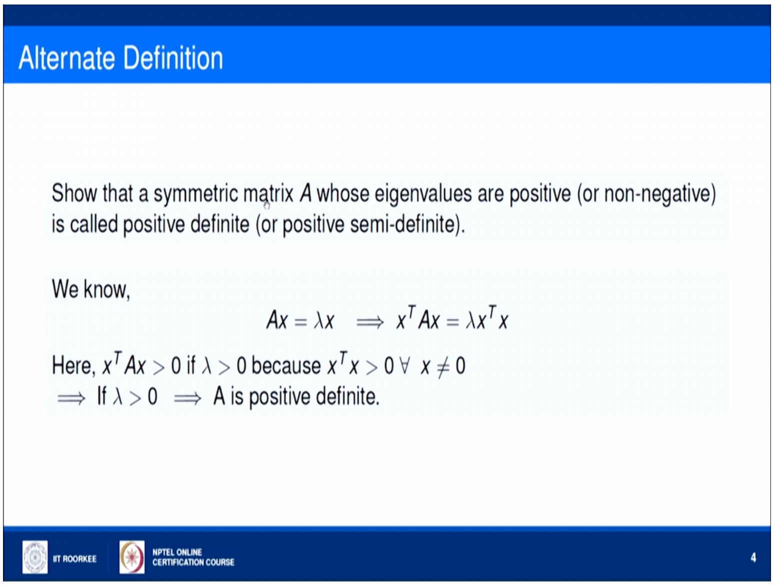

So, that a symmetric matrix A whose eigenvalues are positive or non-negative is called positive definite. So, here what we need to prove?

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$A_{n \times n}$

Let λ be an eigenvalue of A and x be the corresponding eigenvector.

$$Ax = \lambda x$$

$$x^T A x = \lambda x^T x$$

$$\text{or} \quad \lambda = \frac{x^T A x}{x^T x}$$

$$\text{Now if } \lambda > 0 \Rightarrow \frac{x^T A x}{x^T x} > 0 \Rightarrow x^T A x > 0 \quad \forall \underline{x} \in \mathbb{R}^n$$

$$\Rightarrow A \text{ is a PD}$$

$$\lambda \geqslant 0 \Rightarrow x^T A x \geqslant 0 \quad \forall \underline{x} \in \mathbb{R}^n$$

We are having a matrix A, let us say A n by n matrix, and we have to prove that if all the eigenvalues of A are positive then it is a positive definite matrix, and if all the eigenvalues of A are non-negative, then it is a positive semi definite matrix. So, let us take an eigenvalue lambda, let lambda be an eigenvalue of A and x be the corresponding eigenvector.

So, here lambda is eigenvalue and corresponding eigenvector is x, it means A into x equals to lambda in 2 x. Now, multiply both side by x transpose. So, x transpose A into x will become lambda x transpose into x. Or I can write it lambda equals to x transpose Ax upon x transpose x. Now if lambda is positive, this means x transpose Ax upon x transpose x is positive.

Since denominator will be always positive, so, it means x transpose Ax is positive for all x belongs to; this means A is a positive definite matrix. Similarly, if lambda is greater than equals to 0, this means x transpose Ax is greater than equals to 0 for all x belongs to Rn. And please note here x is eigenvector so, they will be non 0 vectors. So, in this way we can prove that if all the eigenvalues of a matrix are positive, then the matrix is positive definite. If all the eigenvalues of a matrix are non-negative, then the matrix is positive semi definite.

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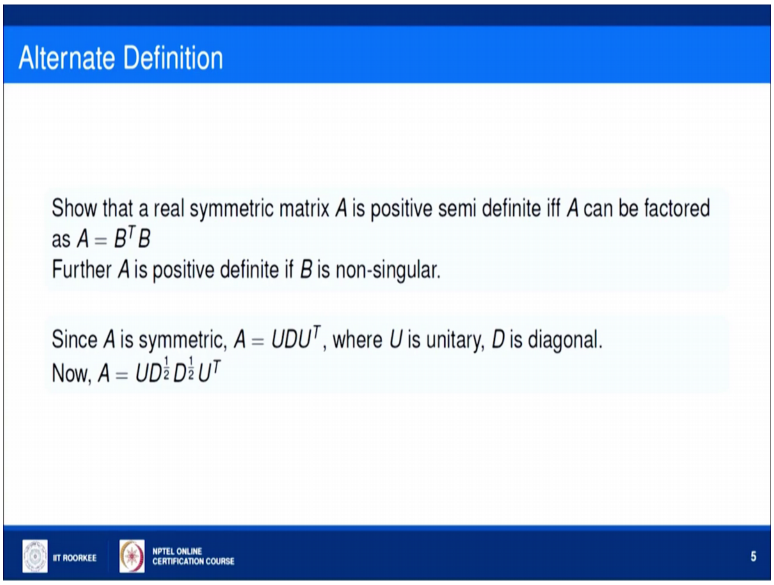

Alternate definition for positive definiteness can be given like this. A real symmetric matrix A is positive semi definite or I will say a positive semi definite, if and only if A can be factored as A equals to B transpose into B. Further A is positive definite if B is nonsingular.

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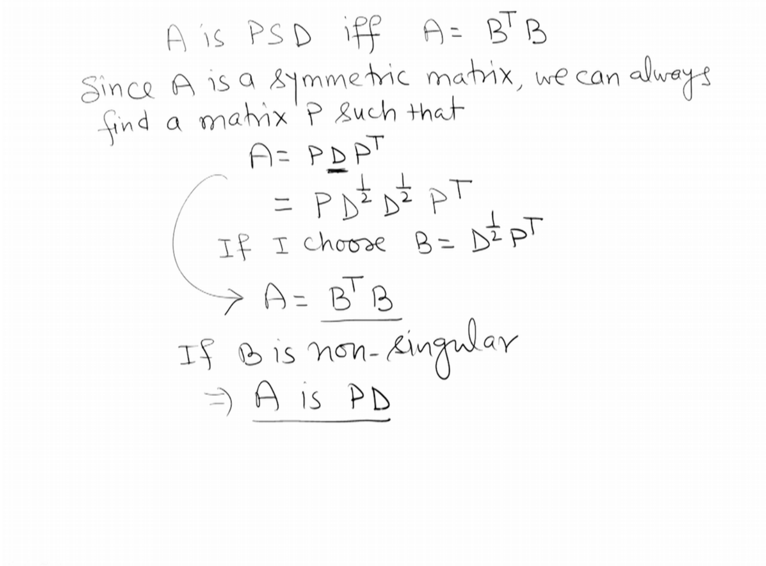

So, the proof can be seen like this so, A is positive definite or positive semi definite, if and only if A can be written as B transpose into B. So, since A is a symmetric matrix, we can always find a matrix P such that A equals to P into D into P transpose. Means, since a symmetric so, it will be always diagonalizable. Now what I need to prove that A can written as h the factor of two matrices that is B transpose into B. Now this can be written as P since A is positive semi definite. So, eigenvalues of a will be non-negative, and hence all the diagonal entries of D will be non-negative. So, I can write this D as D half into D half; where the entries are square root of the original D into P transpose.

So, if I choose B as D half into P transpose, then from here I can write A equals to B transpose into B. If all the eigenvalues are means if B is nonsingular, then all the diagonal entries of B will be strictly positive. And hence A is positive definite in this case so, in this way we can prove this result. There are few more test for check in whether a given matrix is positive definite or not.

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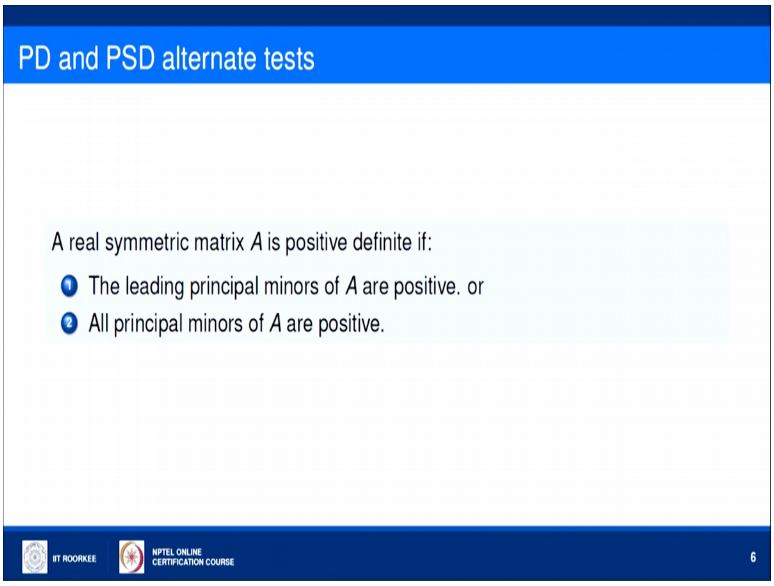

And one of the test is based on the leading principal minors. So, a real symmetric matrix is a positive definite if the leading principal minors of A are positive, or all principal minors of A are positive here positive. Means, having the positive determinates.

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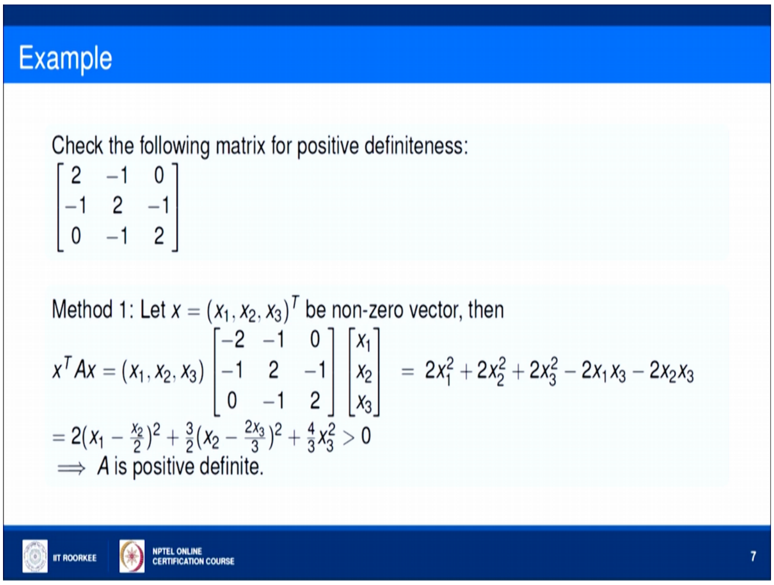

Let us check all these results for a given matrix. So, check the following matrix for positive definiteness. The matrix is 2 minus 1 0 minus 1 2 minus 1 0 minus 1 2. So, the matrix A is given which is given as 2 minus 1 0 minus 1 2 minus 1 0 minus 1 2.

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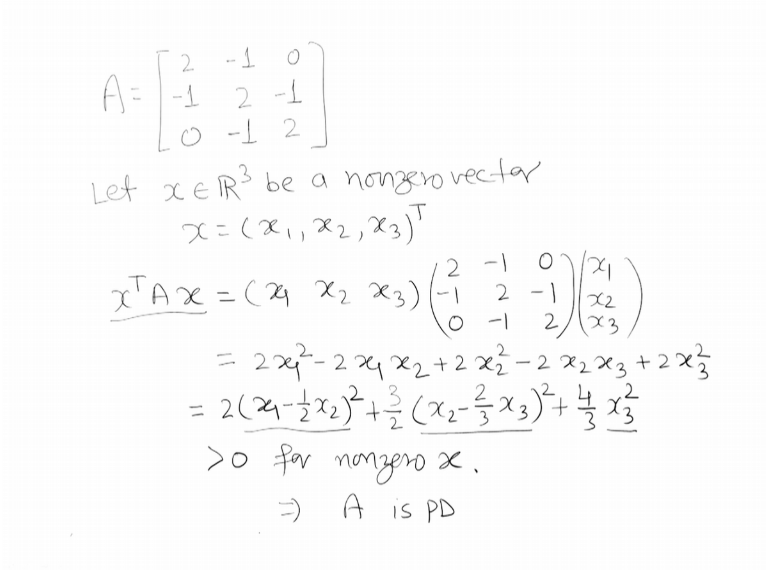

So, we have to check whether this matrix is positive definite or not. So, there are different test so, let us take the first definition, let x belongs to R 3 be a nonzero vector. So, means x equals to x 1 x 2 x 3 transpose. So now, x transpose A x will become x 1 x 2 x 3 into 2 minus 1 0 minus 1 2 minus 1 0 minus 1 2, into x 1 x 2 x 3. This comes out to be 2 x 1 square minus 2 x 1 x 2 plus 2 x 2 square minus 2 x 2 x 3 plus 2 x 3 square. This I can write like this, x 1 so, minus x 1 x 2 so, minus half, x 2 whole square so, if I express it will become x 1 square 2 x 1 square. So, x 1 square is left out here, and then 3 by 2 x 2 minus 2 by 3 x 3 whole square plus 4 by 3 x 3 whole square. So, the above thing can be written in this way, and you know that the final expression is the sum of square terms.

So, this will be always positive for nonzero x. Hence A is positive definite because x T A x is always positive for non-zero vector x, this is one of the way of checking the positive definiteness.

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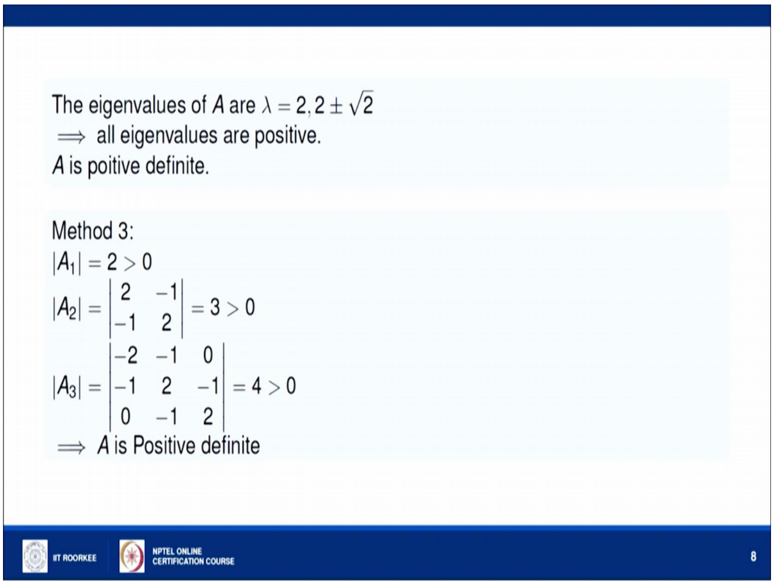

Another way is by calculating the eigenvalues of the matrix A. So, if we check the eigenvalues of this matrix A which is given like this, the eigenvalues comes out to be lambda equals to 2, 2 plus minus root 2.

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$$\lambda = 2 \quad > 0$$
$$\lambda = 2 + \sqrt{2} > 0$$
$$\lambda = 2 - \sqrt{2} > 0$$

All the eigenvalues of A are positive

$\Rightarrow$ A is positive Definite

So, eigenvalues are lambda equals to 2, lambda equals to 2 plus root 2 and lambda equals to 2 minus root 2. If we check it is positive, this is also positive, and this is also positive. So, all the eigenvalues of A are positive this implies that A is positive definite.

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$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A_1 = |2| > 0$$
$$A_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$$
$$A_3 = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2[4-1] + 1[-2+0] = 6 - 2 = 4 > 0$$

$\Rightarrow$ A is positive definite.//

Another test is by checking the minus principal minus. So, matrix is 2 minus 1 0, minus 1 2 minus 1, and then 0 minus 1 2. So, let us take first this minus so, A 1 which is 2 which is greater than 0. Now check the 2 by 2 minor. So, it will be this one so, this is the determinate of 2 minus 1 minus 1 2. This comes out to be 4 minus 1, which is 3 greater than 0. Now check the determinate of A, that is that 3 by 3 minor minus 1 2 minus 1 0 minus 1 2.

So, if I check this determinate this will be 2, 4 minus 1 minus minus plus 1 minus 2 plus 0, this comes out to be 6 minus 2, equals to 4 which is also 0. So, all the principal minors are positive and hence the matrix A is positive definite. So, in this way I have told you 3 different methods to check whether a given matrix is positive definite or not. What are those? The first one just take a non-zero vector x and find out x transpose x, if you can write that thing as a sum of perfect squares, then the given matrix is positive definite. Number 2 find out the eigenvalues, if all the eigenvalues are positive then the given matrix is positive definite.

If all the eigenvalues are non-negative, then the matrix is positive semi definite, means some are positive and some are 0. The third is minor test, if all the principal minors are positive then the matrix is positive definite.

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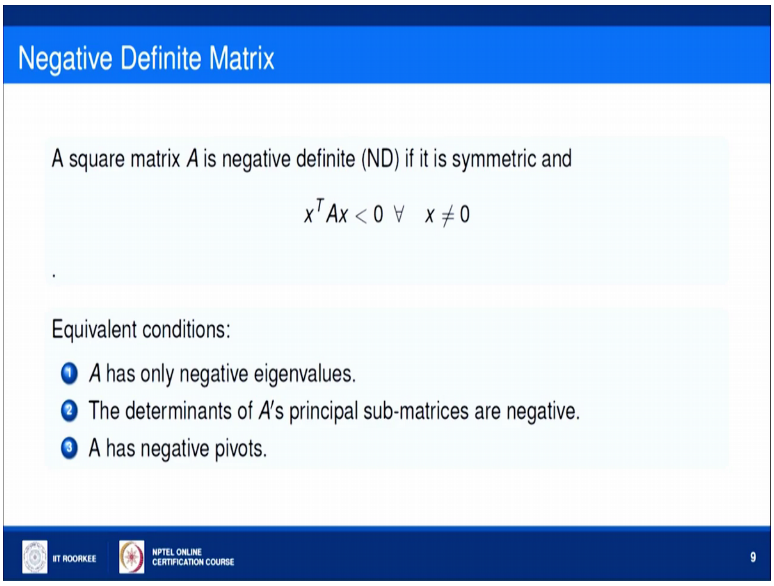

In the same manner we can define the negative definite matrix, A square matrix A is negative definite in short I will say it N D, if it is symmetric and x transpose A x is less than 0 for all x, means, for all non-zero x. The equivalent conditions are A has only non-negative eigenvalues, the determinants of A's principal sub-matrices are negative or A has negative pivots, these are the equivalent conditions.

Like the test we have seen for the previous example just the same test, but in negative sense. To prove a given matrix is negative definite. In the same way we can define the negative semi definite in that symmetric matrix is negative semi definite, if x transpose into x is less than equals to 0, for all x those are not 0.

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A is negative semi definite, if

$$x^T A x \leq 0 \quad \forall \ x \neq 0$$

Then the matrix is called negative semi definite, means, if a matrix are having the eigenvalue 0 and negative, then the matrix is negative semi definite.

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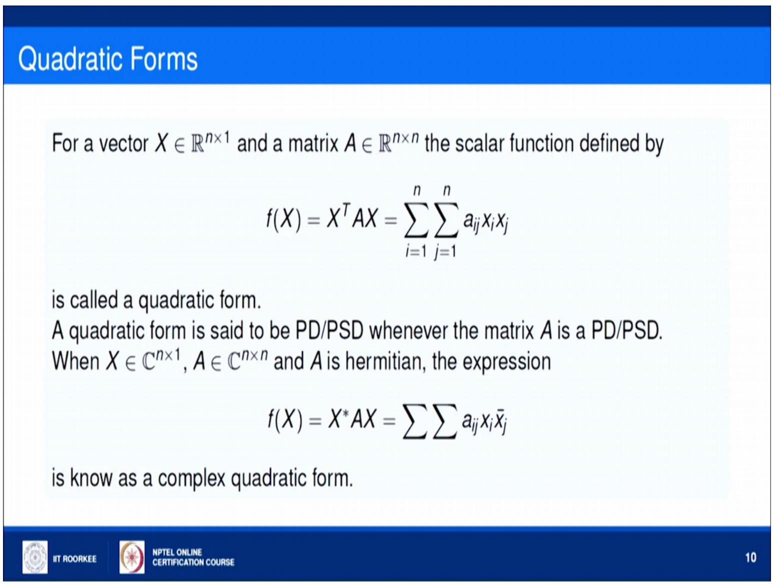

Now, another important definition related to definite positive and negative definiteness is quadratic forms. So, for a vector X which is a real vectors in having n components and a matrix A which is a real matrix of size n by n, the scalar function defined by f X which is X transpose A X, and in summation form if the entries of A is denoted by a i j then it can be written as i equals to 1 to n, j equals to 1 to n a i j x i x j is called a quadratic form.

Further a quadratic form is said to be positive definite, positive semi definite, whenever the matrix A is positive definite or positive semi definite, this is the real quadratic form. In the similar manner we can define the complex quadratic form. So, let X be a complex vector having n components, and A is n by n matrix having complex entries and A is Hermitian. Then the expression f X X star A X which is comes out to be a i j x i x j conjugate is called a complex quadratic form.

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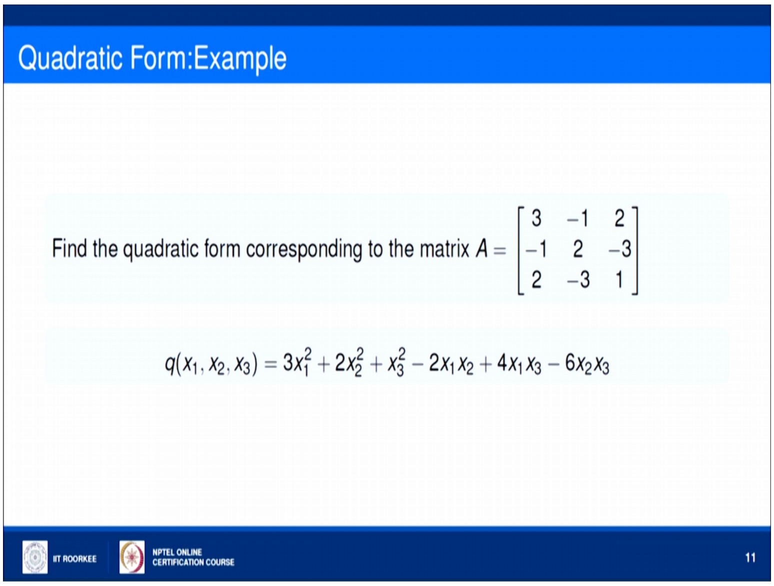

For example, if a matrix is given to you and someone ask find out the quadratic form of this matrix.

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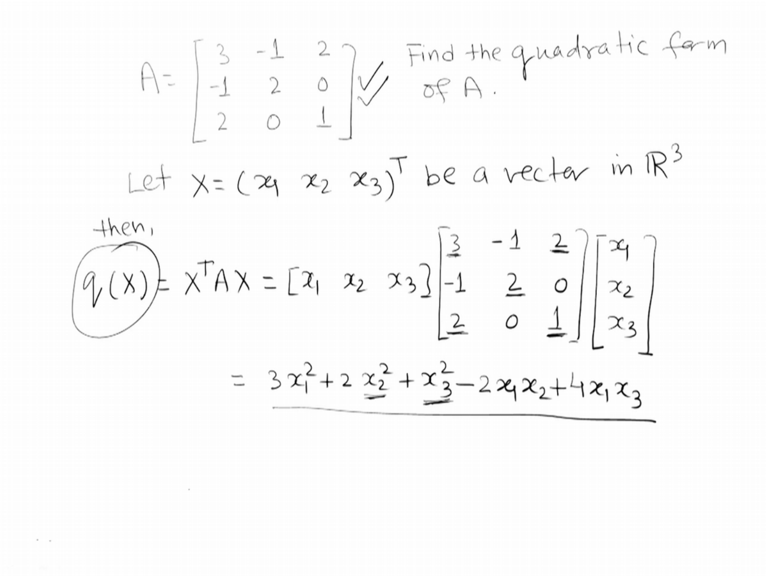

So, let A is given to you which is 3 minus 1 2 minus 1 2 0 and then 2 0 1, find the quadratic form of; let x 1 x 2 x 3 be a vector in r 3, then the quadratic form of A is defined as x transpose A x which is x 1 x 2 x 3 multiplied with matrix 3 minus 1 2 minus 1 2 0 2 0 1 into x, x is x 1 x 2 x 3. So, this comes out to be 3 x 1 square plus 2 x 2 square plus x 3 square, because these square terms will be just multiplied with the diagonal elements.

Now, the diagonal element of first row will be multiple of coefficient of x 1 square, the diagonal entries of second row will be the coefficient of x 2 square, the diagonal entries of third row will be the coefficient of x 3 square. Minus so, minus x 1 x 2 will come from here, and x 2 x 1 will come from here. So, minus 2 x 1 x 2, similarly 2 x 1 x 3 will come from here 2 x 3 x 1 will come here.

So, it will become x 1 x 3 2 plus 2 so, it will be 4 so, 4. The coefficient of x 2 x 3 will be 0 as well as for the coefficient of x 3 x 2. So, this is the required quadratic form of the given matrix. If someone ask to check whether this quadratic form is positive definite or positive semi definite or negative definite or negative semi definite what you have to check you just check this matrix A.

If A is positive definite the quadratic form is positive definite. If A is negative definite the quadratic form is negative definite and similarly for positive semi definite and negative semi definite. Well, in this lecture we have learnt the definition of positive definite matrices, positive semi definite matrices, negative definite matrices and negative semi definite matrices.

Later on we have learnt few test how to check whether a given matrix is positive definite or not. Apart from that we have learnt quadratic form relating to a matrix. In the next lecture, we will learn some other important properties of quadratic form. We will learn the diagonalization of the quadratic form; we will learn some applications of the quadratic form, in particular to draw a given quadratic curve with the help of quadratic form. With this I will end this lecture.

Thank you very much.