Matrix Analysis with Applications
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Lecture - 21
Normal Matrices

Hello friends, so, welcome to the 21st lecture of this course. So, in this lecture I will introduce you Normal Matrices. So, in the last unit of this course you have learned about diagonalization of a matrix. And we have seen that if A square matrix consist a complete set of linearly independent eigenvectors, then we say that the matrix is diagonalizable; that is, if the size of matrix is n by n and you are getting n linearly independent eigenvectors for that matrix, then the matrix is diagonalizable.

There may be case that the matrix is having n eigen values, then corresponding to each eigenvalue you will get a linearly independent eigenvectors, and hence matrix is diagonalizable. In other case when the algebraic multiplicity of any eigenvalue is more than 1, then if you are having the same geometric multiplicity that is algebraic multiplicity equals to geometric multiplicity for each eigenvalue then the matrix is also diagonalizable. However, in this lecture, we will see something more than linearly independent, and that is called the orthogonal set of eigenvectors.

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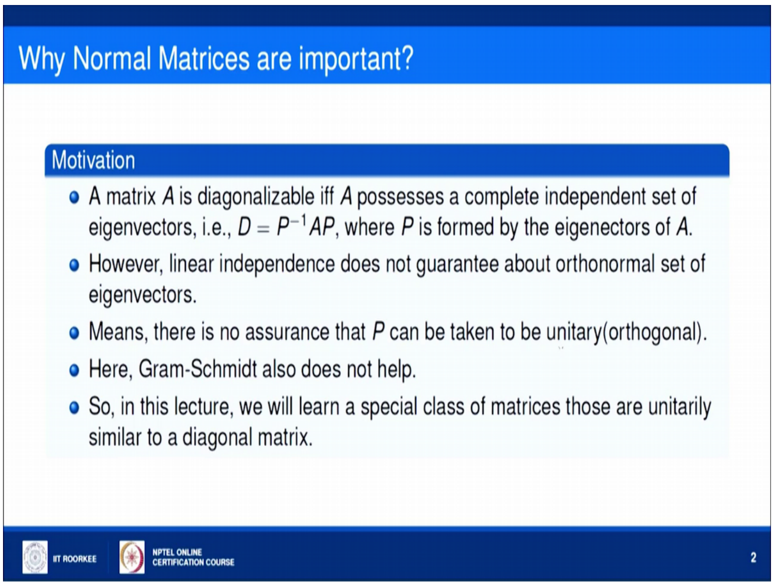

So, linearly independence does not guarantee about orthonormal set of eigenvectors. Means there is no assurance that the matrix P which is called model matrix, and that we are writing from the eigenvectors, means the columns of pr the eigenvectors of the matrix A. So, this matrix can be taken to be unitary, if it is defined with complex entries, or orthogonal if it is having real entries.

So, we are having n linearly independent eigenvectors and from there we are writing P, but we do not have any assurance about the orthonormal or orthogonal property of those eigenvectors. We have learned that gram Schmidt process, from that we can convert a set of n linearly independent vectors into n orthogonal vectors or n orthonormal vectors. Gram Schmidt if you are having n linearly independent vectors and you apply the gram Schmidt process, then you will get a set of n orthonormal vectors.

But there is no guarantee that if the earlier set is n linearly independent eigenvectors, then the resulting set will be having n orthonormal eigenvectors. So, here gram Schmidt also does not help. So, in this lecture we will learn a special class as I told you of matrices those are unitarily similar to a diagonal matrix. So, you know that if I am having a matrix A and it is diagonalizable. So, I can find out a model matrix P such that A equals to P D P inverse; where P is coming from the eigenvectors of A and D is a diagonal matrix. Now unitarily similar means the matrix P is a unitary matrix.

Unitary matrix means, P star that is the conjugate transpose of P into P equals to P into P star equals to identity matrix.

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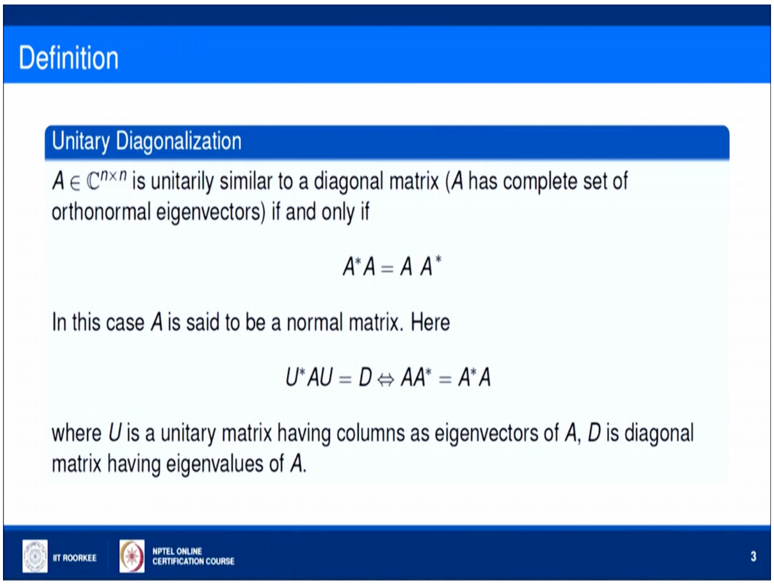

So now, our first definition is unitary diagonalization. So, consider a matrix A of size n by n having complex entries. It is unitarily similar to a diagonal matrix means a has a complete set of orthonormal eigenvectors, if and only if the complex conjugate of A into A equals to A into A star; that is A star A equals to A into A star. In this case, A is said to be normal matrix.

So, what I want to say that a matrix is normal if and only if it is unitarily similar to a diagonal matrix. Means U star AU, where U is a unitary matrix having columns as eigenvectors of A equals to D, where D is a diagonal matrix if U star AU equals to D, then A will be a normal matrix; that is, A star A into A star equals to A star into A. And if A is a normal matrix, then A will be unitarily similar to a diagonal matrix; let us try to prove it.

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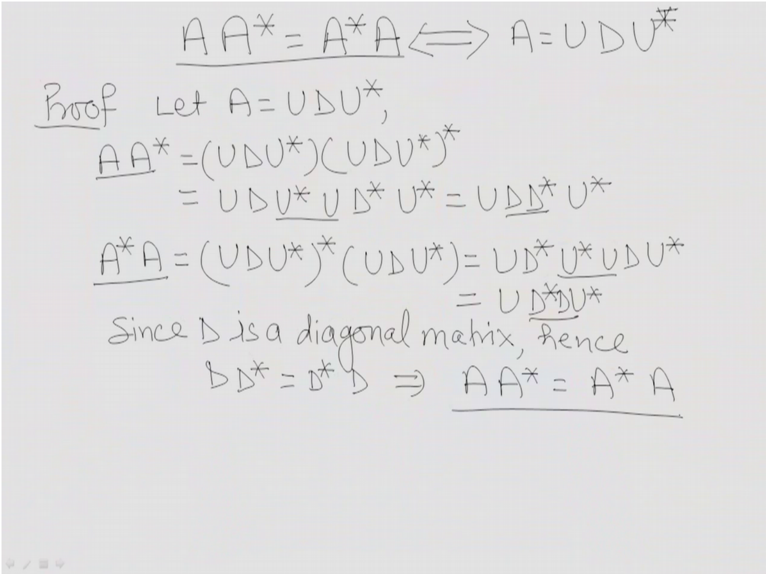

So, here we want to prove that A into A star equals to A star into A means A is normal, that is the most popular definition of a normal matrix, there are alternate definition also. So, A is normal if and only if A equals to U D U transpose; where D is a diagonal matrix consisting with the eigenvalues of A and U is a unitary matrix. So, let us first assume that A is unitarily similar. So, it means let A equals to U D U star.

So, since we are talking about complex matrix. So, instead of transpose I will write star, then if I see A into A star this will become U D U star, and then UD U star star. So, as I told you let me (Refer Time: 07:09) again that is star for the complex transpose, conjugate transpose. So, it will be U D U star and then this will become U D star U star. This equals to, as I told you U is a unitary matrix so, U star U will be identity so, it is U D D star into U star. Now if I see A star into A, then it will become U D U star into U D U star so, this is my A star this is A.

This equals to U and then I will be having D star then U star U D U star. So, this comes out to be U D star again U star U will become identity into U star. So, A into A star will be equals to A star into A only when D into D star equals to D star into D. So, as you know the D is a diagonal matrix so, since D is a diagonal matrix, hence D into D star will be equals to D star into D, and this implies A into A star equals to A star into A.

So, this is the proof of first part that if A is normal A is unitarily similar to a diagonal matrix then A is a normal matrix. Now, see the another part of this so, here we are assuming that A is normal.

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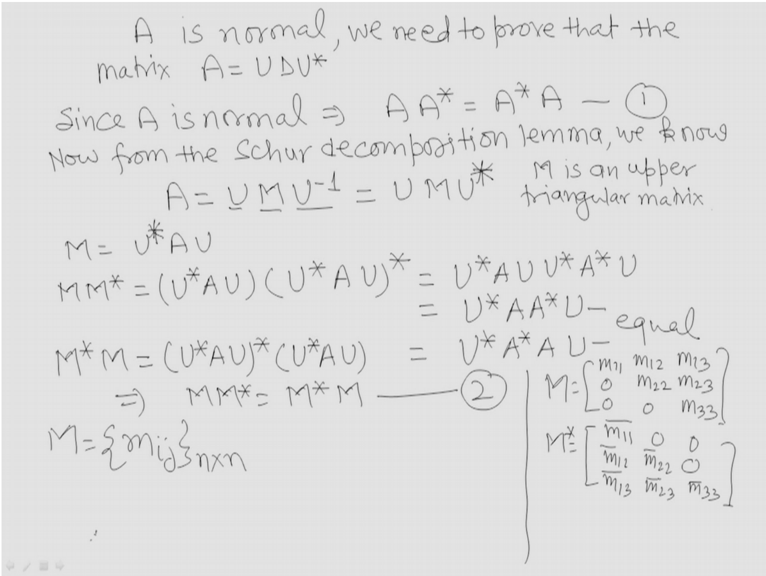

And we need to prove that the matrix A is unitarily similar to a diagonal matrix; that is, A equals to UD U star where U is a unitary matrix. So, since A is normal, it means A into A star equals to A star into A, let us say this lesson number 1. Now from the Schur decomposition lemma, we know that any square matrix of order n can be decomposed is the product of 3 matrices where U M and U inverse, where U is a unitarily matrix unitary matrix. So, U can be written as U M into U transpose, and M is an upper triangular matrix. So now, from here I can write M equals to U transpose A into U.

Now, if I calculate M into M star that is the conjugate transpose of M, it will become U star A into U. So, again it is star because we are talking about complex matrix. So, I will write star here into U star A U star. This comes out to be U star A U that is the first matrix as such. And the second matrix will become U star A star into U. U into U since U is a unitary matrix. So, U into U star will become identity so, it will be U star A into A star into U.

Now, in the same way if I find out M star into M so, M star will be again U star A U star, and then M will become U star A U. This comes out to be U star A star A into U. Now since we have assumed that A is normal. So, these two things are equal. So, from here I can write M into M star equals to M star into M. So, let me write it as lesson number 2. Now the point is we need to prove that A is unitarily similar to a diagonal matrix. Here we have seen that A is unitarily similar to a upper triangular matrix. Now this upper triangular matrix is M. So, only thing we need to prove that this upper triangular matrix is a diagonal matrix. So, what we need to prove that M is a diagonal matrix.

Now, let M is and it is an upper triangular matrix. Now if it is a 3 by 3 matrix and I take M as m 1 1 m 1 2 m 1 3 0 m 2 2 m 2 3 0 0 m 3 3. So, M into M star so, if M is this then M star will become m 1 1 0 0 and conjugate of m 1 1 second row will become m 1 2 m 2 2 m 2 3. Yeah, m 1 2 m 2 2 and m 2 3 is basically 0 so, let me write it 0. And then third row will be m 1 3 conjugate, m 2 3 conjugate, m 3 3 conjugate so, these are the matrices M and M star. So, if I find try to calculate the upper left element of M into M star, then it comes out to be if you see from here, m 1 1 into m 1 1 conjugate m 1 2 into m 1 2 conjugate and so on.

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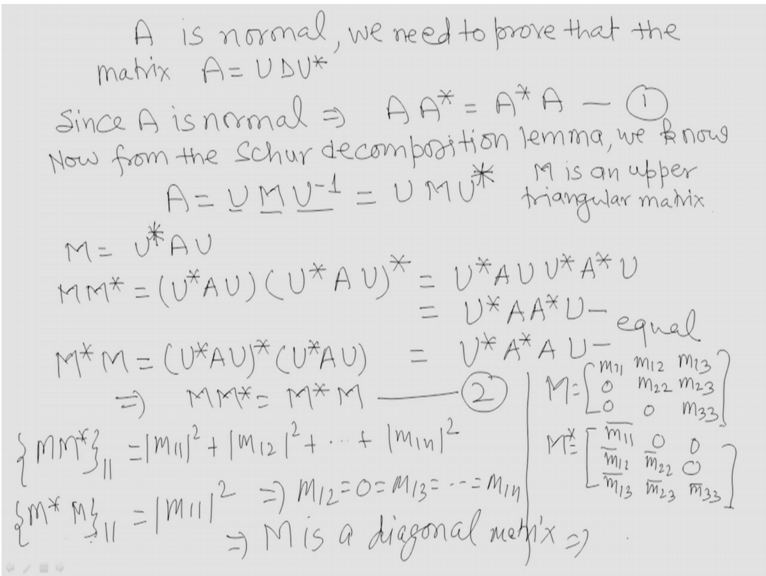

So, the first element of this that is the upper left element will be m 1 1 norm square, m 1 2 norm square up to m 1 1 n norm square. While if I calculate the upper left element of M star into M, then this comes out to be the square of the norm of m 1 1, now what is what is happening?

Since M M star equals to M star into M. So, these 2 elements should be equal, this gives me that m 1 2 equals to 0 equals to m 1 3 up to m 1 n. That is the first row of the matrix

M is having 0 entries except the pivot element that is the first element. Similarly, if I compare the diagonal element of the second row, it will give me that the elements M to j where j is greater than equals to 3 up to n or 0. In this way what I can say that M is a diagonal matrix. And hence A is unitarily similar to a diagonal matrix D which is nothing just M. So, in this way we can prove this result, ok.

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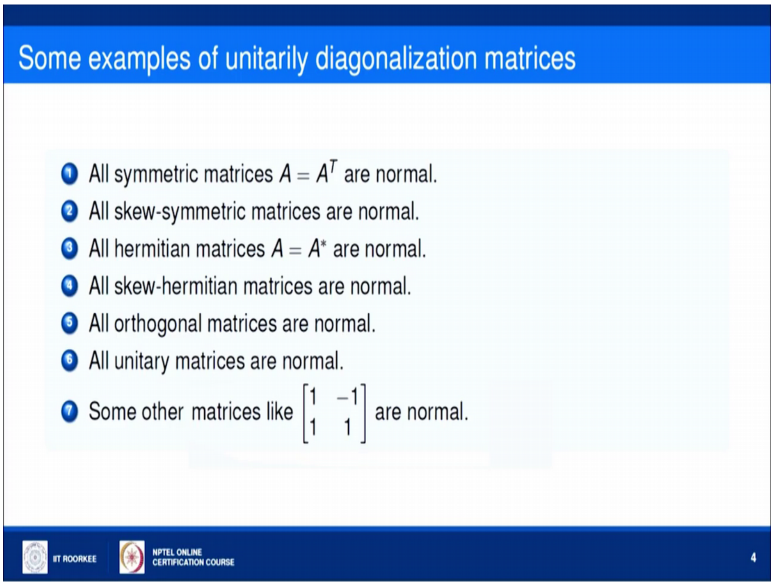

So, after this result let me give you few examples of normal matrices. So, all symmetric matrices are normal. Then all skew symmetric matrices are also normal all Hermitian matrices are normal. All skew Hermitian matrices are normal, in fact, all orthogonal matrices are normal. And if I say about complex vector space, then all unitary matrices are normal. Apart from these examples some other normal matrices like 1 minus 1 1 1 are also normal. There are matrices also those are not normal, we will see.

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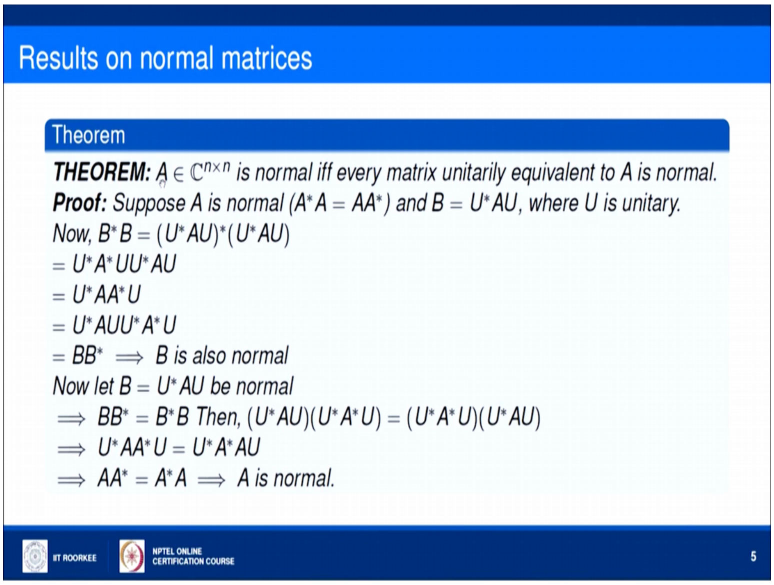

Now, again an important result for normal matrices that a matrix of size n by n having complex entries is normal if and only if every matrix unitarily similar to A is normal; So, the proof of this particular theorem is quiet simple, let me explain here suppose A is normal, it means A star. A equals to A into A star and B is unitarily similar to A; that is, B equals to U star A U where U is a unitary matrix. Now if I calculate B star into B it comes out to be U star A U star into U star A U.

After calculating it comes out to be U star A star U into U star A U U into U star will become identity. So, I can write this expression as U star into A into A star into U. And since U is a unitary matrix so, I can multiply between A and A star by an identity matrix, which is I can write in terms of U and U star is product of U into U star, and which is nothing just B into B star. So, what we have done? B star B equals to B equal into B star, it means B is a normal matrix.

Now, if we prove this from the other way, that B is normal, then B B star will be equals to B star B, then U star A U into U star A star U will be equals to this 1. And from here we can see that A into A star equals to A star into A this implies that A is normal. So, this is the proof of this theorem.

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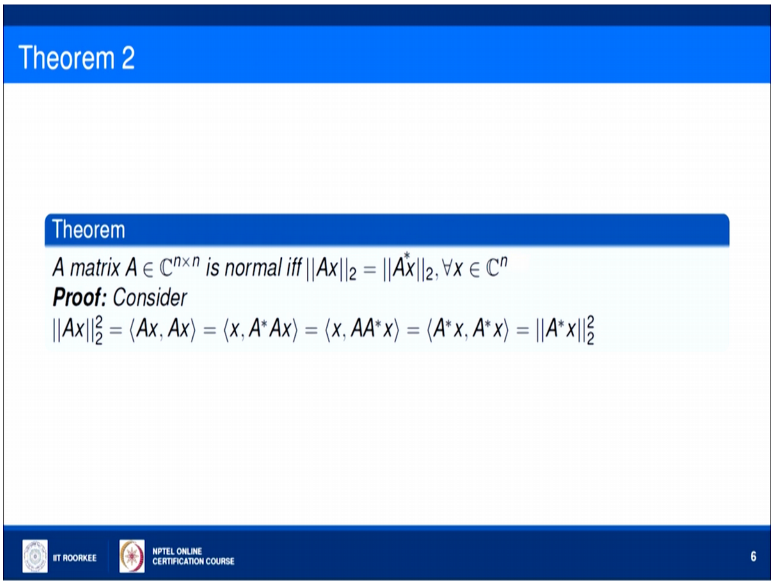

Another important property of normal matrix that a matrix of order n which is a square matrix having complex entries is normal, if and only if the Euclidean norm of A into x for a given vector x, from the n dimensional complex vector space, that norm of Ax equals to norm of A star x.

So, the proof is quite simple from the definition of inner product, that the norm of Ax equals to inner product of Ax with Ax, this can be written as x into A star A into x since A is normal so, A star into a can be written as A into A star. So, x inner product of x with a A star x, this comes out to be inner product of A star x and A star x, which is nothing just norm of A star x so, very simple result one-line proof. Now, come to another important result that is called spectral theorem.

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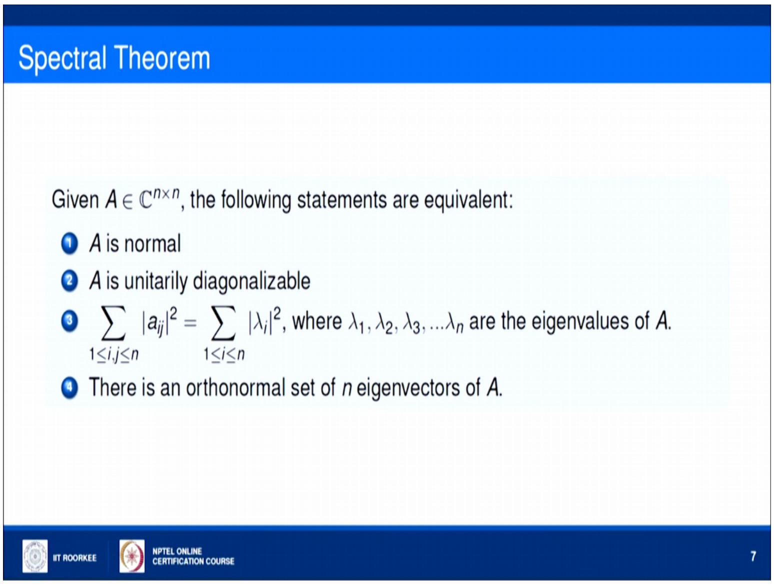

So, given a square matrix of order n having complex entries then the following statements are equivalent. The first one is A is a normal matrix, the second is A is unitarily similar to a diagonal matrix; or in other word we can say, A is unitarily diagonalizable. The third point is the summation of entries of A and square of those equals to since A is means square of the mode of A, that is since a may be the complex and a may contain the complex entries equals to the square of the amplitude of the eigenvalues of A.

That is summation 1 to i to n and mode of lambda i square where lambda 1 lambda 2 lambda 3 lambda n all are the eigenvalues of matrix A. The fourth statement is there is n orthonormal set of n eigenvectors of A. Means A consist a complete set of orthonormal eigenvectors; that is, the corresponding matrix be unitary. And that is some of them we have already proved that A is normal, then I have shown you that A is similar to a unitary similar to a diagonal matrix.

So, first second and second one already we have done, second to fourth is obvious four to second is obvious, and since first to second then we can go from second to fourth. And similarly we can go for third. So, this theorem I am doing without proof, although some of proof we have done in the previous slide.

Now, let us take some numerical examples. So, the first example is a which is given by this 5 plus i minus 2 i 2 4 plus 2 i, a normal matrix or not? So, let me do it.

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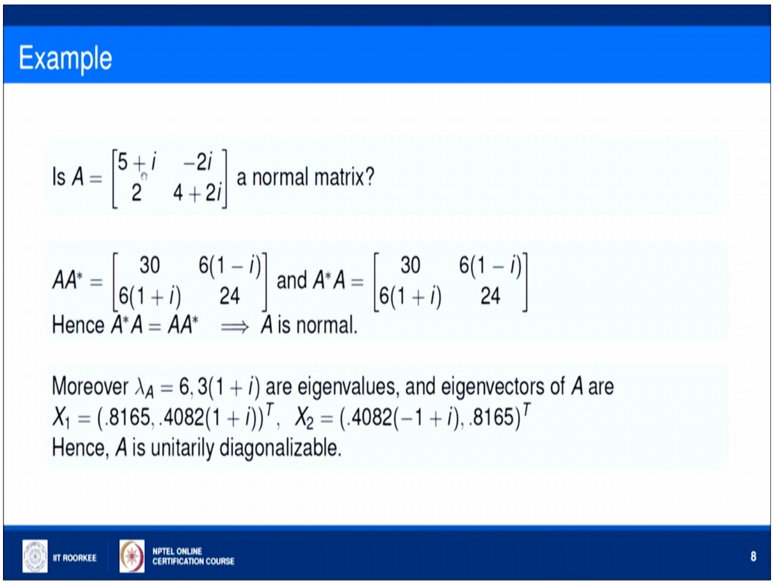

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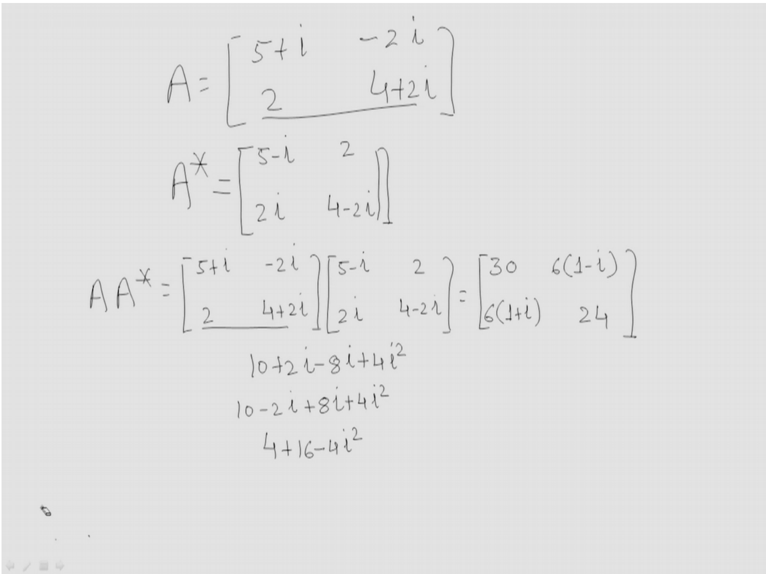

So, A is given as 5 plus i minus 2 i 2, and then 4 plus 2 i. So, we have to check whether this matrix is normal or not. So, first of all we will calculate conjugate transpose of A, that is A star. So, it will become 5 minus i, that is the conjugate of 5 plus i then 2 i minus 2 i will become 2 i, then 2 will remain as such and 4 minus 2 i, ok. Let me calculate A into A star so, A into A star will become 5 plus i minus 2 i 2 and 4 plus 2 i, and then A star will be 5 minus i, it will be 2 year 2 i, 4 minus 2 i. So, this comes out to be 26 plus 4 30.

Similarly, this row will be multiplied with this particular column. So, 10 plus 2 i and then I am having minus 8 i plus 4 i square. So, 10 minus 4 will be 6 1 minus i then this row will be multiplied with this column. So, 10 minus 2 i plus 8 i plus 4 i square so, 10 minus 4 will become 6 so, 6 1 plus i. And the last entry with this row will be multiplied with this column. So, it will become 2 into 2 4 plus 16 minus 4 i square so, this is the A into A star. Similarly, we will calculate A star into A.

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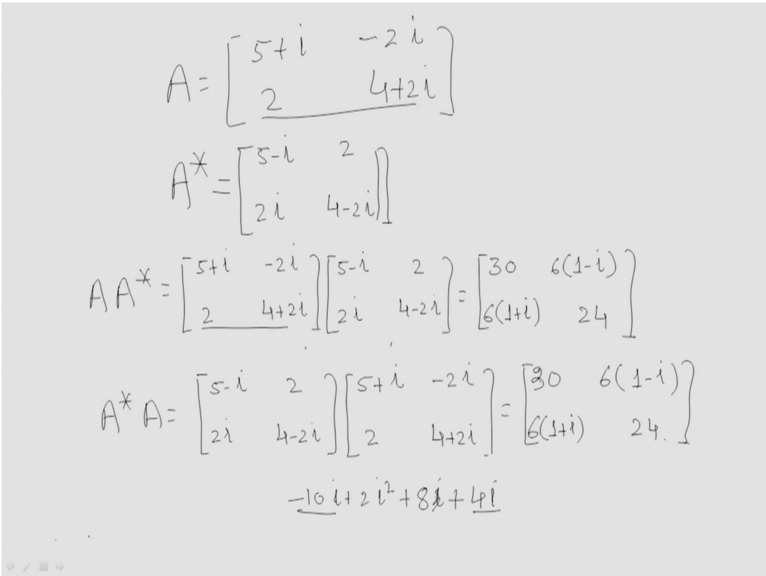

So, A star into a will become 5 minus i 2 2 i 4 minus 2 i multiplied with 5 plus i minus 2 i 2 4 plus 2 i. So, if I see check this so, it will become 25 into 5 26 plus 4. So, 30 this element will become minus 10 i, plus 2 i square plus 8 i 8 plus 4 i. So, if I calculate it minus 10 plus 4. So, it will become minus 6 i and 8 minus 2 so, it comes out to be 6 1 minus i. Similarly, this element will come 6 1 plus i and this comes out to be 24. So, what we are observing here that A into A star equals to A star into A.

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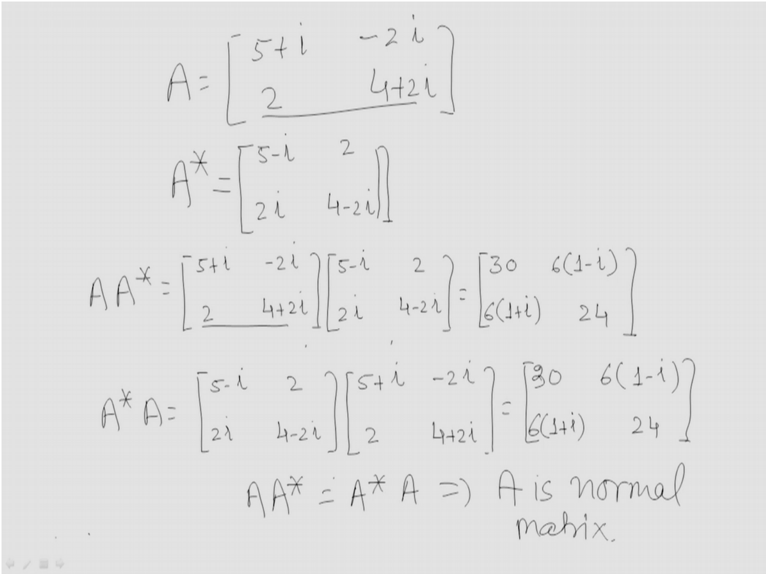

This means A is normal. This is one of the way of checking the whether the given matrix is normal or not. If someone ask that give examples of two distinct classes of normal matrices those are real, but not symmetric.

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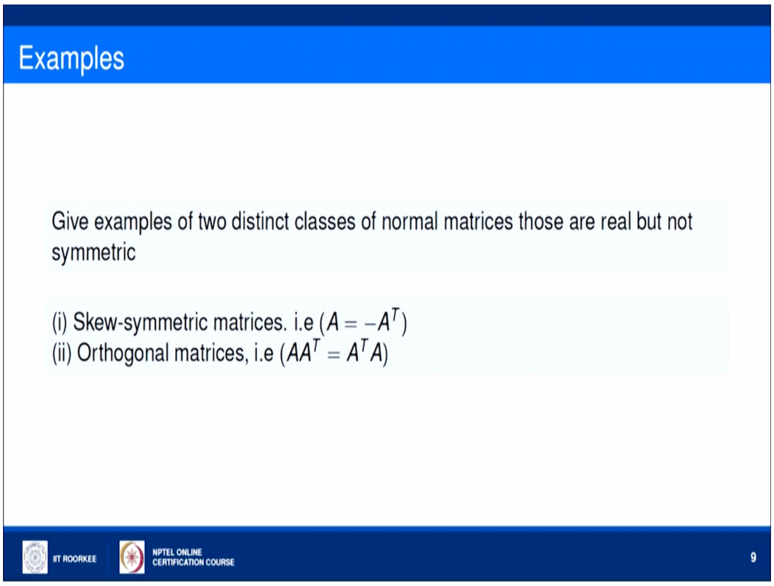

So, I have told you in the beginning itself in the second slide, that such class of matrices may be skew symmetric matrices. That is A equals to minus A transpose and orthogonal matrices; that is, A into A transpose equals to A transpose into A.

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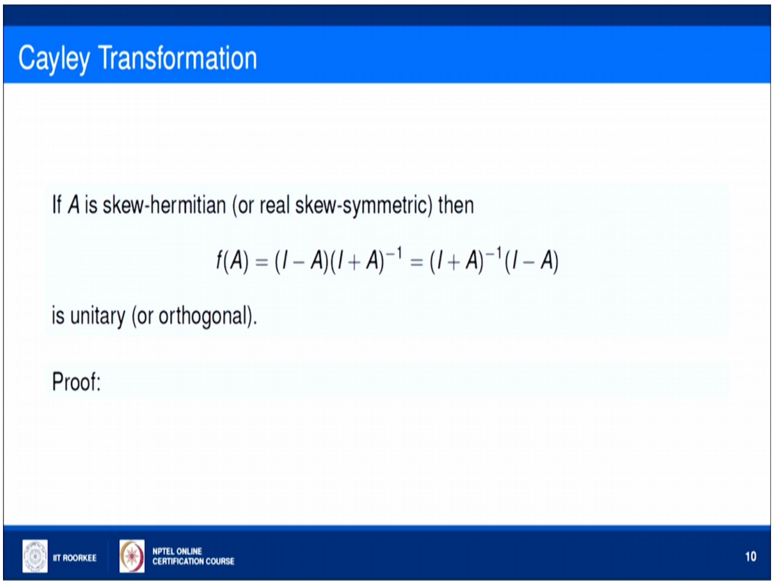

Another important result related to normal matrices are Cayley transformation, and it is saying that if A is skew Hermitian or real skew symmetric, then the function of matrix said defined as I minus A into I plus A inverse equals to I plus A inverse into I minus A is unitary. Or if it is really skew symmetric then it is orthogonal.

Let me try to prove it.

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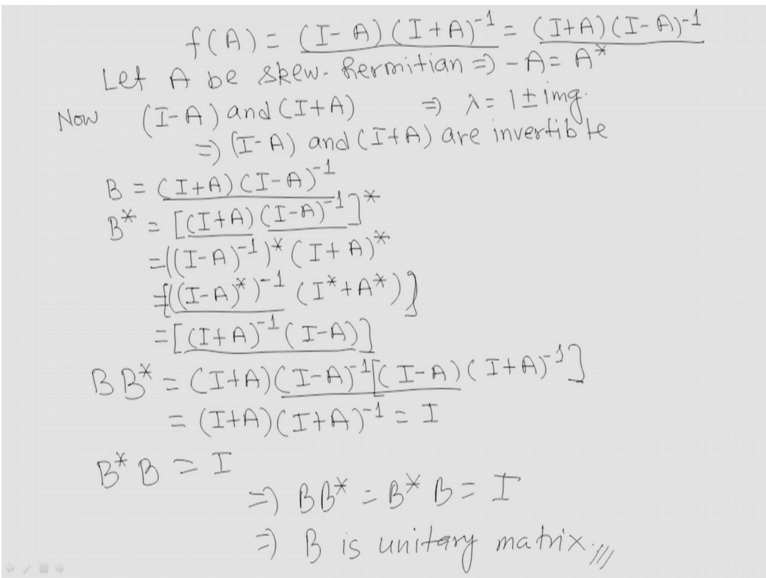

So, what I need to do I am having a function of matrix step that is of A. And it is define as I minus A I plus A inverse and this is equals to I plus A I minus A inverse. And what I need to prove that this particular function that is f, which is a matrix of the same size as A is unitary or orthogonal according to the given matrix A if it is skew Hermitian or skew symmetric. So, let us prove it for in, let A be skew Hermitian. Then what we are having? A equals to or minus A equals to A star.

Now, if I see this matrix I minus A, since A is a Hermitian matrix, and I plus A then the eigenvalues of I minus A or I plus A will be of the form 1 plus minus some imaginary number. Because the eigenvalues of skew Hermitian matrix are purely imaginary or 0 so, eigenvalues will be either 1 or 1 plus minus some imaginary. This gives that I minus A and I plus A are invertible. Means inverse is well defined, so, the above function f is well defined.

Now let us assume B equals to I plus A into I minus A inverse. Then if I calculate B star, B star will become i plus A i minus A inverse, and A star of this matrix so, these are 2 matrix and B. So, A B star will become B star A star so, it will become I minus A inverse star into I plus A star. This I can write in this way; I minus A star into inverse.

So, I am interchanging inverse and star here I star plus A star. Now this will be I minus A star so, I star minus A star and A star equals to I minus A. So, this will become I plus A inverse this whole thing, and this will become I minus A. So, what I am having? That the 2 functions I minus into I plus A inverse and I plus A I minus A inverse both are the complex conjugate or conjugate transpose of each other.

Now, if I take the matrix B into B star it comes out to be I plus A I minus A inverse this is my B into I minus a I plus A inverse, this is B star. So, I plus A, now if I multiply this are I minus A inverse into i minus A. So, becomes identity I plus A inverse, this is equals to I. Similarly, I can show that B star into B is also equals to i, because it is not very difficult because B star is given like this, and B is given like this.

So, I can always find out this thing. So, B B star equals to B star into B equals to I it means B is unitary matrix, which we need to prove. So, this is the proof of this Cayley transformation. Now, next result is so that a triangular matrix is normal if and only if it is diagonal.

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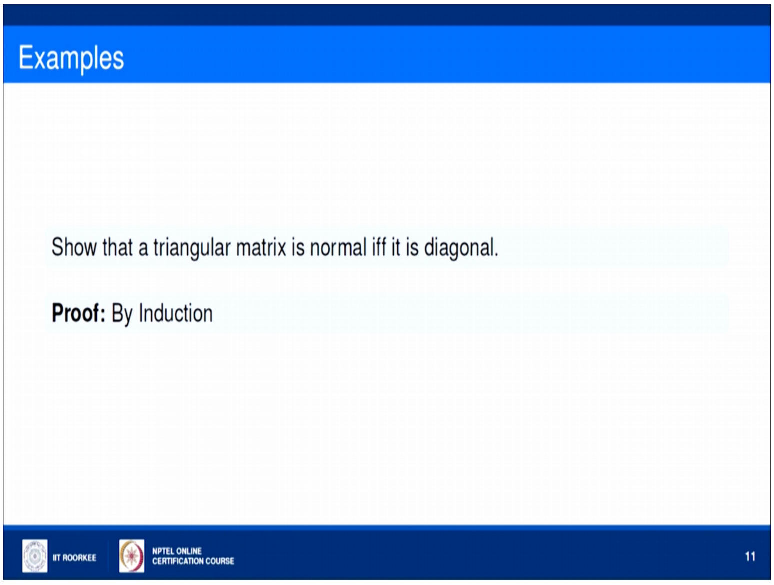

And that we have already proven in the very first definition, where I have shown you by taking the skew decomposition lemma and a matrix M, and I have shown that M M star equals to M star M only when M is diagonal. And the same proof can be used here, the same concept.

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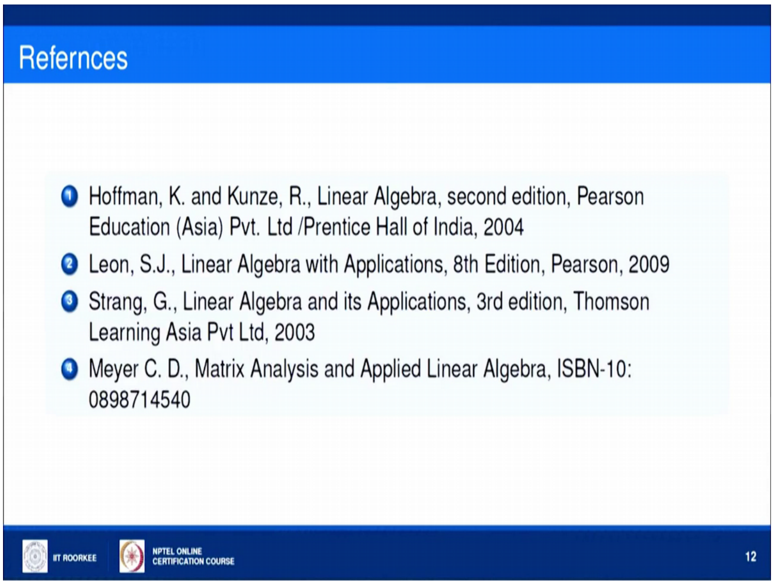

These are few references for this lecture. So, in this lecture we have learned about normal matrices, in the next lecture we will learn another very important class of matrices, those are called positive definite matrices. So, with this I close this lecture.

Thank you very much.