Matrix Analysis with Applications
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Lecture – 20
Gram Schmidt Process

Hello, friends. Welcome to lecture series on Matrix Analysis with Applications. So, today lecture is Gram Schmidt process, what Gram Schmidt process is and how it is used to find an orthogonal set of vectors.

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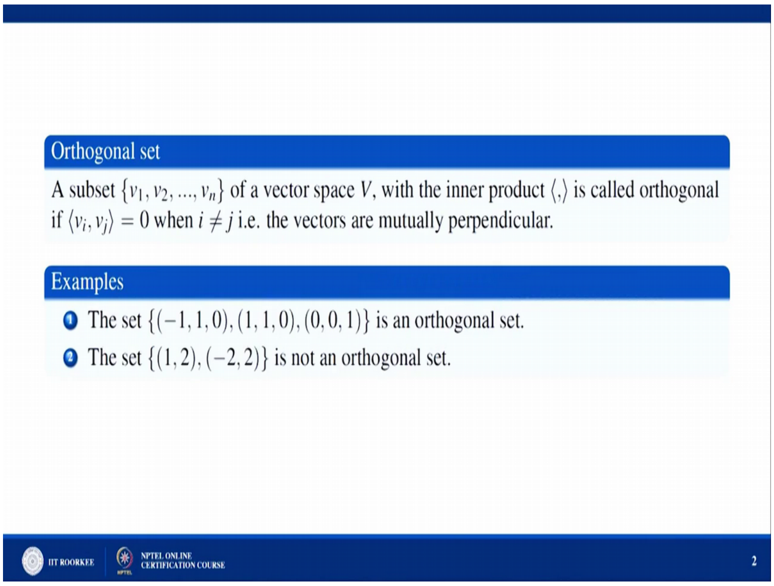

So, first let us understand let us recall our definition orthogonal set. We know that if a subset v 1, v 2 up to v n of a vector space V with inner product defined like this is called an orthogonal, if inner product of v i, v j is equal to 0 when i is not equal to j, that is for any two distinct vectors in this set any two distinct vector in this set are orthogonal I mean are perpendicular.

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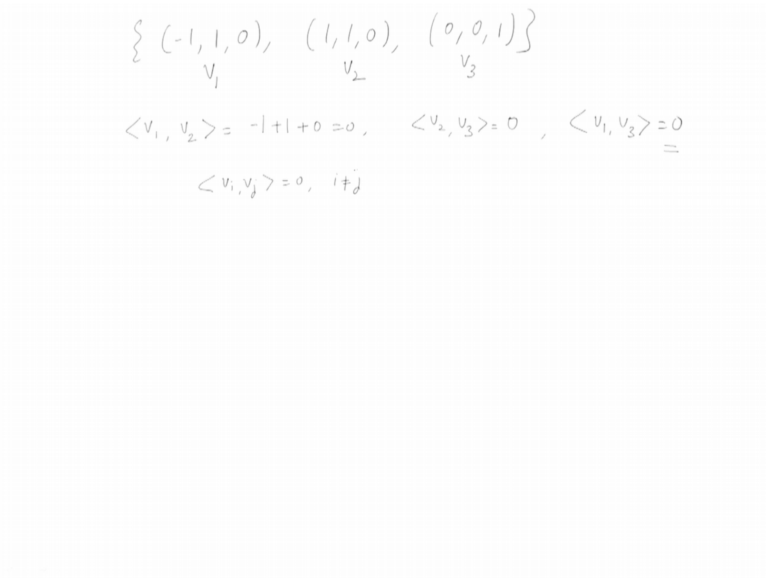

For example, you see this set if you see this set if you take the inner product of these two say the set is simply you see the set is minus 1, 1, 0 then 1, 1, 0 then 0, 0, 1. Suppose, this is V 1 this is V 2 this is V 3. You take the inner product of v 1 and v 2 the usual inner product. The usual inner product between two vectors in real dimensional vector space is simply the standard dot product. So, it would be a dot product of these two vectors.

So, minus 1 into 1 is minus 1, 1 into 1 is 1 and 0 into 0 is 0 which is 0. Similarly, you take inner product of v 2 and v 3, it is simply you take 1 into 0 is 0, 1 into 0 is 0 and 0 into 1 is 0, so, it is 0. Similarly, inner product inner product of v 1 and v 3 is also 0. You can easily verify this is minus 1 into 0 is 0, 1 into 0 is 0 and 0 into 1 is 0. So, it is also 0. So, we can say that v i and v j for i is equal to 0 for i not equal to j. So, means this vector is this set of vector is I mean orthogonal.

Now, if you take the inner product between these two vectors it is minus 2, plus 4 which is 2, which is not equal to 0; that means, this set is not an orthogonal set.

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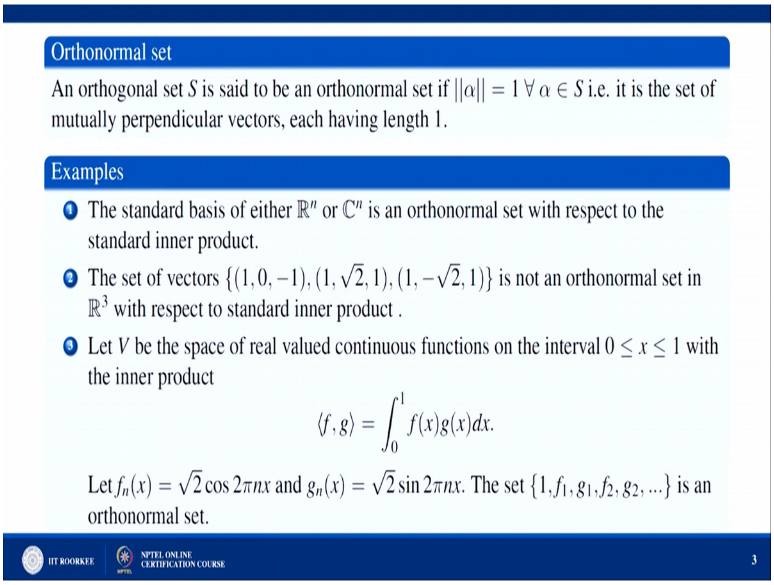

Now, beside an orthogonal set if it also satisfy that norm of each vector is 1; norm means, how we define a norm of vector?

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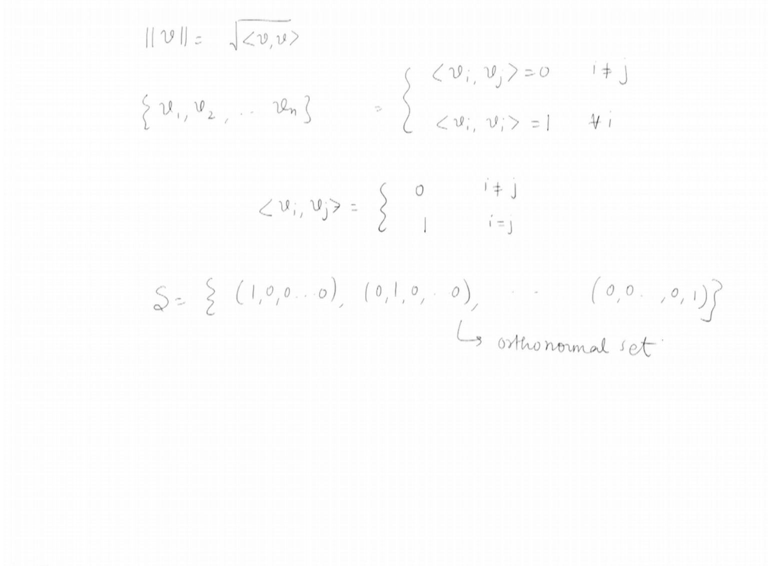

Norm is simply if we want to define norm then norm of a vector v is nothing, but under root of inner product of v with itself this is simply a length of a vector or a norm of a vector. So, if we have a set of vector say v 1, v 2 up to say v n so, this set is called an ortho normal set, if inner product of v i, v j is 0 for i not equal to j and inner product of v i for v i is equal to 1, for all i. Or, we can say that inner product of v i, v j is equal to 0 for i not equal to j n is equal to 1 for i equal to j and if it is hold for all i and j then the set of vectors are called orthonormal set.

So, suppose we have various examples to verify that the set of vectors are an orthonormal sets. Suppose, you have a standard basis of R n, if you have a standard basis of R n what are standard basis of R n? We know standard basis of R n is simply 1 0 and so on up to 0 it is 0, 1, 0 and so on up to 0 and similarly, it is 0, 0 and so on 0 comma 1. If you take the inner product of any two distinct vectors in this set so, it is 0. You can simply verify 1 into 0 is 0, 0 into 1 is 0. So, you can simply verify that if you take inner product of any two I mean distinct vectors in this set it is 0 and the norm of this any vector in this set any vector is 1. So, we can say that this set of vectors are is orthonormal set.

Similarly, if you take the second example, now second example is not an orthonormal you see. If you take the inner product of these two it is 1 into 1 is 1, 0 into root 2 is 0, minus 1 into 1 is minus 1, which is 0. The inner product of these two is 0. If you verify the inner product of these two it is 1, minus 2, plus 1 again 0. If we verify the inner product of first vector and the third vector it is 1, plus 0 and minus 1, again 0; that means, this set of vectors is orthogonal.

Now, the norm of this now the norm of this vector is under root 2 which is not 1; that means, this set of vectors is orthogonal, but not orthonormal.

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$\{v_1, v_2, \ldots, v_n\} \longrightarrow$ orthogonal set

$\left\{\dfrac{v_1}{\|v_1\|}, \dfrac{v_2}{\|v_2\|}, \ldots, \dfrac{v_n}{\|v_n\|}\right\} \longrightarrow$ orthonormal set

u_1

$$\|u_1\| = \left\|\frac{v_1}{\|v_1\|}\right\| = \frac{1}{\|v_1\|}\|v_1\| = 1$$

So, how can we construct an orthonormal set from an orthogonal set you simply divide each vector by it is norm. You see if you have a set of vectors like this say v 1, v 2 and so on up to v n, and it is you know that this set is an orthogonal set. Orthogonal means that inner product of any two distinct vector is 0, and now you want to construct an orthonormal set from this set.

So, how you can do that? You can simply divide each vector by it is norm. You see v 1 upon norm of v 1, v 2 upon norm of v 2 and v n upon norm of v n, it will be some different set from this set, but it will be an orthonormal set. So, always we can construct an orthonormal set from an orthogonal set by dividing each vector with it is norm. You can easily verify now you see that norm of each vector is 1 now, what is the norm of the first vector if it is U 1, what is the norm of U 1? Norm of U 1 is norm of v 1 upon norm of v 1, it is a scalar quantity. So, we can take it out it is 1 upon norm of v 1, norm of v 1. So, it is 1.

Similarly, we can verify for this vectors also and if you take any two distinct vector in this set because norm is only a scalar quantity it will take out it will comes out from the inner product and this set is an orthogonal set. So, this will also be an orthogonal.

Now, similarly if you take the third example where V is a space of real valued continuous functions on the interval x varying from 0 to 1 with the inner product define like this, where f n is defined like this is a under root 2 cos pi 2 pi n x and g n is this then this set is an orthonormal set, it is very easy to show.

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$$f_i(x) = \sqrt{2}\cos 2\pi i x, \quad g_j(x) = \sqrt{2}\sin 2\pi j x$$

$$\langle f_i, g_j \rangle = \int_0^1 f_i(x)\, g_j(x)\, dx, \quad i \neq j$$

$$= 2\int_0^1 \cos 2\pi i x \,\sin 2\pi j x \, dx$$

$$= \cancel{2} \int_0^1 \Big(\sin (2\pi)(j+i)x + \sin\big((2\pi)(j-i)x\big) \Big) dx$$

$$= -\left(\frac{\cos 2\pi (i+j)x}{2\pi(i+j)} \right)\Bigg|_0^1 - \left(\frac{\cos 2\pi (j-i)x}{2\pi(j-i)} \right)\Bigg|_0^1$$

$$= 0$$

$$\langle f_i, f_i \rangle = 1, \quad \langle g_j, g_j \rangle = 1 \qquad \langle 1, f_i \rangle = 0 \;\; \forall i \qquad \langle 1, g_j \rangle = 0 \;\; \forall j$$

You can take you can take any two you see you take f i x; f i x is simply under root 2 cos 2 pi i x you take g j x g j x is simply under root 2 sine 2 pi j x and if you take the inner product of f i with g j which is given by a 0 to 1 f i x into g j x I am taking i naught equal to j here.

So, it will be equal to 0 to 1, if you take these two it is 2 times cos 2 pi i x into sine 2 pi j x d x and when you take 2 times when you take simply you see it is 2 of sine a x into cos b x which is 0 to 1 sine sine a plus b that is 2 pi will come out j plus I times x minus sine a minus b 2 sine a cos b. So, it is plus and it is 2 pi and it is j minus i times x whole d x and when you take the when you take the integration it is simply minus cos of 2 pi I plus j times x upon 2 pi i plus j from 0 to 1, again it is minus cos 2 pi j minus i times x upon 2 pi j minus x j minus i 0 to 1.

Now, when you take x equal to 1 it is 2 multiple of 2 pi and cos multiple of 2 pi is 1 and again from the lower limit also it is 1. So, 1 minus 1 is 0. Here also because by the same result it is 0. So, 0 minus 0 is 0 and similarly, we can verify that inner product of f i with itself is one and similarly inner product of g j which itself is 1, this we can easily verify using the same concept.

So, we can say that this set this set is an orthonormal set because if you take any two different vectors any two different functions. In fact, if you take one function as 1 and you multiply with any f i, I mean I want to say in the same if you take inner product of 1 with f i this is also 0 for all i and inner product of one with g, j is also 0 for all j this also we can verify using the same definition of inner product. So, hence we can say that this set is an orthonormal set.

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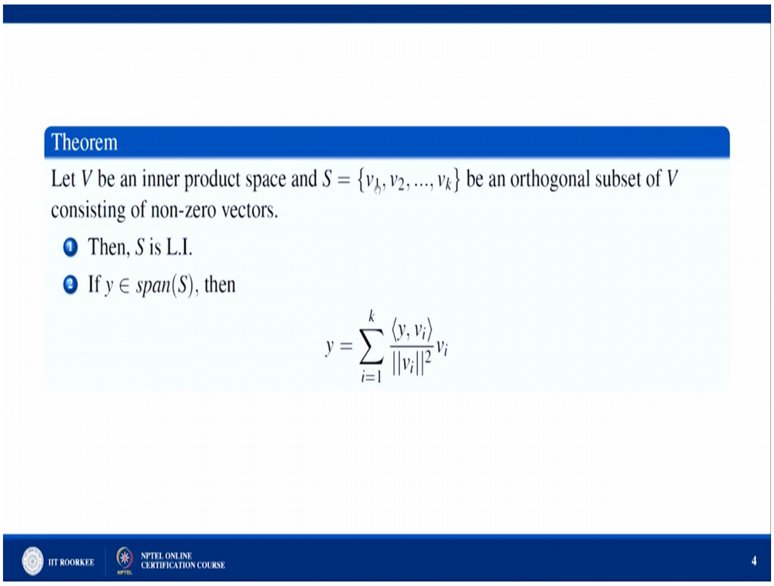

Now, the next result is let V be an inner product space, and the set which is given as v 1, v 2 up to v k be an orthogonal subset of V consisting of non-zero vectors it is it is an orthonormal orthogonal subset and it consist of non-zero vectors. So, the first result is this set is always LI; that means, set of non-zero vectors which is an orthogonal is always linearly independent. How we can show this?

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$$\{v_1, v_2, \ldots, v_k\} \rightarrow \text{orthogonal} \rightarrow \langle v_i, v_j \rangle = 0 \ \forall i \neq j$$

$$v_i \neq 0 \ \forall i$$

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$$

$$\langle \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k, v_i \rangle = \langle 0, v_i \rangle, \quad \text{for any } 1 \leq i \leq k$$

$$\alpha_1 \underbrace{\langle v_1, v_i \rangle}_{0} + \alpha_2 \underbrace{\langle v_2, v_i \rangle}_{0} + \cdots + \alpha_i \langle v_i, v_i \rangle + \cdots + \alpha_k \underbrace{\langle v_k, v_i \rangle}_{0} = 0$$

$$\alpha_i \|v_i\|^2 = 0 \Rightarrow \alpha_i = 0$$

$$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$

$$\Rightarrow LI$$

Very easy to show; you see we have the set v 1, v 2 up to v k. It is given to us that the set is orthogonal orthogonal means inner product of v i with v j is 0 for all i not equal to j. Now, in order to show that this set is and we also know that v i is not equal to 0 for all i because it consist of non-zero vectors.

Now, in order to show that this set is linearly independent, take a linear combination of these vectors, put it equal to 0 and try to show that each scalars are 0 ok. So, take a linear combination of these vectors, and put it equal to 0. Now, we have to show that each alpha is 0 in order to show that this set is linearly independent.

Now, say this vector is v the linear combination of these vectors say the linear combination of these vector is v which is of course, equal to 0. Now, you take the inner you take the inner product of this v with say any v any v i or what you can do you take the inner product of this vector v with any v i inner product of alpha 1 v 1 plus alpha 2 v 2 and so on up to alpha k v k with any say v i here also 0 with v i for any i any i from 1 to k. This i may be 1, this i may be 2 or this i may be k.

Now, this is of course, 0 inner product of 0 with any vector is always 0. Now, when you take the inner product definition of inner product of on this v i; so, this will be alpha 1 times inner product of v 1 with v i plus alpha 2 times inner product of v 2 with v i and so on. In between you will get some alpha i also we are we are having v i, v i and plus and so on alpha k will come out inner product of v k with v i and which is which is equal to 0 which is equal to 0 because the right hand side is 0.

Now, since this is an orthogonal set that means, for any two distinct vectors in this set the inner product is 0. So, that means, this is 0 ; that means, this is 0 ; that means, this is 0 if i is not equal to k it will be, it will be it will have some value only when this i is equal to i. So, that means, it is alpha i times norm of v i square equal to 0 because all others are 0 and since v i is not 0, for all i this means this is not equal to 0, and; that means, alpha i equal to 0.

Now, you vary alpha you vary i, this i may be 1; that means, alpha 1 is equal to 0, this i may be 2; that means, alpha 2 equal to 0 and if you vary this i over k, so; that means, that means, alpha 1 equal to alpha 2 equal to and so on, alpha k equal to 0 and this means set are set is linearly independent. Or you can understand it is like this you take you take the linear combination of this vector and suppose it is V you first take the inner product of this vector with v 1 this will give alpha one equal to 0.

Now, you take the inner product of this vector v with v 2 this will give alpha 2 equal to 0 and similarly, you will take inner product of this v with alpha k this with alpha k equal to 0 and hence we will obtain all alpha is are equal to 0 that mean the set is linearly independent.

The second the second point is the second part of the theorem is if y belongs to span of S span means some linear combination of these vectors of S then y can be expressed as this sum also it is also easy to show you see.

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$$y \in \operatorname{span}\{v_1, v_2, \ldots, v_k\}$$

$$\Rightarrow y = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k$$

$$\langle y, v_p \rangle \qquad 1 \leq p \leq k$$

$$= \langle \beta_1 v_1 + \cdots + \beta_k v_k , v_p \rangle$$

$$= \beta_1 \underbrace{\cancel{\langle v_1, v_p \rangle}}_{0} + \beta_2 \underbrace{\cancel{\langle v_2, v_p \rangle}}_{0} + \cdots + \beta_p \langle \underline{v_p, v_p} \rangle + \cdots + \beta_k \underbrace{\cancel{\langle v_k, v_p \rangle}}_{0}$$

$$= \beta_p \|v_p\|^2$$

$$\Rightarrow \beta_p = \frac{\langle y, v_p \rangle}{\|v_p\|^2}$$

$$y = \sum_{p=1}^{k} \beta_p v_p = \sum_{p=1}^{k} \frac{\langle y, v_p \rangle}{\|v_p\|^2} v_p .$$

Now, this y it belongs to span of this span of S. S is what? S is v 1, v 2 up to v k. If this y belongs to span of this means there will exist some scalars such that y can be expressed as linear combination of those scalars with these vectors ok; that means, this implies that y will be equals to some beta 1 v 1 plus beta 2 v 2 and so on up to beta k v k because this y is in the span of these vectors. So, now, we have to find out beta beta betas beta 1, beta 2 up to beta like this we have to find out.

Now, if you take inner product of this y with suppose v p ok, where p is where p is any value between k and 1 again p may be 1, may be 2, may be k any p then this is equal to inner product of beta 1 v 1 and so on up to beta k v k with v p. Now, applying the definition of inner product this is beta 1 inner product of v 1 with v p plus beta 2 inner product of v 2 with v p plus and so on, beta p inner product of v p with v p plus and so on, in beta k inner product of v k with v p and since this set is an ortho orthogonal set so, all these terms are 0 only this term left. So, this means this is equals to beta p into norm of v p square. So, that means, beta p is nothing, but inner product of why with v p upon norm of v p square.

So, what is beta 1? You replace p by 1. What is beta 2? You replace p by 2 and similarly other betas. So, what we can say about y? Now, this y can be written as summation p from 1 to k beta p v p. So, this can be written as p from 1 to k beta p is inner product of y with v p up on norm of v p square times v p. So, hence we obtain this. So, the same result instead of I am having p no problem with that.

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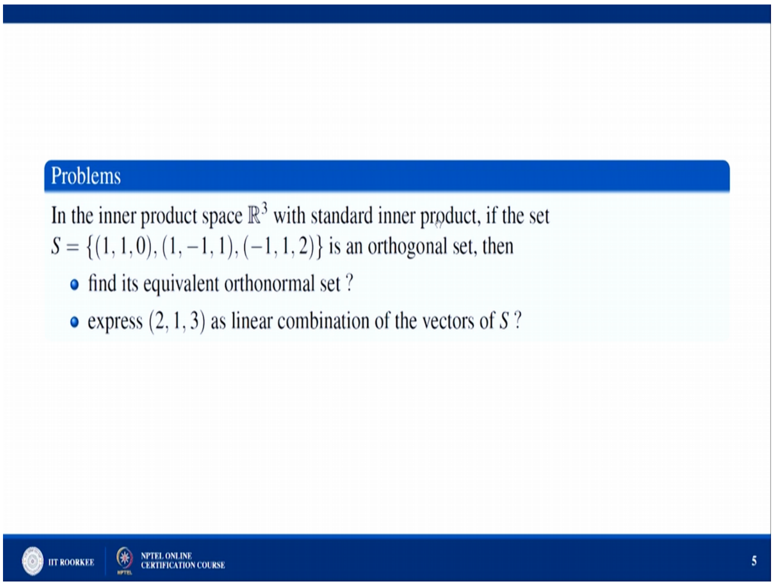

Now, in this problem you see in the inner product space R 3 with the standard inner product, this set is an orthogonal set. You can easily verify, you see you take the inner product of any two distinct vectors of this set you will find that it is 0. So, this set is an orthogonal set.

Now, how can we how we can find it is equivalent orthonormal set? This I have already discussed that in order to form an orthonormal set from an orthogonal set, simply divide each vector by its norm. So, what is the norm of this vector? It is under root 2, norm of this vector under root 3 norm of this vector is under root 1 plus 1 plus 4 that is under root 6.

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$$S' = \left\{ \frac{(1,1,0)}{\sqrt{2}}, \frac{(1,-1,1)}{\sqrt{3}}, \frac{(-1,1,2)}{\sqrt{6}} \right\}$$

$$(2,1,3) = \sum_{i=1}^{3} \frac{\langle (2,1,3), v_i \rangle}{\|v_i\|^2} v_i$$

So, what is an equivalent orthonormal set of this the equivalent orthonormal set of this is suppose S dash which is the first vector is minus 1, 1, minus 1, 1, 0 upon under root the second vector is minus 1, minus 1, 1 1, minus 1, 1 upon under root 3; the third vector is minus 1, 1, 2 upon under root 6. So, this will be an equivalent first element is 1, 1, 0 the first element is 1, 1, 0. So, this is an equivalent orthonormal set.

Now, again the next problem we have to express 2, 1, 3 as the linear combination of vectors of S. So, how can we can how we can express 2, 1, 3 as a linear combination of elements of S. So, we can use the previous result you see. In the previous result this set is an orthogonal set and if y belongs to span of S then this y is some expression like this.

So, this so, here you can see that this because this is an orthogonal set and its and all vectors are linearly independent. So, this will be a basis of basis of R 3, any three linearly independent set of R 3 will be a basis of R 3. So, this is a basis of R 3 means this 2, 1, 3 will belongs to span of this span of these three vectors v 1, v 2 and v 3 and since 2, 1, 3 belongs to a span of these three vectors.

So, we can use the previous result sorry we can use a previous result here and this y we can write as like this y we can write it like this it is k from i from 1 to it is we can write here i from 1, 2, 3 because these are three vectors inner product of 2, 1, 3 which is y with v i's upon norm of v i's square into v i's.

So, now, you vary now, you vary i here you can see in this example v 1 is this, v 2 is this, v 3 is this you can simply substitute in that expression to find out that to find out the expression in which 2, 1, 3 can be expressed as linear combination of elements of or vectors of S.

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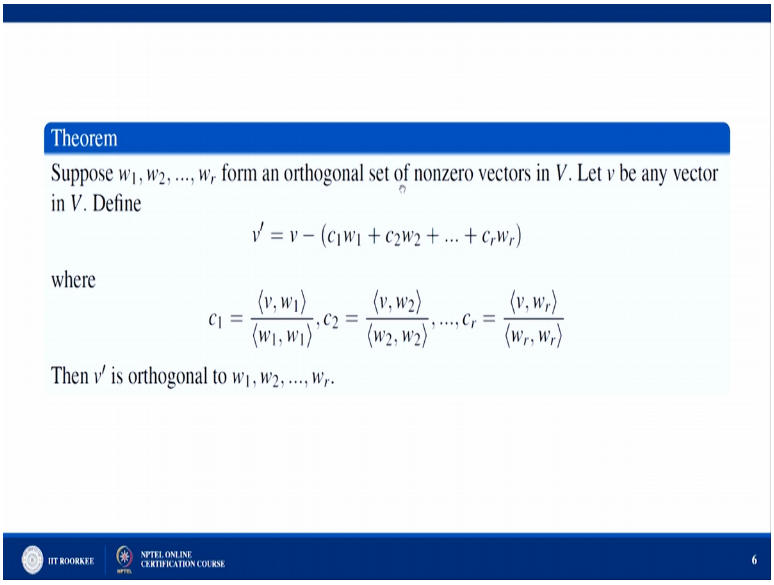

Now, the next result is supposed w 1, w 2 up to w r form an orthogonal set of nonzero vectors in V, it is an orthogonal set again and if we write v which is an any vector in V and if we define v dash as this where c i is the defined like this expression, orthogo orthano inner product of v with w 1 upon norm of w 1 square and similarly other c i’s then this v dash is always orthogonal to w again it is easy to show you see you see what is what is v dash here you see.

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$$v' = v - \sum_{i=1}^{r} c_i w_i \quad , \quad c_i = \frac{\langle v, w_i \rangle}{\|w_i\|^2}$$

$$\langle v', w_i \rangle = 0 \quad \text{for all } i$$

$$\begin{aligned}
\langle v', w_i \rangle &= \left\langle v - \sum_{i=1}^{r} c_i w_i \,,\, w_i \right\rangle \\
&= \langle v, w_i \rangle - \langle c_1 w_1 + c_2 w_2 + \cdots + c_r w_r, w_i \rangle \\
&= \langle v, w_i \rangle - c_i \langle w_i, w_i \rangle \\
&= \langle v, w_i \rangle - \frac{\langle v, w_i \rangle}{\cancel{\|w_i\|^2}} \cancel{\langle w_i, w_i \rangle} \\
&= 0
\end{aligned}$$

What is v dash here? v dash is a v minus it is sum of i from 1 to n it is c i w i r, ok, it is up to r. So, you can take it up to r and sorry it is v and what are c i'sl c i's are nothing, but this c i's are given to us that the c i is inner product of v with w i upon norm of w i square and we have to show that this v dash this v dash is orthogonal to w 1, w 2, up to w r any w. So, that means, we have to show that this inner product of v dash with any w i is equal to 0 for all i, this we have to prove.

So, take the inner product of v dash with w i's. This is inner product of what is v dash? v minus some from i from 1 to r c i w i with w i. This is inner product of v with w i minus. Now, when you open this sum it will be what? c 1 you see minus will come out its inner product of c 1 w 1 plus c 2 w 2 and so on up to c r w r with w i this will remain as it is.

Now, since this w 1, w 2 up to w r is an orthogonal set ; that means, w 1 with w i is 0 if i is not equal to 1, ; that means, this will this will be having some nonzero value when w i with inner product with w i only ; that means, it will value only one coefficient itself c i, w i with w i. You see when you apply a definition of inner product in this expression w 1 inner product with w i. So, in between we are having some c i w i also. So, only that expression will be here all other for all other for other i's it will be 0.

Now, this is v comma w i's minus c i is given by this expression is given to us when you substitute it over here, and this cancels out and this is minus this is 0. So, we have shown that we inner product of v dash with w i is 0 for all i, because i is an arbitrary I mean you can put it 1 or 2 or any i. So, hence we can we have shown that v j v dash is orthogonal to w i's.

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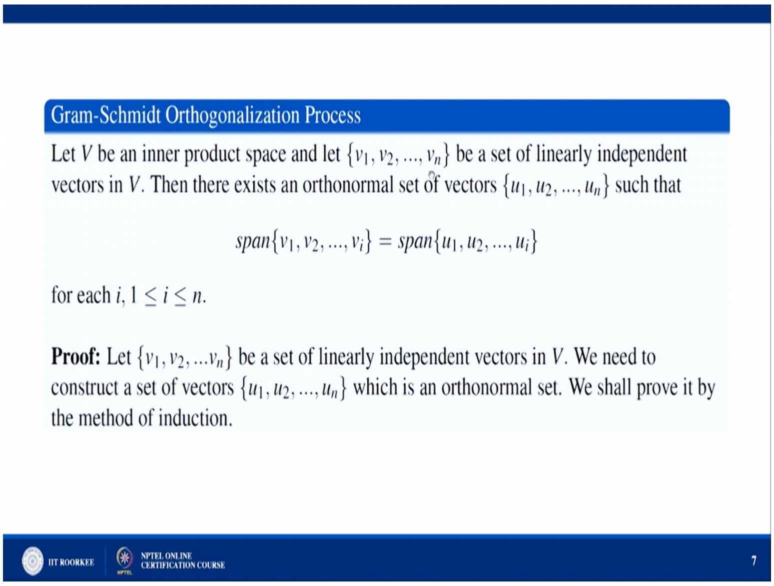

Now, we are having important process to orthogonalize the given set of vectors. Suppose a set of vectors v 1, v 2 up to v n is a linearly independent set, and you want to find out an orthonormal or orthogonal anyone anything orthogonal orthonormal set of vectors u 1, u 2, up to u n such that the span of v 1, v 2 up to v i is equal to span of u 1, u 2 up to u i for each i; i varying from 1 to n. So, this is basically this property or this result is called Gram Schmidt orthogonalization process.

Now, what is the proof of this? The proof is basically we can obtain the proof using mathematical induction. So, we have to construct a set of vectors u 1, u 2 up to u n such that this set is an orthonormal set and the span of u 1, u 2, up to u i is equal to a span of v 1, v 2 up to v i for each i; i is varying from 1 to n this we have to show.

So, in the mathematical induction first we take i equal to 1, ok. First you take this i equal to 1 and then we take then we assume for i equal for this result for i hold for i minus 1 and try to show that it is also true for each i for i that means, we have done using mathematical induction.

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$\underline{i=1}$ $\quad u_1 = \frac{v_1}{\|v_1\|}$ $\quad \|u_1\| = 1$

$$\|u_1\| = \left\|\frac{v_1}{\|v_1\|}\right\| = \frac{1}{\|v_1\|}\|v_1\| = 1, \quad v_1 \neq 0$$

$$v_1 = \|v_1\| u_1$$

$$\text{span}\{v_1\} = \text{span}\{u_1\}$$

$i > 1,$ $\quad \underline{i-1}$

$$w_i = v_i - \langle v_i, u_1 \rangle u_1 - \langle v_i, u_2 \rangle u_2 \cdots - \langle v_i, u_{i-1} \rangle u_{i-1}$$

$$u_i = \frac{w_i}{\|w_i\|}, \quad w_i \neq 0$$

$$\{u_1, u_2, \ldots u_{i-1}, u_i\}$$

$$\langle u_i, u_p \rangle$$

Now, for i equal to 1 we can easily see you see if we take i equal to 1. So, you can always set u 1 as v 1 upon norm of v 1 you have to set v 1 u 1, u 2 up to u n such that this set is an orthonormal set and the span of v 1, v 2 up to v i is equal to span of u 1, u 2 up to u i for each i. So, first we are taking i equal to 1 this is a first step for i equal to 1 i I assume that u 1 is like this, of course, you can easily verify that norm of u 1 is 1 you can simply see the norm of this is what the norm of u 1 is what the norm of u 1 is the norm of v 1 upon norm of v 1.

And norm is a scalar quantity it can be taken out norm of v 1 and which is 1 and of course, this norm is well defined I mean this v 1 is u 1 is well defined because v 1 is not equal to 0 because it is given to us that v 1, v 2 up to v n is an linearly independent set and if any vector is 0 it will become LD set linearly independent set. So, that means ; that means, each v i is not equal to 0. So, this expression this expression is well defined.

And, now, since v 1 equal to norm of v 1 times u 1 so, we can easily say that a span of v 1 is equal to a span of u 1 because it is some scalar multiplication of u 1. So, is since is a scalar multiplication of u 1. So, whatever span this v 1 is creating the same span will span by the vector u 1. So, we have we have proved this result for i equal to 1.

No, we will assume that it hold for i equal to 1 say i minus 1 it holds for i, i equal to i minus 1. Now, we have to show that it also hold for i.

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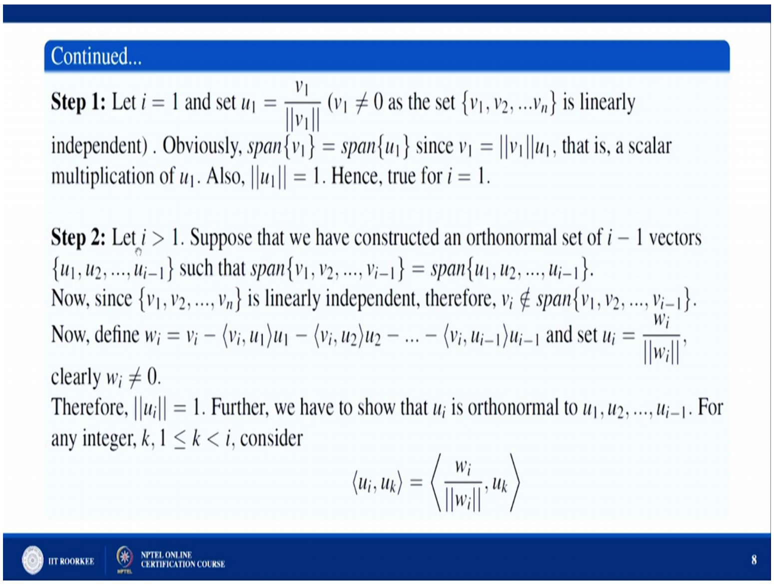

So, basically what i assumed in the step 2 that we have basically I have assumed that we have constructed an orthonormal set of i minus 1 vectors. u 1, u 2 up to u i minus 1 such that the span of these i minus 1 v i's is equal to a span of these u i's i minus 1 u i's.

Now, first of all since these vectors are linearly independent so, of course, any v i's cannot be expressed as linear combination of i minus 1 v i's. If it is not, then this v i will if this v i belongs to a span of this means this mean the set is linearly dependent, because this v i belongs to span of this; that means, there exist a vector v i in this set such that this that v i can be expressed as linear combination of elements of this set. So, of course, this v i will not belongs to span of this number one.

Now, we defined our self we defined w i like this you see we defined w i as we defined w i as v i minus inner product of v i with w 1 times w 1 minus inner product of v i with u 2 with u 2 and so on inner product of v i with u i minus 1 times u i minus 1 because we are considering i minus 1 set of u i's. So, we have we our self have constructed a w i satisfying this expression this we have constructed our self.

Now, we have constructed we have set u i as a inner as a w i upon norm of w i. Now, again the norm of u i is equal to 1 for each i that is easy to verify and again this expression is well defined because w i is not equal to 0, because if w i equal to 0 ; that means, v i is equal to linear combination of u i's and linear combination of u i's u i's with i minus 1 vector is same as linear combination of v i with i minus 1 vectors that may the set v 1, v 2 up to v n will become LD. So, which is not possible hence w i will not equal to 0.

Now, now, we can now, we have to show that this set is an orthogonal set orthonormal set basically we have to show that this u i is orthogonal to any u p, where p is varying from 1 to i minus 1. So, the thing is very easy to show u p or u k you can take anything, where k is varying from 1 to i minus 1. Now, what is u i here u i is u i is w i upon norm of w i, as we have already discussed. This expression of norm of w i can be taken out by a definition of inner product can be taken out. This w i is given by you can see here this w i this expression we can substitute it here with u k.

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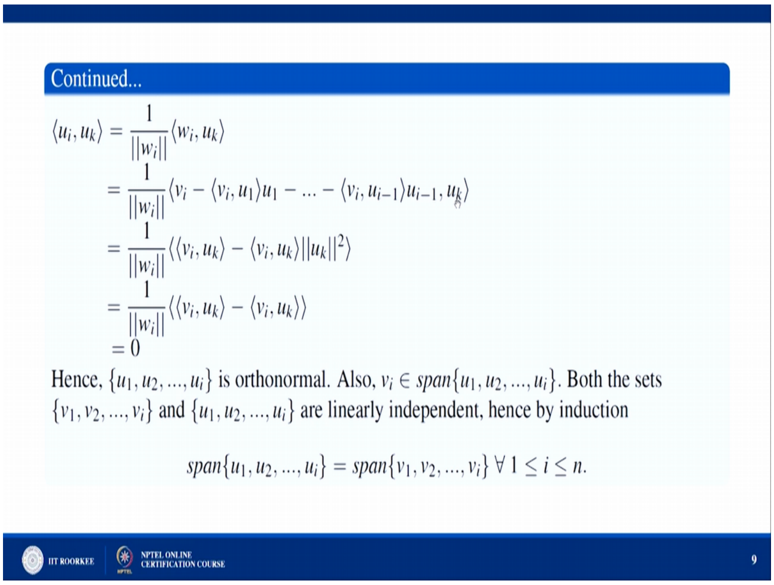

Now, u 1, u 2 up to u i minus 1 is an orthogonal set i mean orthonormal set this is by our assumption assumption of mathematical induction. So, that means, if you take the inner product of v i with u k it will be remain as it is and this u k for any k between 1 to i minus 1 this will this will be 0 for all u i’s other than i equal to k.

Though it will exist only for i equal to k and for i equal to k the inner product of u k with u k will be norm of u k square and it will v i into u k you see in some term you will be having some v i u k inner product of v i with u k with u k. Now, a inner product of u k with u k will be norm of u k square and this will come out since it is a orthonormal set. So, this norm is 1 and this will cancel out. So, it is 0.

So, we have shown that the set of now, i vectors u 1, u 2 up to u i's becomes an orthonormal set, we have set it like this only and also v i belongs to span of this also we have seen from above. So, both the vectors are linearly independent. Hence by induction we can say that span of this equal to span of this for all i's. So, hence we have proved Gram Schmidt process.

So, basically what in what happens in Gram Schmidt process that if you are having a any linearly independent set of vectors say u 1, u 2 up to u k then you can always find an orthonormal or orthogonal set of vectors v 1, v 2 up to v k such that span of u i's u i's u 1, u 2 up to u i is equals to a span of v 1, v 2 up to v i for each i this is a main concept of Gram Schmidt process.

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$$\{u_1, u_2, \ldots, u_k\} \to LI$$

$$\hookrightarrow \{v_1, v_2, \ldots, v_k\} \to \text{orthogonal}$$

$$v_1 = u_1$$

$$v_2 = u_2 - \frac{\langle u_2, u_1 \rangle}{\|u_1\|^2} u_1$$

$$v_3 = u_3 - \frac{\langle u_3, u_2 \rangle}{\|u_2\|^2} u_2 - \frac{\langle u_3, u_1 \rangle}{\|u_1\|^2} u_1$$

$$v_p = u_p - \sum_{i=1}^{p-1} \frac{\langle u_p, u_i \rangle}{\|u_i\|^2} u_i$$

$$\langle v_1, v_2 \rangle$$
$$= \langle u_1, u_2 - \frac{\langle u_2, u_1 \rangle}{\|u_1\|^2} u_1 \rangle$$
$$= \langle u_1, u_2 \rangle - \frac{\overline{\langle u_2, u_1 \rangle}}{\cancel{\|u_1\|^2}} \cancel{\langle u_1, u_1 \rangle}$$
$$= \langle u_1, \cancel{u_2} \rangle - \cancel{\langle u_1, u_2 \rangle}$$
$$= 0$$

Now, how we can apply Gram Schmidt process in some in various problems? You see say u 1, u 2 up to u k which is linearly independent given to you. Now, you want to construct an orthogonal set of vectors using Gram Schmidt process from this set of vectors. So, how we can do that? You so, suppose that set of vectors is v 1, v 2, up to v k of course, the same number of elements will be there.

So, we have to construct this orthogonal set. If you obtain orthogonal then orthonormal can be find out by dividing each vector by its norm, because all vectors will be definitely nonzero vectors because this vectors are linearly independent.

So, how we can find v 1; v 1 you take it equal to u 1, v 2 will be equals to u 2 minus norm of u 2 with v 1 upon norm of u 1 square with u 1. You can easily verify you can easily verify the norm of inner product of v 1 with v 2 will be 0 because inner product of v 1 with v 2 is what is u 1 with u 2 minus inner product of u 2, u 1 upon norm of u 1 square with u 1 which is this into this inner product of u 1, u 2 minus this quantity is I mean this quantity is a real quantity.

So, it can be taken out into inner product of u 1 with u 1. So, this cancels out ok, this bar will be here, yeah. Here will be a bar because this scalar is coming from the second term. So, this is inner product of u 1 with u 2 and this we again u 1 with u 2. So, this cancels out and this is 0. So, we can easily verify that we are hence we are constructing an orthogonal set of vectors.

Similarly, v 3 will be similarly u 3 minus inner product of u 3 with u 2 upon norm of u 2 square with u 2 minus inner product of u 3 with u 1 upon norm of u 1 square with u 1. So, if you want to construct say v p. So, v p will be u p minus i if varying from 1 to p minus 1 inner product of u p with u i upon norm of u i square with u i. So, in this way we can construct an orthonormal orthogonal set of vectors and if further we want to find orthonormal set of vector using this so, we can divide each vector by its norm.

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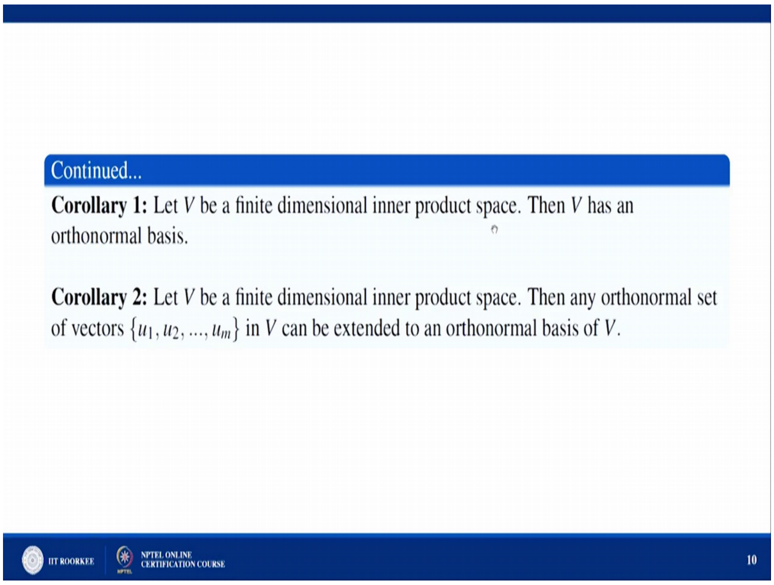

Now, these are some result from this if V be a finite dimensional inner product space then it has always having a orthonormal basis, very easy to show you see. If we are having a finite dimensional vector space so, it will be having a finite basis, say it is u 1, u 2, up to u n if it is having and if it is a end dimensional vector space and using Gram Schmidt process we can always find an equivalent orthonormal set of vectors. So, that means, there exists an orthonormal basis of a finite dimensional vector space inner product space.

The next is if V is a finite dimensional inner product space then any orthonormal set of vectors any in V can be extended to form an orthonormal basis of V.

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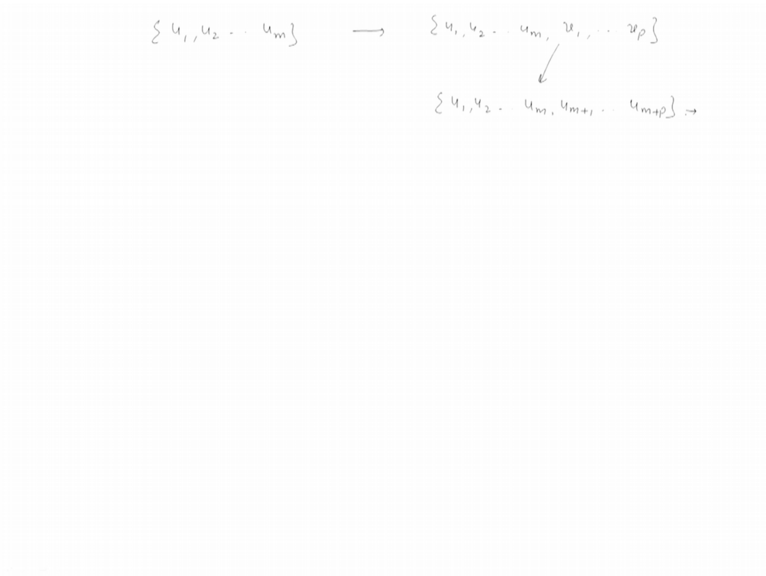

If it is an orthonormal, if it is an orthonormal set of vectors if it is a orthonormal set of vectors you see this is an orthonormal set of vectors. So, we can always extend this to find an basis say to find an basis of V if basis dimension of basis is m plus p, and using using Gram Schmidt orthogonalization process we can always find its equivalent orthonormal basis using the corollary one. So, that means, u 1, u 2 up to u m and it is u m plus 1 and so on up to u m plus p that will be the equivalent orthonormal ortho orthonormal basis of that that finite dimensional vector space. So, these are few examples.

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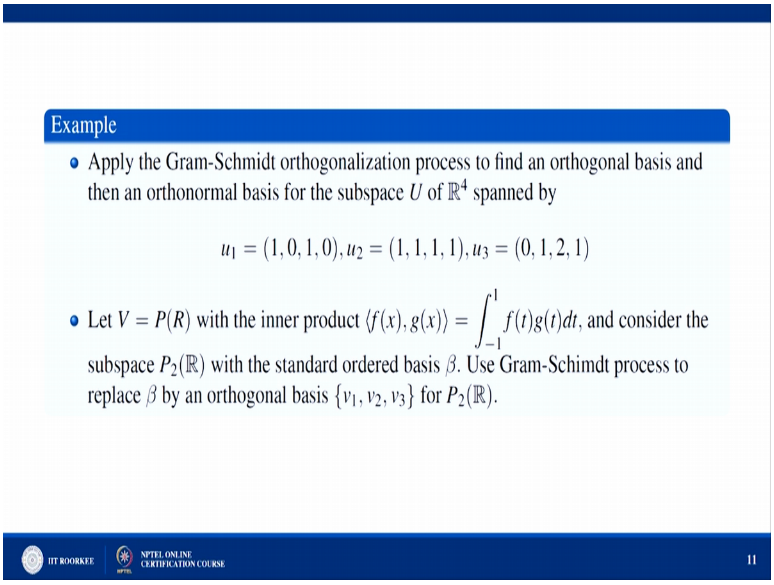

Based on this let us discuss the first one using the second one can be obtained similarly. So, what basically Gram Schmidt process is? You see you have given a set of linearly independent vectors and how to find an equivalent orthogonal or orthonormal set of vectors from it such that the span of u 1, u 2 up to u i is equal to span of v 1, v 2 up to v i for each i.

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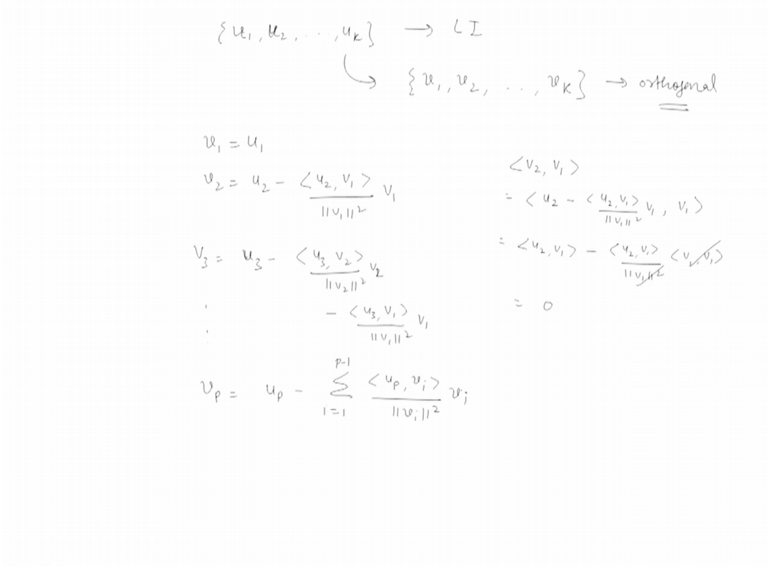

That means suppose you are having a set of vector say the set of vectors are v 1, v 2 or u 1, u 2 and so on up to u k, these are linearly independent and you want to construct an equivalent orthogonal set of vectors from this using Gram Schmidt orthogonalization process. So, suppose it is v 1, v 2 and so on up to v k, number of elements will be same of course, so, that is orthogonal. These orthogonal set of vectors we want to construct from this.

So, how we will form it, how will how we will obtain this? You set v 1 is equal to u 1. Now, v 2 can be obtained as v 2 minus inner product of u 2 with v 1 upon norm of v 1 square into v 1 you can easily verify you see in that proof also we are doing the same thing, but other way out, you see if you find the inner product of u 2 with v v 2 with v 1 what is this? This is inner product of u 2 minus u 2 v 1 upon norm of v 1 square times v 1 with v 1. So, this is inner product of u 2 with v 1 minus this is a scalar quantity will comes out by the definition of inner product and this is a inner product of v 1 with v 1. So, this is a squares cancels out and this is 0.

Similarly, we can verify for others also. If you want to find v 3 this is v u 3 minus inner product of u 3 with v 2 upon norm of v 2 square into v 2 minus inner product of u 3 with v 1 upon norm of v 1 square with v 1 it is v 2 and similarly, if you want to find say any v p it is u p minus summation i from 1 to p minus 1, inner product of u p with v i's, norm of v i is square times v i's.

So, this is how we can easily find out set of orthogonal vectors and once we obtain an orthogonal set of vectors from this linearly independent set using Gram Schmidt process so, orthonormal set can be find out by dividing each vector with it is norm.

So, now, let us discuss one example based on this before discussing an examples let us discuss these two results you see that if V; V is a finite dimensional inner product space then V is an orthonormal basis you see if it has a finite dimensional inner product space; that means, it is having a finite dimensional basis and using Gram Schmidt process you can always find it is equivalent orthogonal orthonormal set which is of course, a basis because a span a span is equal a span of both the sides are equal by the Gram Schmidt process and is independent also. So, by the Gram Schmidt process we can always find an orthonormal basis of any finite dimensional inner product space V.

Now, if V is a finite dimension inner product space then any orthonormal set of vectors v u 1, u 2, up to u m can be extended to form an orthonormal basis of V.

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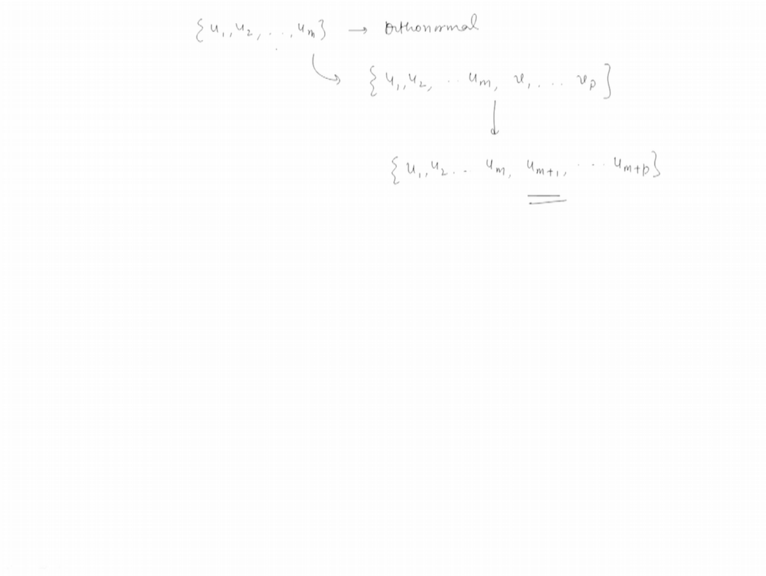

This is also very easy to show you see you are now having a set of vectors u 1, u 2, up to u m. This is a orthonormal set. Now, you can always extend up any set to form the basis of V finite dimensional vector space V. So, suppose we extended it up to p suppose dimension of the vector space v is m plus p and now, using Gram Schmidt process Gram Schmidt orthogonalization process you can always convert this set of vectors into its equivalent orthonormal set of vectors. So, hence we have extend this orthogonal set to an orthogonal basis of this vector space V.

Now, these are some problem based on this and there are two problems can be easily obtained. Let us discuss the first problem, the second can be solved on the same lines, the definition of inner product is different other things are same.

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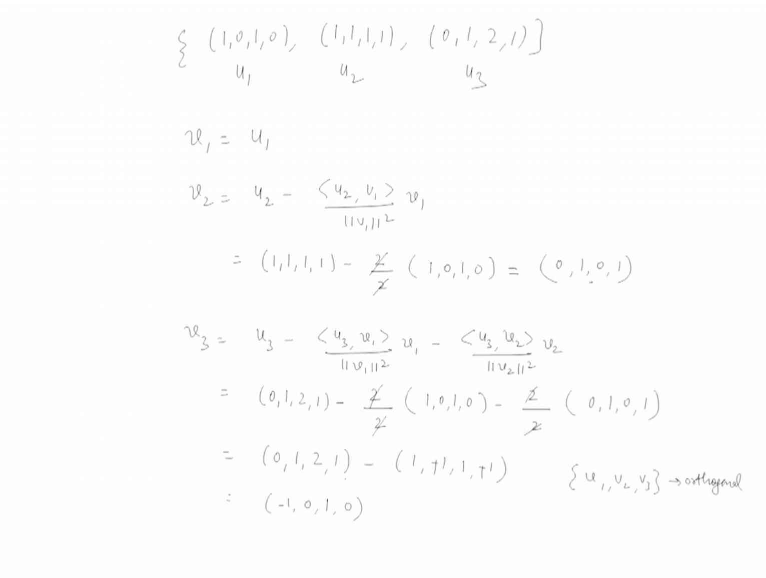

You see here we are having in this example you are having the set of vectors as 1, 0, 1, 0, the second is 1, 1, 1, 1, the third is 0, 1, 2, 1 this is u 1, u 2, u 3. So, v 1 is equals to u 1 by the Gram Schmidt process, v 2 with be equals to u 2 minus inner product of u 2 with v 1 upon norm of v 1 square times v 1 which is 1, 1, 1, 1 minus; inner product of u 2 with v 1 is what? Is 1 plus 1, 2 upon norm of v 1 square, v 1 is u 1 and norm of this is 2 and v 1 is u 1 which is 1, 0, 1, 0. So, this is equals to basically 2, 2 cancels out it is 0, 1, 0, 1.

Now, if you take the inner product of this with v 1; v 1 is u 1; that means, 0 plus 0 plus 0 plus 0 which is 0. So, that means, we are going on the right direction, how we find v 3 now? v 3 is again u 3 minus inner product of u 3 with v 1 upon norm of v 1 square times v 1 minus inner product of u 3 with v 2 upon norm of v 2 square times v 2. So, this is you see u 3 is 0, 1, 2, 1 minus.

Inner product of u 3 with v 1 u 3 with v 1, v 1 is u 1; the inner product of u 3 with u 1 is 1 into 0 is 0 0 this is 2, this is 0, that is 2 inner product of v 1 square is again 2, v 1 is 1, 0, 1, 0 minus, inner product of u 3 with v 2 u 3 with v 2 v 2 is this vector. So, this is 0, this is 1, this is 0, this is 1, 1 plus 1 is 2 and norm of u 2 square, norm of u 2 square is again 2 and v 2 is 0, 1, 0, 1 this 2, 2 cancels out 2, 2 cancels out it is 0, 1, 2, 1 minus when you subtract these two it is 1, minus 1, 1, minus 1 and if you take the you have to add these two sorry. So, it is 1 this is a 0 plus 1 is 1, this is 1 plus 0 is 1, this is 1. Now, you subtract these two it is minus 1 it is 0, it is 1, it is 0. Now, if you take inner product of these two, these two it is 0, 0, 0, 0 you have 0 inner product of these two is minus 1 plus 1, 0, yeah.

Now, this v 1, v 2, v 3 this set of vectors v 1, v 2, v 3 is an orthogonal set and if you want to find out the corresponding orthonormal set, then divide each vector by its norm. Norm of this is 2, norm of this is 2 and norm of this is also I mean norm of this is under root 2, norm of this is under root 2 and norm of this is also under root 2. So, divide each vector by it norm, you will get an equivalent orthonormal set of vectors. Now, similarly we can discuss this solve this examples.

So, hence we have seen this in this lecture that if we have a if you are given a set of a linear independent vectors how we can find it is in equivalent orthogonal or orthonormal set of vectors.

So, thank you.