Matrix Analysis with Applications

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Lecture – 02
Echelon form of A matrix

Hello friends, welcome to lecture series on matrix analysis with applications. So, in the first lecture we have seen that how we can apply elementary row operation to get A row echelon form of A matrix A ok. Now we will see echelon form of A matrix what do you mean by echelon form and how we can obtain equal echelon form by applying elementary row operations, before the starting echelon form we will first we will see elementary matrices what do you mean by elementary matrices?

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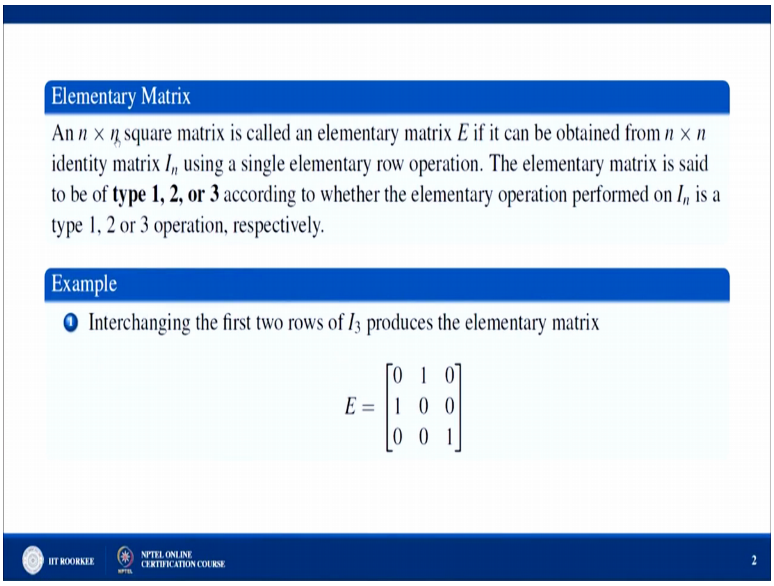

And n cross n square matrix is called an elementary matrix E if it can be obtained from n cross n identity matrix I n using single elementary row operations. We have already seen the last lecture that there are 3 basic elementary row operations, the first is you multiple replace any row by multiplying it by non zero scalar, the second elementary row operation is you replace any row R say R th row by R th row plus c times some s th row ok, and you can always interchange any 2 rows, so these are the basic 3 elementary row operations.

Now, if you apply an elementary row operation in many matrix A and the same elementary row operation you apply in the identity matrix I, then the then that matrix which is obtained from I is called an elementary matrix ok. Now the elementary matrix said to be of type 1, 2 or 3 according to whether the elementary operation performed on I n is type 1, 2 or 3 operation respectively. the 3 operation which we are defined if we are applying the first operation of the identity matrix, we are saying it is the elementary its elementary matrix of type one.

Similarly, type 2 and type 3 now for example, interchanging the first 2 rows of I 3 produces the elementary matrix is this you see, if you identity matrix and you interchange if you interchange first 2 rows. So, this matrix is obtained and this matrix we call as elementary matrix ok.

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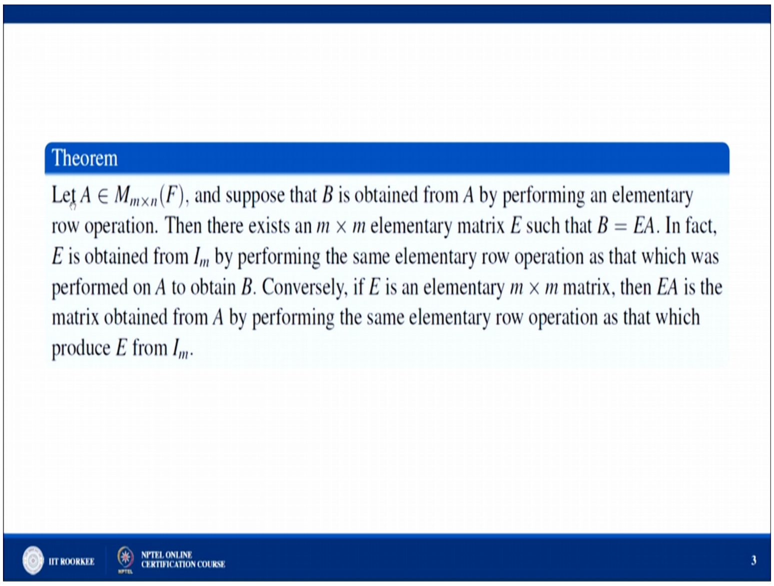

Now, let this is A result let A is A m cross n matrices over the field F, and as and suppose that B is the obtained from A by performing an elementary row operation ok. Then there exist and m cross m elementary matrix E such that B is equal to E. So, there always exist in elementary matrix E such that B is equal to EA.

In fact, E is obtained from I am by performing the same elementary row operation as that what was performed on A to obtain B ok. If you if you obtain B from A by applying some elementary row operation and same elementary row operation you applied on I m, so we obtain E. Conversely if E is an elementary m cross A matrix then E A is the matrix obtained from A by performing the same elementary row operation as that which produce E from I m. So, convert, so the statement is also true, so let us discuss it by A few examples.

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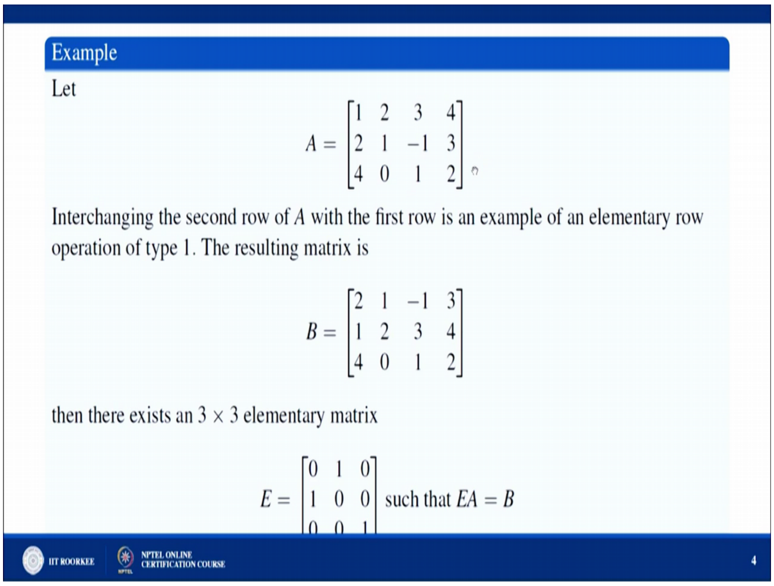

See A matrix A here A is this matrix of order you see it is 3 cross 4 3 rows, 4 columns. Now in this matrix you suppose interchange the second row of A with the first row you apply elementary row operation you interchange R 2 by R 2 by R 1. So, the new matrix is obtained which is row equivalent to A, which is 2 1 minus 1 3 the second row, the first row will be into second row now 1 2 3 4 4 0 1 2 ok. And if you have an identity matrix here is order 3 cross 3 and you interchange the first and the second row.

So, this array this matrix is obtained and this matrix is A elementary matrix here ok, and E A is always equal to B when you multiply E with B E with A sorry E with A this E with A, then you E is obtained v this you can easily check ok, now let us take second example you see here A is this matrix ok.

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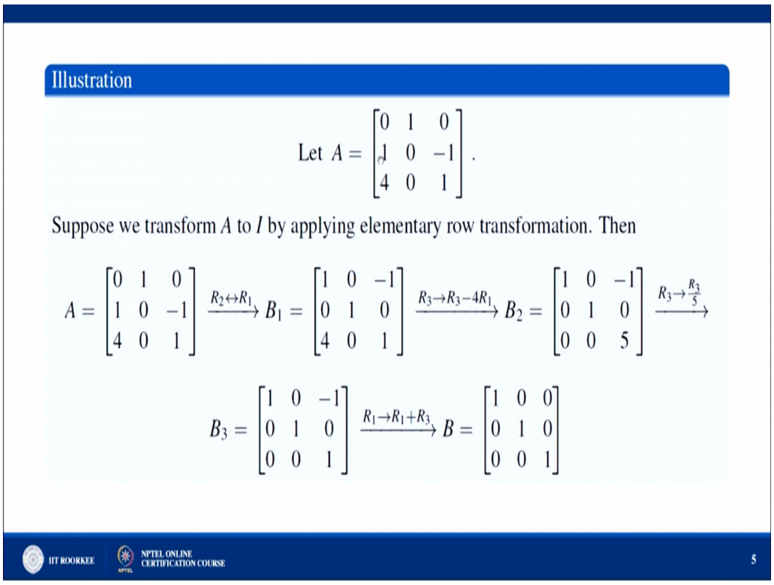

Now, suppose we want to transform this A into I by applying elementary row operations ok, then now how you how you will make this as the identity matrix by applying elementary row operations you see the first elementary is 1 here which is non-zero element you first interchange, this row with the first row that is you interchange R 2 and R 1 you will obtain A new matrix B 1 which is 1 0 minus 1, this row will come here and this row will come here 0 1 0 and this 4 0 1.

Now, you make 0 here with the help of this, so this minus 4 times us. So, replace R 3 by R 3 minus 4 times R 1 you get new matrix B 2 and B 2 will be equal to the first 2 rows remain the same and the third row, this minus 4 times this will be 0, this minus 4 times this will be 0, this minus 4 times this will be 5. Now you make 1 here to make 1 here you would simply divide this row by 5 you obtained this matrix B 3 it is 1 0 minus 1 0 1 0 0 0 1 now to complete identity you have to make 0 here with the help of this. So, this plus this that is you replace R 1 by R 1 plus R 3. So, this will give an identity matrix B ok.

Now, now to obtain this matrix not to obtain this matrix from this matrix what is what is the elementary matrix you see, we have we have interchange R 2 by R 1. So, same elementary row operation you apply on identity matrix of order 3 cross 3 you replace R 2 by R 1. So, which matrix we will obtained the matrix which will obtained as this matrix E 1 0 1 0 1 0 0 0 0 1 which is the obtained by interchanging R 2 and R 1 of identity matrix ok.

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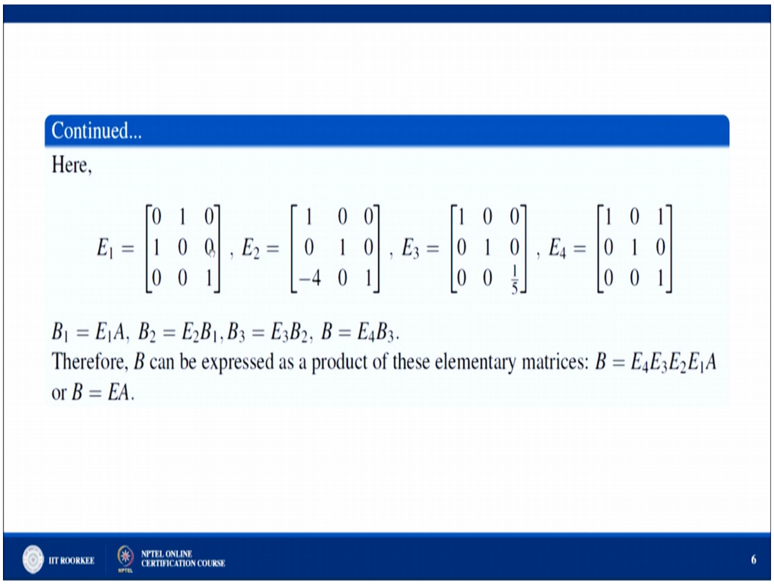

Now similarly to obtain B 2 from B 1 we have what you have done we have replace R 3 by R 3 minus 4 times R 1. Now you apply A same elementary row operation is the identity matrix to get the new elementary matrix E 2 ok.

So, so here if you take the identity matrix and replace the third row by third row minus 4 times the first row then you get minus 4 0 1. So, this is the elementary matrix 2 similarly now you take the third I mean B 3 how B 3 is obtained from B 2 to obtain B 3 from B 2 you have replaced you have divide simply the third row by 5, you apply the same elementary row operation is the identity matrix to get the element to get the elementary matrix cross point to this row operation.

So, what will be that the simply divide the third row by 5 all the 2 rows are same. So, this is E 3 and similarly the E 4 can be obtained by simply adding the first row by the third row, the same you apply the same elementary row operation in the identity matrix you apply the same elementary row operation identity matrix to get E 4.

Now, now by the theorem which we have just stated to get B 1 from A we have the elementary row operation E 1 ok. So, this can be written as, so B 1 can be written as E 1 times A because E 1 is the elementary row matrix cross point to this row operation, similarly B 2 will be E 2 into B 1 B 3 will be E 3 into B 2. And similarly the last matrix B which is an identity matrix of course is E 4 into B 3. Now what we have obtained if you if you substitute B 3 over here B you replace substitute B 3 here, this is E 3 into B 2 then you then you replace B 2 by E 2 into B 1 and then B 1 by E 1 into A.

So, what finally, we have obtained finally, we have obtained B is equals to E 4 E 3 E 2 E 1 into A so; that means, that means if here we have applied 4 number of elementary row operations and for each elementary row operation we obtained and I elementary matrix ok, to get the final matrix which is the row equivalent to A matrix A they simply E 4 into E 3 into E 2 into E 1 and the first matrix initial matrix A ok, or we can say that B equal to E A where B E is simply multiplication of all elementary matrices in this order E 4 E 3 E 2 E 1.

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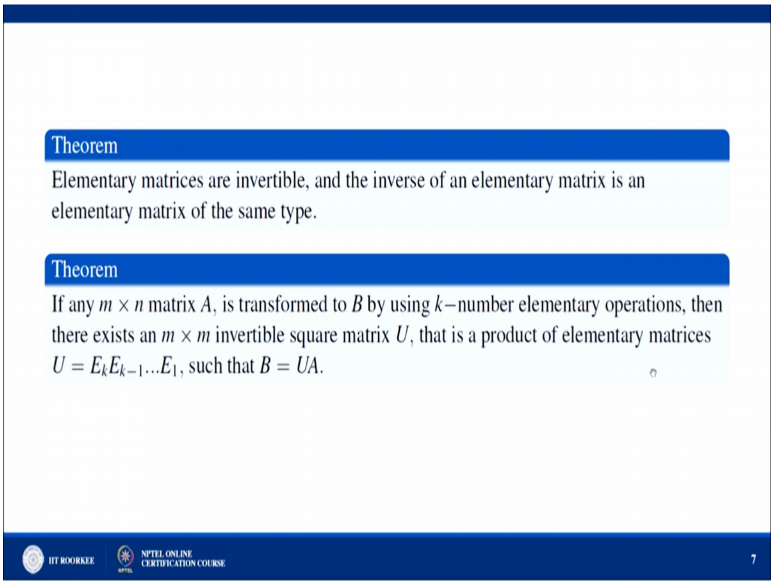

So, that means, what I want to say that if any m cross and matrix A is transformed to B by applying k number of elementary row operations then there exist and m cross m invertible square matrix U that is A product of elementary matrices, U equal to E k E k minus 1 E k minus 2 up to E 1 such that B is equals to U into A ok.

So, that matrix always exist and it is inevitable and why elementary matrix are invertible you see elementary matrices are obtained are obtained from identity matrix by applying some elementary row operations. And applying elementary row operation you see if you if you take if you see ah matrix A.

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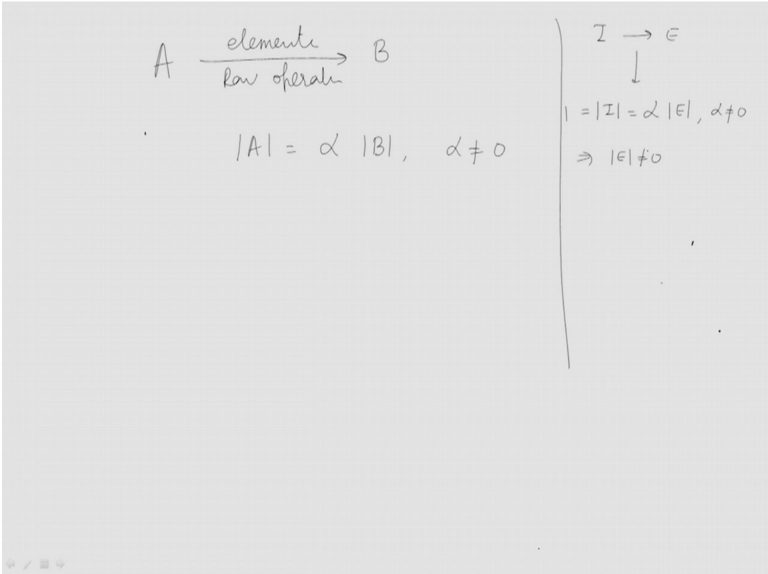

You apply a some elementary row operation on this matrix, elementary row operation and you got a new matrix B which is the row equivalent to A ok. So,, so this matrix can be obtained either by interchanging 2 rows or 2 multiply this A by and any row of this matrix A by non-zero scalar or you replace some ith row by ith plus c times gth row where I not equal to j ok.

So, we can simply say that determinant of A will be some alpha time determinant of B where alpha is not equal to 0, because applying elementary row operation will not change the will not change the nature of the matrix I mean if it is invertible remains invertible ok. It simply change the it is simply change the determinant of the matrix by some A scalar because we can multiply a row by A non-zero scalar.

So, if so elementary matrix are obtained from identity matrix by applying elementary row operations. So, since identity matrices invertible it has a determinant 1. So, elementary matrix are also invertible because determinant of identity matrix will be some alpha time determinant of E where alpha is not equal to 0, and E this is 1. So, this imply determinant of E is not equal to 0 ok, so invertible. So, elementary matrix are always invertible and then there product also invertible because product of invertible matrix also invertible ok.

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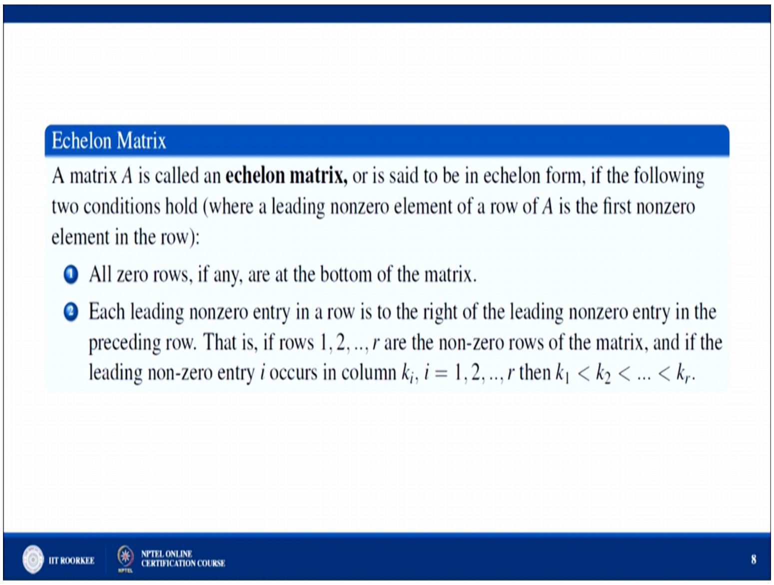

Now, come to echelon form of a matrix A matrix A is called an echelon form or is said to be in echelon form if the following 2 conditions hold. Where a leading non-zero element of a of row of A is a first non-zero element in the row ok, we are taking the leading element as the first non zero element in that row ok. So, what are 2 properties the first properties all 0 rows if any are at a bottom of the matrix? If there is a row continuing all 0 that must be at the bottom of the matrix number 1 number 2 each leading non-zero entry in a row leading non-zero entry means in the first non-zero entry in a row is to the right of the leading non zero entry in the preceding row.

If you have the second row the first non zero entry in the second row must be on the right side of the first non zero entry in the first row ok, that is if rows are 1 2 up to R are the non-zero rows of a matrix and if the leading non-zero entry i occurs in the column k i, i from 1 to r then k 1 is less than k 2 is less than k r. So, let us discuss this by some examples.

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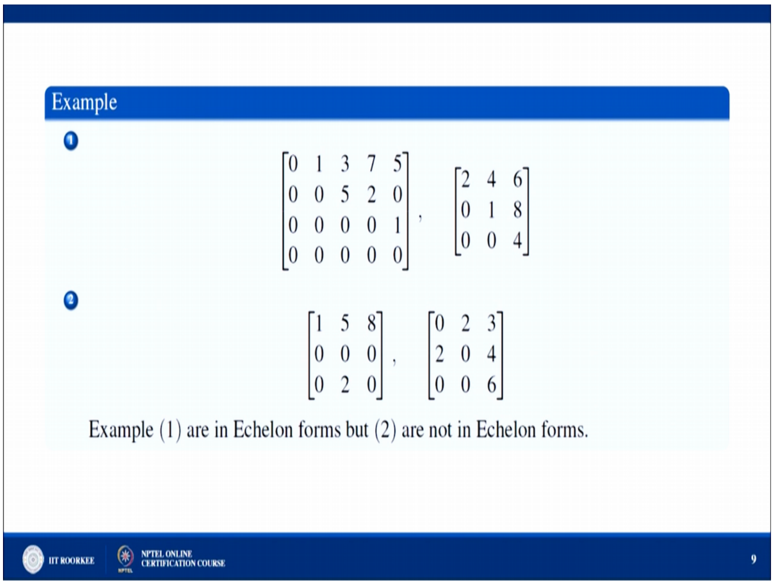

Now, you see here now you focus on the first matrix you see first of all the 0 rows are at the bottom. The row containing all 0 elements this is the bottom now first property holds, now you see the first leading element leading means the first on 0 element is here this is A 1 2 the first leading non zero entry here is this 1.

The first non zero entry here is this 1 now this element the first non zero entry of this row is to right of this row. The first non-zero entry of this you should right of you should right of this row the next row right of this row. So, this satisfy the properties of the echelon form of a matrix and hence we can say that this is then echelon form.

Now this is a second example, the second example you see that there is no row containing all 0 ok. Now this is the first non zero element in this row this is the second 1 0 element in this is the first non zero element in the second row and this is the third, this is the non zero element here third row. And all the non zero rows this non zero row this leading element of this row is a right of this leading element of this row is should right of this. So, this is a nucleon form this is a echelon form of some matrix.

However if you see here you see this is not a echelon form because this row is an containing all 0 element, but it is not at a bottom of the matrix. So, this not a echelon form here if you see in this example this is a first non-zero element or the first row. And this is the first non zero element of the second row, but this non zero element is on the left of the first non zero element of the preceding row. So, this is not A nucleon form because it must be on the right side. If it is 0 if it is 0 then we have to make 0 here to make it a echelon form. So, this is not a echelon form now let us try this example that how we can make it a echelon form of the matrix.

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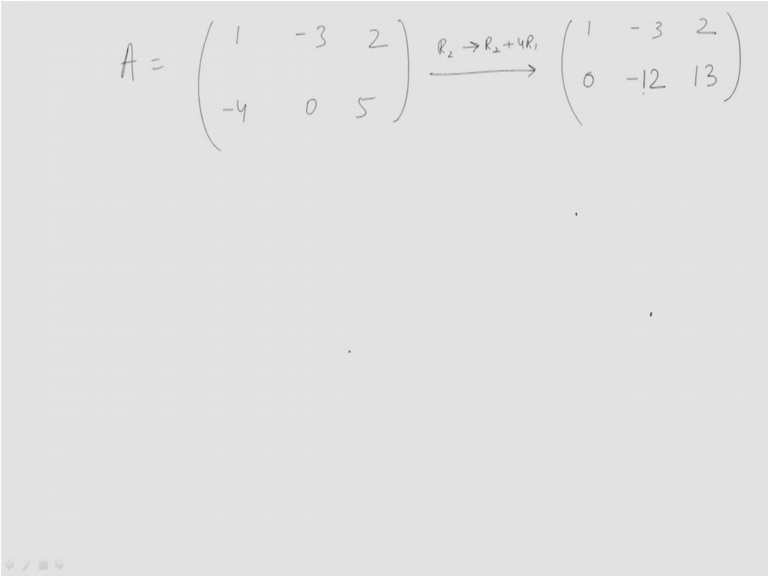

So, what is the problem is 1.

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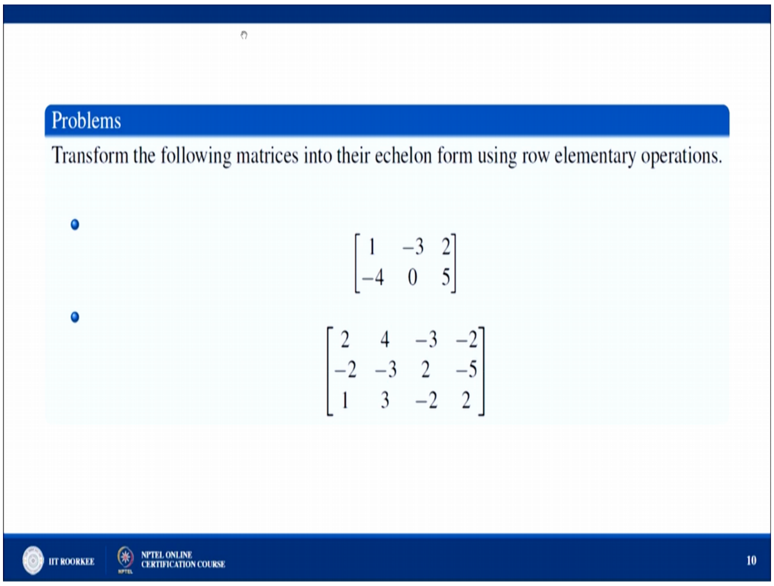

It is minus 3 2 then it is minus 4 0 5, now suppose by applying elementary row transformation you want to find out its echelon form. So, so the first non-zero element 1 who live it as it is, now in this row to keep all the non zero entries on the right, you see if we have in the echelon form each leading non zero entry, it should a write compared to the preceding row, to satisfy that condition you take the first non zero element in the first row and make 0 a below in that column all the elements must be 0 in that column ok.

So, you apply elementary row operation to make 0 here you take R 2 at replace R 2 by R 2 plus 4 times R 1 this is make this element in operation it is minus 1 3 2 it is 0. Now, it is this plus 4 time this is minus 12 this plus 4 time this is 30. Now this is the first non zero element of this row which is 1, this is the first non zero element of second row which is minus 12, and it is to the right of this row there is no row we continuing all the 0 elements. So, we can say that this is the echelon form of this matrix A. So, in this way we can convert the, we can find out the echelon form of a matrix.

Now, let us consist a second example, the second example can be framed you see you have to form, form the echelon form of this matrix. So, how we can do that now a what is the matrix now the matrix says.

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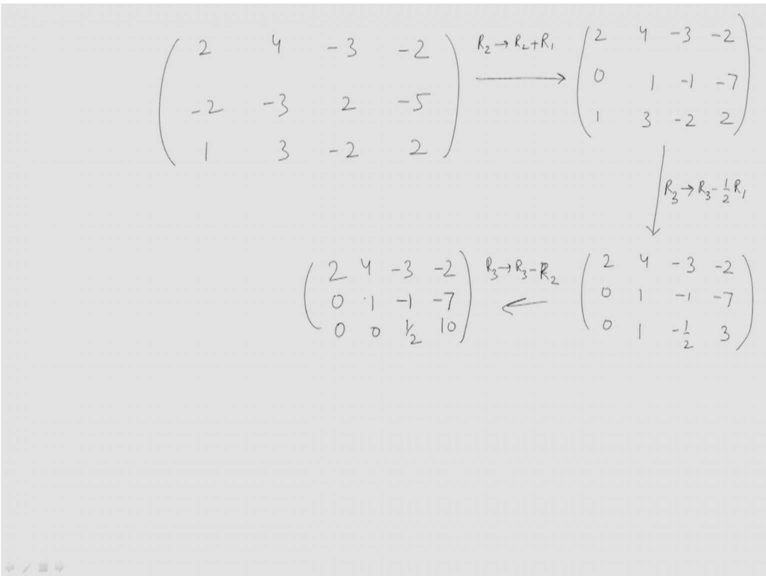

It is 2 minus 2 4 minus 3 minus 2 then it is minus 2 minus 3 2 minus 5 1 3 minus to 2 now you want to frame its echelon form. So, you leave first non zero element as it is make 0 here with the help of this make 0 here with the help of this. So, we apply elementary row operations, so you replace R 2 by R 2 plus R 1.

So, this is 2 4 minus 3 minus 2 it is 0 it is 1 it is minus 1 it is minus 7 you leave it as it is the third row would give as it is, now you make 0 here with the help of this. So, you take R 3 by R 3 minus 1 by 2 R 1 by 2 R 1. So, it is 2 4 minus 3 minus 2, it is 0 1 minus 1 minus 7 it is 0 it is 3 minus 2 is 1, then it is minus 2 and it is plus 3 by 2 that is minus 1 by 2.

Then it is 2 plus 1 that is 2 plus 1 is 3. Now this is and this is with the right of this side, now below this element make all the elements 0 you pick out the first non leading element make 0 in that column leaving this element make 0 to all the elements below this. And the second non-zero second leading element I mean second non-zero element in that column in that row make 0 below. So, here you make 0 you simply replace R 3 by R 3 minus R 2.

So, this will be simply 2 4 minus 3 minus 2 this is 0 1 minus 1 minus 7 this is 0 0 this is 1 by 2 and this is 10 ok. So, this is and this is now yeah now it is you see that there is there is no row containing all 0, and all the you if you take that this first non zero row this first non zero entry all the elements below these are zero, first non zero entry second row all the element below these are 0 and their third element, so this is likely on form of this matrix ok. So, this is this is all about echelon form we will see some more illustrations of echelon form and its applications in the next lecture.

Thank you.