Matrix Analysis with Applications
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Lecture – 19
Vector and Matrix Norms

Hello friends welcome to lecture series on Matrix Analysis with Applications. So, today we will deal with Vector and Matrix Norms. What do you mean by a vector norm and how can you find out a matrix norm. So, that we discuss in this lecture, first is what you mean by norm of a vector.

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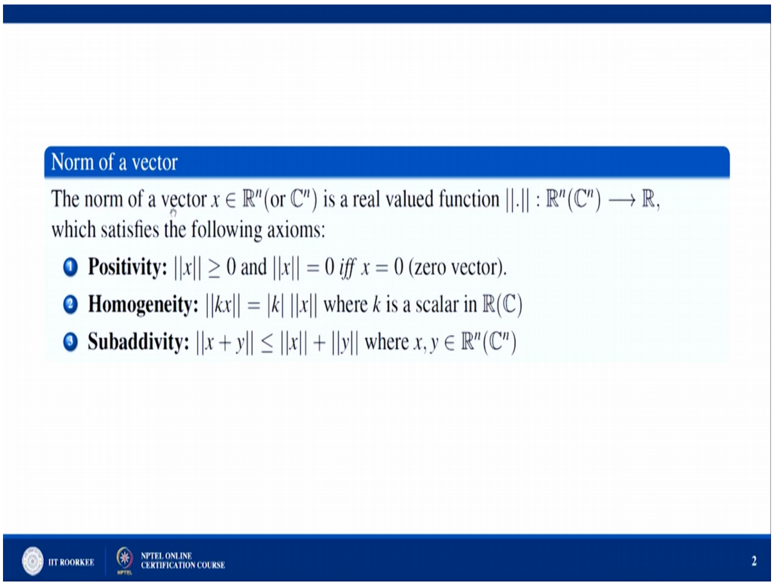

The norm of a vector x which may be in R n or in C n, I mean C n means complex tuple is a real valued function, which is defined from R n or C n R R it is always a real value which satisfy the following axioms. The first axiom is positivity, positivity means norm of any vector x is always greater than equal to 0 for any x.

And norm of x is equal to 0 if and only if x is itself 0 second property is homogeneity; that means, norm of k x where k is a scalar, which is in R or C is equal to mod of k into norm of x. The third property is subaddivity which is norm of x plus y is less than equals to norm of x plus norm of y something like triangle or triangular inequality where x y belongs to R n or C n.

So, if vector x satisfying these three properties then that is called norm of that vector that is a norm induced in that vector; So, there are different type of norms which can be induced on R n or C n, now let us discuss these things by an examples suppose.

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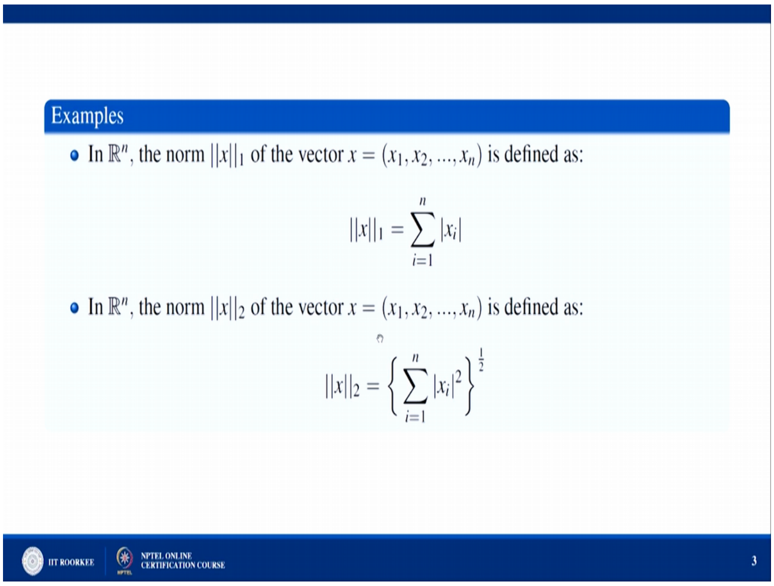

Suppose you are in R n, then the norm 1 we are got calling is norm of x 1 norm 1 of vector x which is x 1 x 2 up to x n is defined as sum of x i's mod of x i's where i is varying from 1 to n, so let us discuss this thing.

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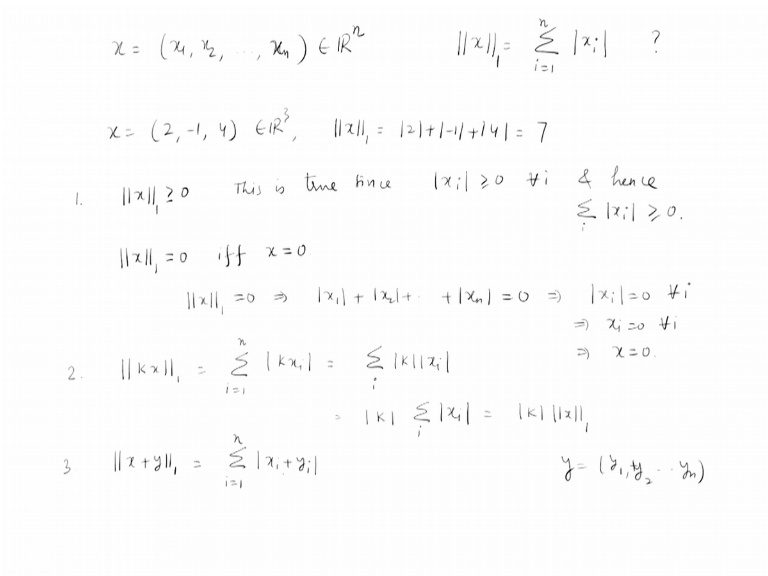

So, x is in R n x which is x 1 x 2 up to x n up to x n is in R n. How we are defining norm of this x here? The norm of this x we are defining as this is you are calling as norm 1 it is summation i from 1 to n it is mod of x i, I mean sum of mod of this, mod of this, mod of their sum of all the mods all the absolute values of this set. If you take x as suppose 2 minus 1 4 which is an R 3 then norm 1 of this vector is simply mod of 2 plus mod of minus 1 plus mod of 4 which is 7.

Now, is it a norm first of all is it a norm how can we show that it is a norm. So, in order to show that this is a norm we have to show the 3 axioms which we have studied earlier. Now, the first is the first axiom is norm of x is always greater than equal to 0, which is true because sum of absolute values cannot be less than 0 we are taking the modulus and then we are adding them it cannot be less than 0. So, which is true this is true since mod of x i is always greater than equal to 0 for all i and hence, sum will be sum over i of mod of x i will also greater than equal to 0, so this is then this is true.

Now, norm of x 1 equal to 0 if and only if x is equal to 0, now if x equal to 0 suppose this vector is a 0 vector then of course, the sum of the absolute values is 0. So, the first part is over now if norm of x 1 is equal to 0 then this means mod x 1 plus mod x 2 and so on up to mod x n is equal to 0.

If sum of positive values all are positive values is equal to 0 this means what this means each x i is equal to 0 for all i, because this is positive this is positive this is positive and sum of positive quantities is equal to 0 this means each quantity is 0. And mod of x i equal to 0 means what x i equal to 0 for all i. So, this means x equal to 0, so we have shown that first property holds which is positivity.

Now, let us try to show a second property second property is norm of k x; that means, you multiply this vector x by k by a scalar k. So, what will be the norm of this by definition this will be mod of summation i from 1 to n mod of k x i, which is equals to summation over i mod of k mod of x i by the definition this k is a scalar quantity can be taken out. So, it is mod of k sum over i mod of x i and this is simply mod of k into norm of x norm 1 of x. So, we have shown that norm of k x is equals to mod k times norm of norm 1 of x, so property holds.

Now, the third axiom third axiom is mod of x plus y norm of x plus y is less than equals to norm of x plus norm of y. How we can show this is equal to sum of i from 1 to n mod of x i plus y i, here x is x 1 x 2 x 3 up to x n and y is y 1 y 2 y 3 up to y n x is this quantity and y is y 1 y 2 up to y n, now mod of this quantity now we can easily see.

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$$\|x+y\|_1 = \sum_{i=1}^{n} |x_i + y_i|$$

$$|x_i + y_i| \leq |x_i| + |y_i| \quad \forall i$$

$$\Rightarrow \sum_i |x_i + y_i| \leq \sum_i |x_i| + \sum_i |y_i|$$

$$\Rightarrow \|x+y\|_1 \leq \|x\|_1 + \|y\|_1$$

Now, here it is norm of x plus y norm 1 of x plus y is equal to sum of i from 1 to n mod x i plus y i. Now mod of x i plus y i is always less than equals to mod x i plus mod of y i for all i, this is by the property of modulus i mean mod. So, this implies sum will also be less than equal to, sum over i mod x i plus sum over i mod y i and this implies norm of x plus y norm 1 of x plus y is less than equals to norm 1 of x plus norm 1 of y.

So, we have shown all the three properties all the three properties which are here all the three properties. So, we can say that the definition given by here of the norm is correct i mean this is a this is a norm, so this is how we define norm 1 of vector x.

Now, let us come to the second part norm 2 of in R n norm 2 of vector x oh in R n can be defined like this it is the square of all the vectors sum of square of vectors and the under root of that expression that is a norm 2 of a vector x. So, it is also it is it also defines a norm, so how we can show this.

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$$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2}$$

1. $\|x\|_2 \geq 0$ $\quad$ $\|x\|_2 = 0$ iff $x = 0$.

if $x = 0$ then $\|x\|_2 = 0$.

let $\|x\|_2 = 0 \Rightarrow \sqrt{x_1^2 + \cdots + x_n^2} = 0$

$\Rightarrow x_1^2 + \cdots + x_n^2 = 0$

$\Rightarrow x_i^2 = 0 \ \forall i$

$\Rightarrow x_i = 0 \ \forall i$

$\Rightarrow x = 0$

2. $\|kx\|_2$

$= \left(k^2 x_1^2 + \cdots + k^2 x_n^2\right)^{1/2} = \left(k^2\right)^{1/2}\left(x_1^2 + \cdots + x_n^2\right)^{1/2}$

$= |k| \, \|x\|_2$

Now, here x norm of x 2 we are taking as sum from i from 1 to n x i is square and whole under root that is x 1 square plus x 2 square and so on up to x n square whole power 1 by 2. So, the first property is that norm of vector is always greater than equal to 0 which is true because you see all this quantities are nonnegative. So, under root is defined and under root is always greater than equal to 0. So, this is; obviously, true by the definition this is; obviously, true norm of norm 2 of x is always greater than equal to 0 for any x this is true by the definition itself.

Now, we have to show that norm of x 2 is equal to 0 if and only if x equal to 0. Now if x is 0; that means, x 1 x 2 up to x n all are 0, so if x is 0 the so this expression is 0. So, norm of x 2 is; obviously, 0 if x is 0 then ah; that means, norm of x 2 is; obviously, 0.

Now, let norm of x norm 2 of x is equal to 0, so this implies under root of x 1 square plus and so 1 up to x n square is equal to 0. So, this implies x 1 square plus and so on up to x n square is equal to 0, and since it is a sum of square of positive quantities and this is equal to 0 this means x i square equal to 0 for all i and this means x i equal to 0 for all i this means x equal to 0. So, we have shown that if norm is equal to 0; that means, x itself equal to 0, so the first property we have shown.

Now, second property second property is norm of k x must be equals to mod of k into norm of x. So, norm of x 2 which is by the definition; that means, we are multiplying each x i's by k. So, what is the norm of this quantity then by a definition it is k square x 1 square plus k square x n square whole power half 1 by 2. So, a k square 1 by 2 will come out it is k 1 square and so on up to x n square whole raise to power 1 by 2 and this is mod k into norm of x norm 2 of x. So, second property also holds true. Now, you have to show a third property, so how we can show a third property.

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$$\|x+y\|_2 = \left(\sum_{i=1}^{n} (x_i+y_i)^2\right)^{1/2}$$

Consider:

$$(x_1z+y_1)^2 + (x_2z+y_2)^2 + \cdots + (x_nz+y_n)^2 \geq 0$$

$$\Rightarrow z^2\left[\sum_i x_i^2\right] + 2\left(\sum_i x_iy_i\right)z + \sum_i y_i^2 \geq 0$$

$$\Rightarrow Az^2 + Bz + C \geq 0$$

$\Rightarrow$ It has atmost one real root

$\Rightarrow D \leq 0$

$$\Rightarrow \cancel{4}\left(\sum_i x_iy_i\right)^2 - \cancel{4}\left(\sum_i x_i^2\right)\left(\sum_i y_i^2\right) \leq 0$$

$$\Rightarrow \left(\sum_i x_iy_i\right)^2 \leq \left(\sum_i x_i^2\right)\left(\sum_i y_i^2\right)$$

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$$

$$\left(\sum_{i=1}^{n} (x_i+y_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} (x_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n} y_i^2\right)^{1/2}$$

$$\sum_{i=1}^{n} (x_i+y_i)^2 \leq \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 + 2\left(\sum_i x_i^2\right)^{1/2}\left(\sum_i y_i^2\right)^{1/2}$$

$$\Rightarrow \sum_{i=1}^{n} x_iy_i \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}\left(\sum_{i=1}^{n} y_i^2\right)^{1/2}$$

The third property is norm of x plus y is less than equals to norm of x plus norm of y this we have to show, in order to show that this is this is a norm. So, this will be equal to by the definition it is summation i from 1 to n each element of x plus y will be x i plus y i you make it square sum these sum all the components sum of a square of all the components to the power 1 by 2 that will be x i plus y i whole square whole raise to power 1 by 2.

So, we have to we have to show that we have to show that normal of x plus y 2 will be less than equals to a norm of x 2 plus norm of y 2, this we have to show. So, what is expression is this expression by this from here is summation i from 1 to n x i plus y i whole square whole raise to power 1 by 2 is less than equals to summation i from 1 to n x i square whole raise to power 1 by 2 plus summation i from 1 to n y i square whole raise to power 1 by 2, we have to show that this inequality is true.

Now, let us square both the sides if you square both the sides what we will obtain we will obtain sum from i from 1 to n x i plus y i whole square is less than equals to x i square sum of x i square i from 1 to n plus sum of y i square i from 1 to n plus 2 times 2 times these two expressions that is sum of i of x i square whole raised to power 1 by 2, and sum of i y i square whole raised to power 1 by 2.

And you were when you further simplify this what you will obtain you see a plus b whole square is a square plus b square plus 2 a b. So, square, square will cancel out square, square will cancel out. So, 2 will cancel from this 2. So, what we will obtain finally, from here after simplification this expression we will obtain.

Now, if any how we prove that this is true then means this is true and this means this is true so; that means, we have done. So, any how we have to prove that this expression is true. So, now, let us try to prove that this expression is true and this is sometimes this is also called Cauchy's Schwarz inequality. So, how we will prove Cauchy's Schwarz inequality for this real case let us see.

Now, you see it is simply can be proven using quadratic expression how we can do that you simply consider this polynomial consider you see. It is x 1 z plus y 1 whole square plus x 2 z plus y 2 whole square and so on, up to x n z plus y n whole square, this is always greater than equal to 0. This is very obvious sum of square sum of square quantities sum of non-negative expressions always greater than equal to 0 that is very true.

Now, what we will obtain from here from obtain if you simplify this what we obtain it is z square times sum over i of x i squares plus twice of it is sum of sum over i x i y i into z plus sum of y i square it is greater than equal to 0, you simply open each square and you simplify you will get this expression. So, this is something like a a z square plus b z plus c greater than equal to 0 where a is this expression b is twice of this expression and c is this expression.

Now, if a polynomial of degree 2 is greater than equal to 0 what does that mean for any z z is a arbitrary like variable. If z is a arbitrary variable and for any arbitrary variable this expression is always greater than equal to 0 this means a parabola, because a quadratic polynomial always represent a parabola.

So, this, so this parabola for any arbitrary z is always greater than equal to 0 this means parabola is always on the on the upper side. Either this or this for any z; that means, it has it has at most 1 root it has at most 1 real root. So, this means it has at most 1 real root, so this imply discriminant of this polynomial will be less than or equal to 0. And what is the discriminant, discriminant b square b is here b is this expression b square that is this, whole square minus 4 a c a is this c is this a is less than equal to 0.

Now, 4 4 will cancels out with 0 4 will go with 0, so what we obtain from here that summation over i x i y i whole square is less than equal to sum over i x i square into sum over i y i square.

So, these are same expression which, which is here if you take under root both the sides we get the same expression over here if this is true this is true, and if this is true then this is true and hence this is true. So, we obtain the third inequality over here. So, hence we can see that the norm which is which is induced in this manner, I mean the definition of norm is true the next is, so this is an norm 2 of a vector x.

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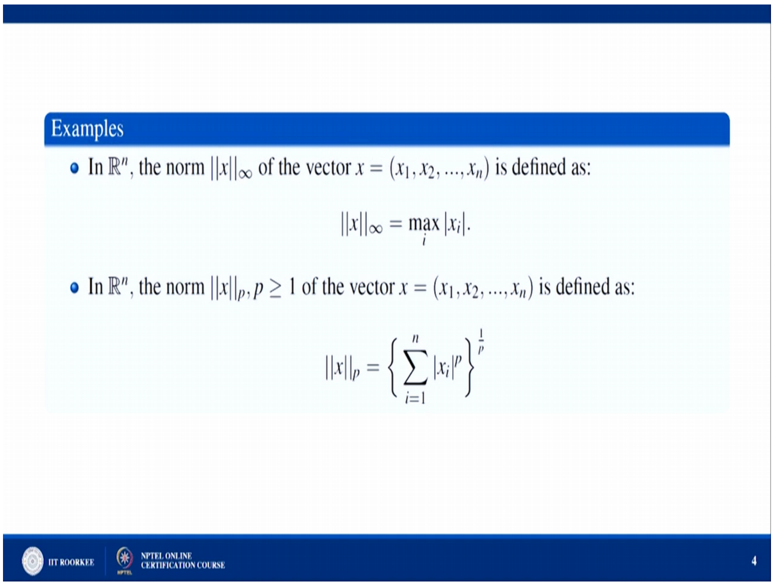

Now, the norm infinite norm of a vector x is given as maximum of mod of x i this simply a notation infinite norm. This is now it, it is also a norm induced in a vector x this is very easy to show again we, we have to show all the three properties you see.

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$$\|x\|_\infty = \max_i |x_i| \qquad x = (x_1, x_2, \ldots, x_n)$$

1. $\|x\|_\infty \geq 0 \qquad \|x\|_\infty = 0 \Rightarrow \max_i |x_i| = 0 \Rightarrow x = 0$

$$|x_j| \leq \max_i |x_i| = 0$$

$$|x_j| \leq 0 \ \forall j$$

$$\Rightarrow x_j = 0 \ \forall j$$

$$\Rightarrow \underline{x = 0}$$

2. $\|kx\|_\infty = \max_i |kx_i| = |k| \max_i |x_i| = |k| \, \|x\|_\infty$

3. $\|x+y\|_\infty = \max_i |x_i + y_i| \leq \max_i |x_i| + \max_i |y_i| = \|x\|_\infty + \|y\|_\infty$

$$|x_i + y_i| \leq |x_i| + |y_i|$$

Infinite norm of vector x is maximum over i of mod x i, where x is x 1 x 2 up to x n. Now, the first property the first property is norm is always greater than or equal to 0 which is; obviously, true because you are taking maximum of absolute values which cannot be negative number 1. So, first property that norm of this quantity is always greater than equal to 0 it is; obviously, true by the definition itself.

Now, if x is equal to 0 of course, norm infinite norm will be 0 because the maximum will be 0. And if x if norm f infinite is equal is equal to 0 this means maximum over i of mod x i equal to 0. And each x j each mod x j is always less than equals to maximum over mod of x i's, you see because x j is always less than equal to maximum over mod x i is always true and it is equal to 0 by this; that means, mod x j is less than equal to 0 for all j if it is less then equal to 0 for all j means what 0.

So, because it cannot be less than 0 it is a mod value cannot be less than 0. So, the so x j will be equal to 0 for j and; that means, x equal to 0. So, we have we have shown that x is equal to 0 if this happens. So, the first property holds. Now, second property is norm of k x we have to show. So, it is maximum of it is easy to show again norm of k x i's and it is mod of x times maximum over i mod of x i which is equals to mod k times norm x infinite. So, a second property; obviously, hold.

Similarly, we can easily show the third property also this case norm of x plus y is equal to maximum over i mod of x i plus y i. And since mod of x i plus y i is less than or equals to mod x i plus mod y i you take the maximum both the sides and you can easily obtain that this is less than or equal to maximum over i mod of x i plus maximum over i mod of y i which is equals to norm infinite x plus norm infinite y. So, hence we have shown all the three properties for this norm. So, this is the norm induced by induce of a vector x.

The last 1 here is the norm p norm on a vector x where p is greater than equal to 1 if p is equal to 2. So, it will be a norm 2 norm or norm 2 of a vector x is defined like this. So, we can prove all the three properties of this norm also for the third one, we have to use generalised Cauchy's Schwarz inequality.

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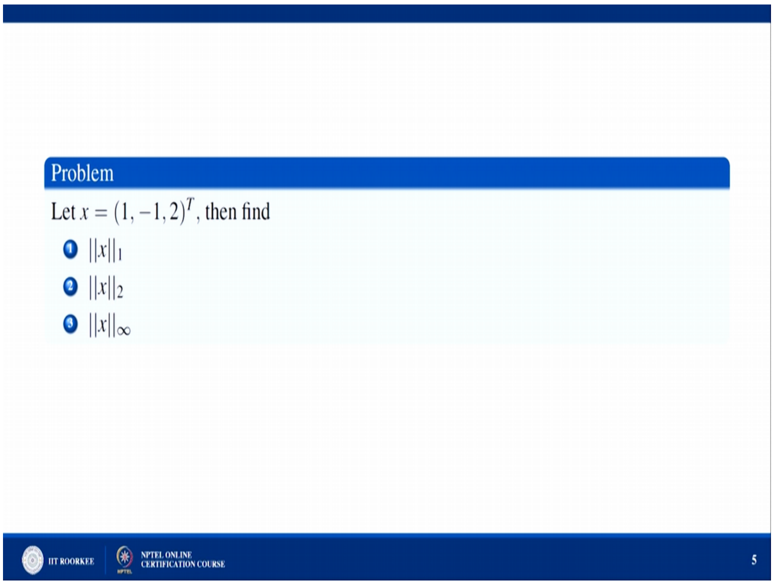

So, suppose you are having this vector 1 minus 1, 2 and you want to find out 1 norm of this vector, so was is the first norm of this vector.

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$$x = (1, -1, 2)$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{3} |x_i| = |1| + |-1| + |2| = 4$$

$$\|x\|_2 = \left((1)^2 + (-1)^2 + (2)^2 \right)^{1/2} = \sqrt{6}$$

$$\|x\|_\infty = \max_i |x_i| = \max\{ |1|, |-1|, |2| \} = 2$$

What is x here x is 1 minus 1 2, now norm 1 of this vector we have already defined it is i from 1 to n mod of x i’s norm 1 and here n is 3. So, it is summation i from 1, 2 3 mod of x i’s which is mod 1 plus mod minus 1 plus mod 2 which is 4.

What is norm of this norm 2 is simply square of each x i’s and the under root of this power half that will be it is 4 plus 1 plus 1 6 that is under root 6 norm. Infinite norm of this will be maximum of maximum over i of mod x i.

Now, mod of first quantity is 1 mod of second quantity is 1 mod of third quantity is 2 that is maximum of mod 1 mod minus 1 mod 2 which is 2. So, this is norm infinite norm of this vector, so in this way we can find out norm 1 norm 2 norm infinite or p norm for any p greater than equal to 1.

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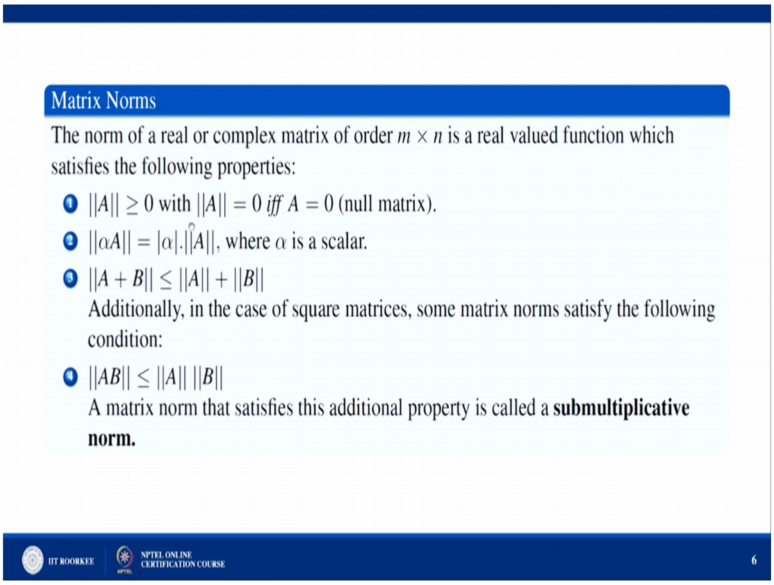

Now, how can you define matrix norm, norm of a matrix. So, let us see here suppose we are having a m cross n real or complex matrix. So, norm of a real or complex matrix of order m cross n is a real valued function which satisfy the following properties. What are those properties? The first property is a same as a vector norm that is norm of a matrix is always greater than equal to 0 and this is equal to 0 if and only if A is a 0 matrix or a null matrix the first property is same. The second property is also same that is norm of alpha A is equal to mod of alpha times norm of A where alpha is a scalar maybe the (Refer Time: 23:48) are complex.

Now, norm of A plus B is less than equals to norm of A plus norm of B. So, these three properties are similar to the those properties which we have discussed for finding norm of a vector. Instead of here instead of vector we are having a matrix A. Additionally in case of a square matrices some matrix norm satisfy this condition also that is norm of A plus B A into B is less than equals to norm of A into norm of B. And the matrix norm that satisfy this addition property is called sub multiplicative norm, that is true for square matrices may be true for a square matrices.

Now, what are the various methods to induce a norm in a matrix. So, what are various things? The first is maximum column sum which we are denoting by a norm 1 of a matrix A.

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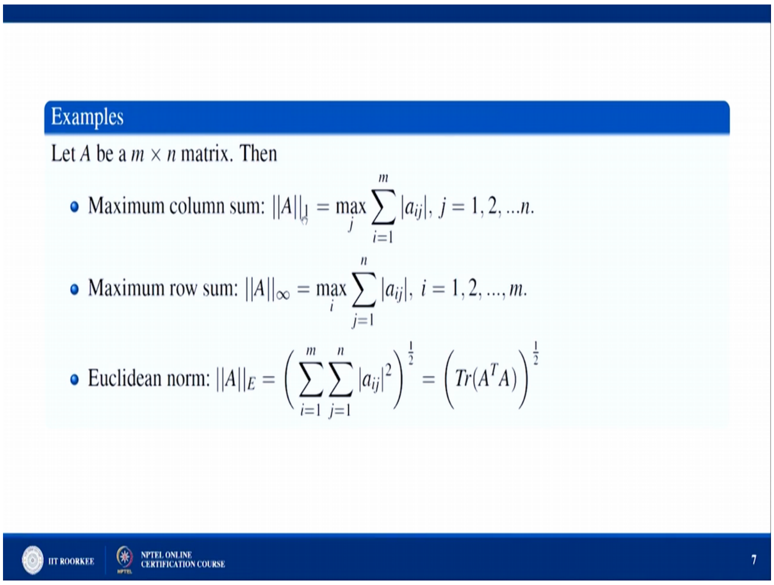

What is this is you sum all the mod of all this columns and you take the maximum over j that is the maximum column sum let us let us take this thing then it should be clear.

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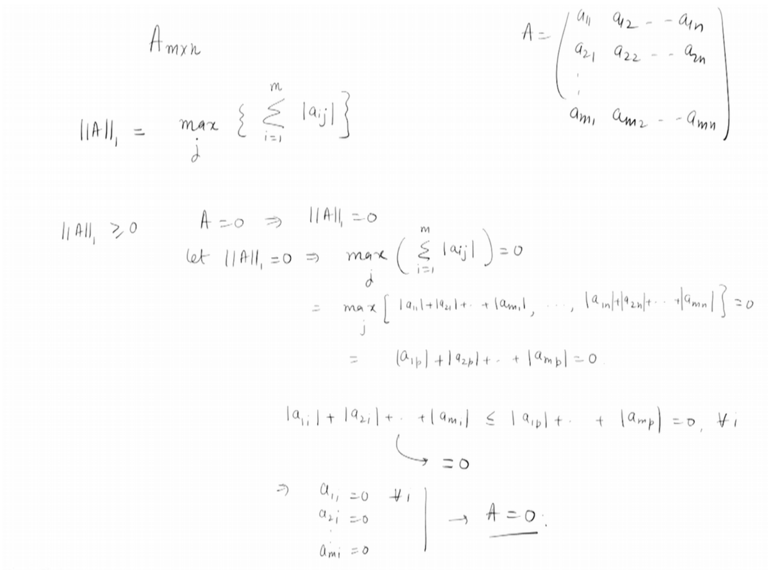

You see you are having a matrix of order m cross n matrix A may be a 1 1 a 1 2 a 1 n a 2 1 a 2 2 and so on up to a 2 n a m 1 a m 2 and so on up to a m n this is a matrix A.

Now, what is norm 1 of this matrix norm 1 this we are taking as maximum you are adding all these columns, mod of this column mod of this element plus mod of this element mod of this element the first element. Similarly, mod and some of this second column third column and then you are taking the maximum; that means, what; that means, you are taking the some mod of a i j, here i is varying from 1 to n and the maximum over j. I think that is the matrix here it is i from 1 to m i is varying from 1 to m.

Now, let us try to prove that this is a norm first of all; that means, satisfy all the three properties. So, the first property is norm of norm 1 is always greater than equal to 0 which is of course, true because you are adding the adding the non-negative quantities. And then taking the maximum which cannot be negative, so it is; obviously, true by the definition itself.

Now, if a is a null matrix then this is if a is a null matrix if a is equal to 0 then of course, norm 1 of matrices equal to 0. Now let norm 1 of this matrix is equal to 0 then we have to show that a only possibility is a is 0, I mean null matrix then norm is equal to 0 this means what this means maximum over j sum i from 1 2 m mod of a i j is equal to 0. Now, maximum of a i j maximum of sum of this is equal to 0 means what you see this means maximum over j the first element is mod a 1 1 plus mod a 2 1 plus and so on, mod a 1 m 1 the second and similarly the last element is mod a 1 n plus mod a 2 n plus and so on, mod a m n is equal to this is equal to 0 maximum of this is equal to 0.

Now, maximum of this is equal to 0 means what each element, I mean suppose maximum is obtained at j equal to p; that means, this suppose this is equals to a mod a 1. Suppose it is obtained at p; that means, mod a 1 p plus mod a 2 p plus and so on, mod a m p is equal to 0 if maximum is obtain at j equal to p. Now, each values each a 1 i 1 i plus a 2 i plus and so on, a m i will be less than equals to a 1 p plus and. So, on a m p because this is a maximum value and which is 0 for all i, and this sum of non-negative quantity is less than equal to 0, it cannot be negative because it is a sum of non-negative expressions.

So, this means it is equal to 0 this means it is equal to 0 and if it is equal to 0 this means each a i j each a 1 i equal to 0 for all i a 1 i equal to 0 a 2 i equal to 0 so; that means matrix A is equal to 0. So, we have shown that the first property holds for norm 1 of a matrix A. Now, if you go to a second property if using this definition if you go for a second property.

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$$\|A\|_1 = \max_j \left(\sum_{i=1}^{m} |a_{ij}| \right)$$

$$\|KA\|_1 = \max_j \left(\sum_{i=1}^{m} |k a_{ij}| \right) = |k| \left(\max_j \sum_{i=1}^{m} |a_{ij}| \right)$$

$$= |k| \, \|A\|_1$$

$$\|A+B\|_1 \leq \|A\|_1 + \|B\|_1 \checkmark$$

$$\|A+B\|_1 = \max_j \left(\sum_{i=1}^{m} |a_{ij} + b_{ij}| \right)$$

$$|a_{ij} + b_{ij}| \leq |a_{ij}| + |b_{ij}|$$

$$\sum_i |a_{ij} + b_{ij}| \leq \sum_i |a_{ij}| + \sum_i |b_{ij}|$$

Norm 1 is defined as maximum over j sum i from 1 to m mod of a i j. So, norm of KA 1 will be maximum over j. It is summation i from 1 to m mod of k i j which is mod k times it can be easily taken out because k is a scalar expression a scalar quantity, which is maximum over j sum over i from 1 to m mod of a i j. So, it is mod K times norm A of 1, and similarly we can show the third properties also that norm of A plus B norm 1 of A plus B is less than equals to norm 1 of A this property we have to show. You see you take the left hand side, which is equals to maximum over j some i from 1 to m mod of a i j plus b i j maximum over this.

Now, again you can use mod of a i j plus b i j is less than equals to mod of a i j plus mod of b i j. Now you take the some over i both the sides, if you take the sum of over i both the sides and now you take the maximum both the sides you will get the required expression a required this expression.

So, hence we can say that the definition of this is a norm induced on the matrix A, which we are calling as norm 1. Similarly, we can define maximum rows some in instead of column sum we are taking row sum and then we are taking the maximum. So, this we are denoting by a non-infinite of a matrix A, similarly we can go the proof of this next is Euclidean norm.

Now, Euclidean norm is simply you, you take square of all the elements of the matrix A and the under root of this sum of a square of all the elements of matrix A. And you take the under root this is a Euclidean norm of matrix A and this can also be obtained by a finding trace of A transpose A, trace is sum of diagonal elements. You find the sum of diagonal elements of A transpose A take put it under root new you get the same expression, this you can easily verify and it is a norm this we can prove also. So, these are three different norms which you can be induced on a matrix A, this is norm 1 this is infinite norm and this is Euclidean norm on a matrix A.

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$$A = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 1 & -2 \\ -2 & 0 & 2 \end{pmatrix}$$

$$\|A\|_1 = \max\{|2|+|3|+|-2|,\ |-1|+|1|+|0|,\ |0|+|-2|+|2|\}$$
$$= \max\{7,\ 2,\ 4\} = 7$$

$$\|A\|_\infty = \max\{|2|+|-1|+|0|,\ |3|+|1|+|-2|,\ |-2|+|0|+|2|\}$$
$$= \max\{3,\ 6,\ 4\} = 6.$$

$$\|A\|_E = (4+1+0+9+1+4+4+0+4)^{1/2} = (27)^{1/2}$$

Now, suppose you have this a problem. In this problem what is A? A is 2 minus 1 0 3 1 minus 2 minus 2 0 2.

Now, what is norm 1 of this matrix? Norm 1 of this matrix is simply you find absolute of sum of each column and then take the maximum; that means, maximum of mod of mod of for the 2 is 2 mod of 2 plus mod of 3 plus mod of minus 2, mod of minus 1 plus mod of 1 plus mod of 0 mod of 0 plus mod of minus 2 plus mod of 2. Now, here maximum will be it is 2 plus 2 plus 3 that is 7, then it is 2 then it is 4, so that is 7.

So, a maximum I mean a norm 1 of this matrix is 7. Now, what is norm infinite norm of this matrix, it is maximum of from you take now the columns now the rows sorry. It is mod 2 plus mod minus 1 plus mod 0 mod 3 plus mod 1 plus mod minus 2 mod minus 2 plus mod 0 plus mod 2, it is maximum of 2 plus 1 3 5 and 1 6 and it is 4.

So, maximum is 6 and what is Euclidean norm of this matrix A it is you simply you find square of all the elements sum of square of all the elements it is 4 plus 1 plus 0 plus 9 plus 1 plus 4 plus 4 plus 0 plus 4 and the under root of this, which is it is you can easily find. It is 4, 8, 12, 12, 14, 18, 18 plus 9 is 27 ok, again check it is 5 5 plus 10 is 15 15 plus 12 is 27, so it is 27 under root of 27. So, in this way we can find out norm 1 infinite norm or Euclidean norm of matrix.

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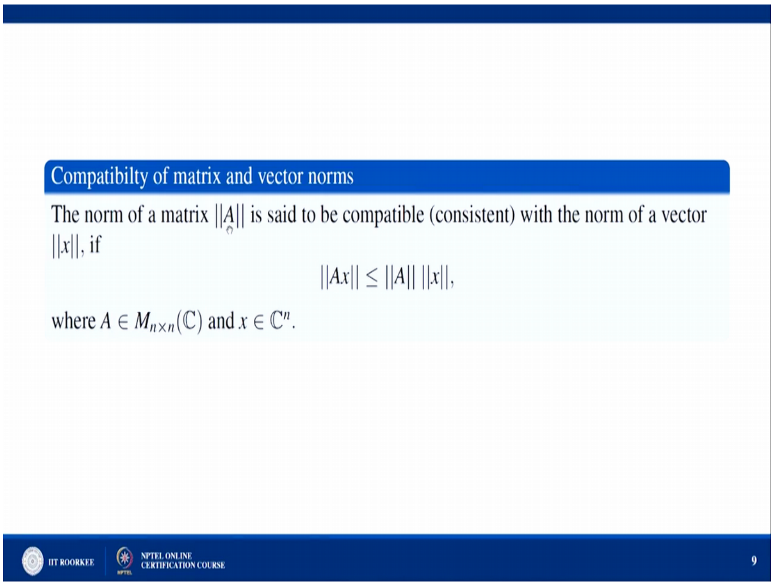

Now, when we can say that matrix is compatible a norm of a matrix which is given by this is said to be compatible or consistent with a norm of vector x if norm of A into x is a here a x is a vector and A is a matrix.

So, norm of A x if it is less than equals to norm of A into norm of norm of x, then we say that norm of a matrix is compatible with the with the norm of vector x. So, if this inequality hold then we say that norm of a matrix A is compatible with a norm of a vector.

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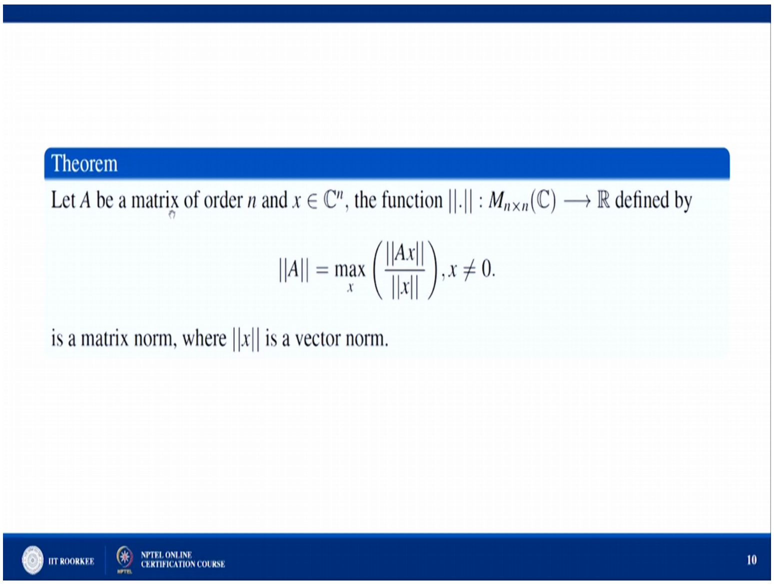

Now, we have again another result let a be a matrix of n and let C belongs to let x belongs to C n then the function, which is a matrix of order n cross n C 2 R defined by this expression, where x is non 0 is a matrix norm it is also matrix norm here x is a vector. So, the proof is very easy one can easily show the proof you have to show all the three properties of a matrix norms this can be easily obtained. So, we can show that we can easily show that this is a norm a spectral norm.

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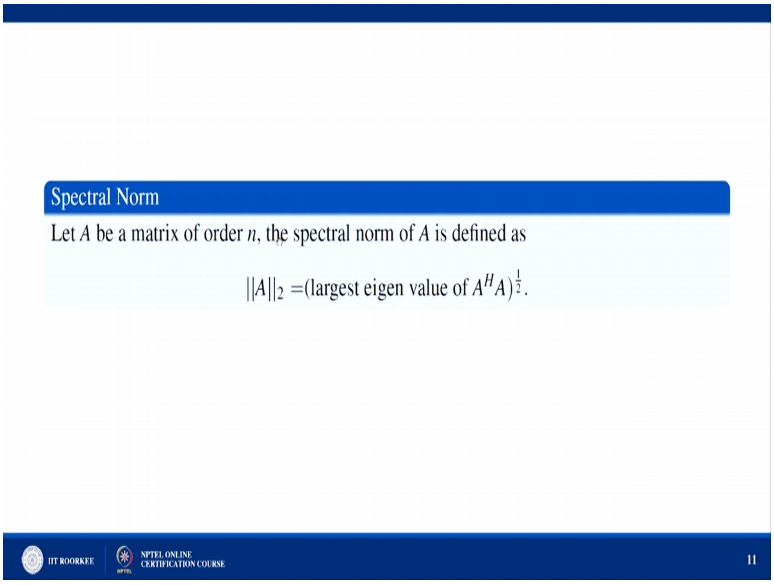

Now, let A be matrix of order n, the spectral norm of a matrix A is defined as largest eigenvalue of a conjugate transpose A whole raise to power 1 by 2. If A is a real matrix it will be A transpose A. If A is a complex matrix, it is A conjugate transpose A whole raised to power half.

So, you find the largest eigenvalue of this matrix take the under root of that, that is the norm 2 of a matrix A, and we are also calling as spectral norm of A. So, for a real matrix spectral norm is simply largest eigenvalue of A transpose A whole raise to power 1 by 2. Suppose you want to find out the spectral norm of this matrix.

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$$A = \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$$

$$A^H A = A^2$$

$$\|A\|_2 = (\text{largest e-values of } A^H A)^{1/2}$$

$$= (\text{'' \quad '' \quad ''} \ A^2)^{1/2}$$

$$= (9)^{1/2} = 3.$$

$$A^H = \bar{A}^T = \begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix} = A$$

$$|A - \lambda I| = 0 \Rightarrow (1-\lambda)(2-\lambda) - (1-i^2) = 0$$

$$\lambda^2 - 3\lambda + 2 - 2 = 0$$

$$\lambda = 0, 3.$$

So, what this matrix is? This matrix is 1 1 plus iota 1 minus iota 2. It is basically a complex matrix, so we will find A conjugate transpose A. So, what is A conjugate transpose, which is A conjugate transpose. What is A conjugate first? A conjugate is 1 1 minus iota 1 plus iota and 2 its transpose which is 1 1 minus iota 1 plus iota and 2. So, it is basically A itself, so it is basically Hermitian matrix; if A is equals to A conjugate transpose means it is a Hermitian matrix.

So, it is basically A square. For norm 2 of A for norm 2 of A, you have to find out the largest eigenvalue of A conjugate transpose A whole raise to power 1 by 2 that is largest eigenvalue of A square whole raise to power 1 by 2. So, you find out the largest eigenvalue of A square it that will be the largest eigenvalue of A square because we know that if lambda is a eigenvalue of A then lambda square will be the eigenvalue of A square.

So, you find out the eigenvalues of this matrix. How to find Eigen values of matrix? You simply put this equal to 0. So, this implies 1 minus lambda 2 minus lambda minus 1 minus iota square will be 0. So, this is lambda square minus 3 lambda plus 2 minus 2 equal to 0, so that is lambda equal to 0 and 3.

So, the largest eigenvalue of A is 3. So, what is the largest eigenvalue of A square? It is 9. So, this will be simply 9 raise to the power 1 by 2 which is 3. So, in this way we can find out the spectral norm of a matrix A or norm 2 of a matrix A. So, this is how we can find out norm of a vector and norm of a matrix.

So, in the next few lectures we will see that how we can what are the applications of norm of a matrix.

Thank you.