Matrix Analysis with Applications
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Lecture – 18
Inner Product Spaces

Hello friends, welcome to lecture series on Matrix Analysis with Applications. Our today's lecture is based on inner product. What inner product is and how we will see some important properties of inner product also.

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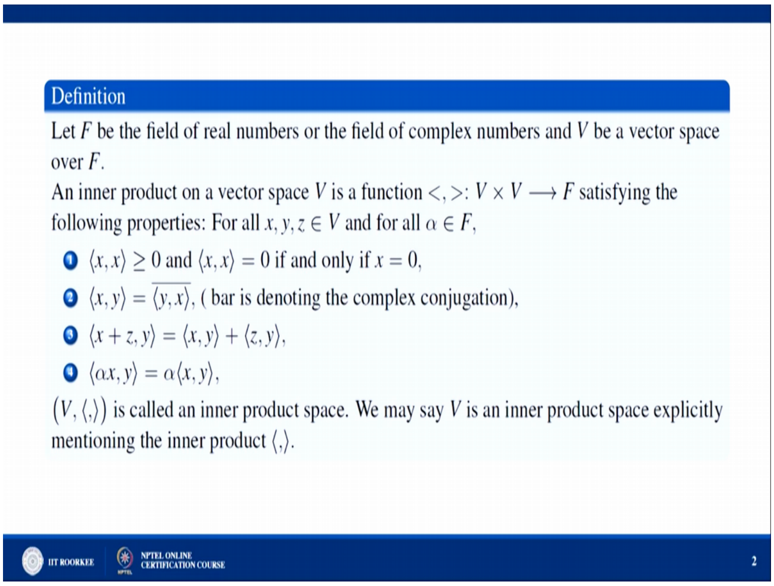

So, what are inner product let us see. So, let F be a field of real numbers or the field of complex numbers and V be a vector space over F. An inner product on the vector space V is a function which is denoted by this from V cross V to F satisfying the following properties. So, these are the four properties which have at inner product must be must hold for every x, y, z in V and for every alpha in field. So, what are these four properties let us see.

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$$1.\quad \langle x, x\rangle \geq 0 \quad \& \quad \langle x, x\rangle = 0 \iff x = 0.$$

$$2.\quad \langle x, y\rangle = \overline{\langle y, x\rangle}$$

$$3.\quad \langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$$

$$4.\quad \langle \alpha x, y\rangle = \alpha\langle x, y\rangle.$$

$$\langle\ ,\ \rangle : V\times V \to F$$

$$\langle x, \alpha y\rangle = \overline{\langle \alpha y, x\rangle} = \overline{\alpha\langle y, x\rangle} = \bar{\alpha}\,\overline{\langle y, x\rangle} = \bar{\alpha}\langle x, y\rangle.$$

The first property is the first property is inner product of x cross x is always greater than equal to 0; and it is equal to 0 if and only if x is equal to 0 is a first property. The second property is inner product of x and y is equal to inner product of y and x whole bar, where bar is a complex conjugate. The third one is inner product of x plus y comma z is same as inner product of x, z plus inner product of y, z. And the four property is the four property is inner product of inner product of alpha x comma y is same as inner alpha times inner product of x comma y where alpha is any scalar in field.

So, if you have a function in which is defined like this from V cross V to F satisfying these four properties then that product is called inner product. And that vector space defined in which this inner product is defined is called inner product space.

Now, one important property which can be seen from the four property you see if you take inner product of x comma alpha y so that will be equal to inner product of alpha y comma x whole bar by second property. So, this can be written as using this property, this can be written as alpha time inner product of y x whole bar which is equals to alpha bar inner product of y comma x whole bar. And this is alpha bar. And the inner product the complex conjugate of this from the second property is equals to inner product of x comma y. So, if this scalar is in the second with the second component, then it is alpha bar; and if it is a first component, then it is alpha.

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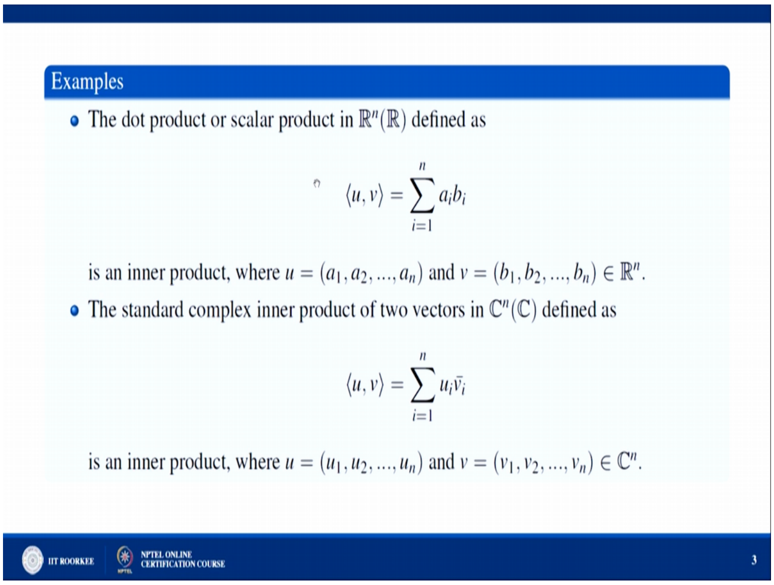

Now, let us discuss first few examples based on this. The first example is the dot product in R n. If we define a dot product in R n which is defined as this then this is a inner product space defines the inner product. Let us see how.

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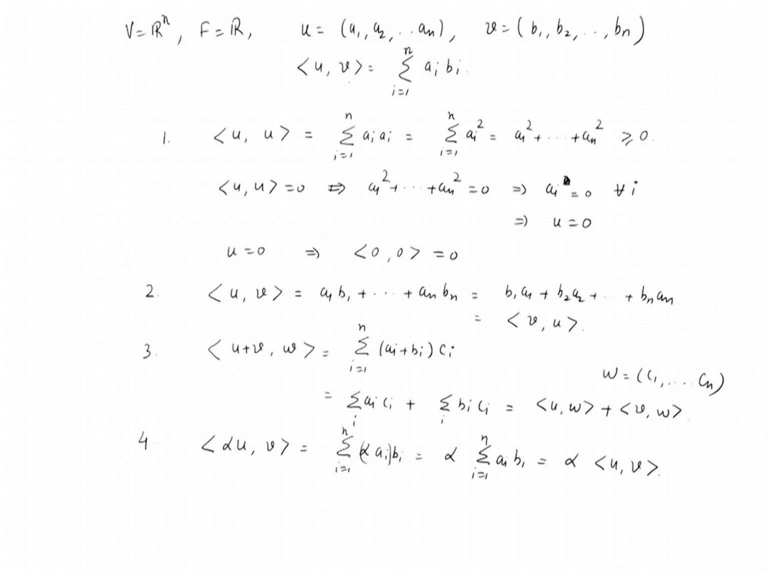

You see you are taking V as R n and field as R ok. And if you take u as a 1 a 2 up to a n; and V as b 1 b 2 up to b n, then the inner product of u and v is simply summation of a i b i, i from 1 to n. Now, this can easily be shown you see. The first property is inner product of u comma u. What are inner product u comma u, it is summation i from 1 to n a i into a i, which is simply summation i from 1 to n a i squares or a 1 square plus a 2 square up to a n square and that is always greater than equal to 0.

And this inner product is equal to 0 if and only if you see if it is equal to 0, this implies sum of a i is equal to 0. And this implies a i equal to 0 for all i; a i equal to 0 for all i. And this means u equal to 0. And of course, if u equal to 0, then this implies inner product of 0 comma 0 is of course 0 by this definition. So, the first property holds.

Now, for second property you can see inner product of u comma v if you write it is a 1 b 1 plus a 2 b 2 and so on up to a n b n, this can be written as b 1 a 1 plus b 2 a 2 and so on up to b n a n. So, this is same as inner product of v comma u, because we are talking in real space I mean real field is a real vector space so bar is itself.

Now, the third property is third property is inner product of u comma v comma u plus v comma w is equals to you can see if you if you take w as c 1 c 2 up to C n then this will be equals to summation i from 1 to n it is a i plus b i times c i by this definition which is summation over i a i c i plus summation over i b i c i. And this is equal to inner product of u and w plus inner product of v and w. So, third property also holds.

Now, the fourth property is inner product of alpha times u comma v is equal to this is summation i from 1 to n. So, it is alpha times a i b i as you can already see. And this is equals to alpha times summation i from 1 to n a i b i which is equal to alpha times inner product of u comma v. So, hence we have seen that all the four properties hold over this definition when we define the inner product by this. All the four properties hold. So, this defines inner product. And the vector space over which this inner product is defined we are calling that vector space as inner product space.

So, there may be some other ways also to define the inner product over R n, this is this is one of the way we are defining the inner product between u and v. Now, if you take C n I mean vector space as C n over the complex field then the inner product may be defined like this. This is again formed an inner product this can easily be verified.

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$$V = \mathbb{C}^n, \quad F = \mathbb{C} \qquad u = (u_1, u_2, \ldots u_n), \quad v = (v_1, v_2, \ldots v_n)$$

$$\langle u, v \rangle = \sum_{i=1}^{n} u_i \bar{v}_i$$

1. $\langle u, u \rangle = \sum_i u_i \bar{u}_i = \sum_i |u_i|^2 = |u_1|^2 + \cdots + |u_n|^2 \geq 0$

$\langle u, u \rangle = 0 \Rightarrow |u_1|^2 + \cdots + |u_n|^2 = 0 \Rightarrow |u_i| = 0 \ \forall i \Rightarrow u_i = 0 \ \forall i$
$\Rightarrow u = 0$

$u = 0 \Rightarrow \langle 0, 0 \rangle = 0.$

2. $\langle u, v \rangle = \sum_{i=1}^{n} u_i \bar{v}_i = \overline{\sum_{i=1}^{n} v_i \bar{u}_i} = \overline{\langle v, u \rangle}$

3. $\langle u+v, w \rangle = \sum_i (u_i + v_i)\bar{w}_i = \sum_i u_i \bar{w}_i + \sum_i v_i \bar{w}_i \qquad w = (w_1, \ldots w_n)$
$= \langle u, w \rangle + \langle v, w \rangle$

4. $\langle \alpha u, w \rangle = \sum_i (\alpha u_i)\bar{w}_i$
$= \alpha \sum_i u_i \bar{w}_i = \alpha \langle u, w \rangle$

You see we are taking V as C n and field as C. Now, inner product of u and V for any u V n C n is defined like this i from 1 to n; it is u i's v i's bar conjugate of this. We are we are taking u as u 1 u 2 up to u n and v as v 1 v 2 up to V n.

So, the first property is that inner product of u comma u it is equals to summation over i u i into u i bar which is summation over i mod of u i squares. This means mod of u 1 square plus mod of u 2 square and so on up to mod of u n square. So, this is always greater than equal to 0 for all u. Now, if this is equal to 0 this implies mod of u 1 square plus and so on mod of u n is equal to 0 and this implies mod of u n is equal to 0 for all i and this implies u i equal to 0 for all i and; that means, u equal to 0. Now, if u equal to 0 of course, inner product of 0 comma 0 is 0 so, the first property holds.

Now, for second property inner product of u comma v if you take is summation i from 1 to n u i v i bar which can be written as summation i from 1 to n v i u i bar whole bar. And this is inner product of v comma u whole bar. So, second property also holds.

Now, the third property is summation inner product of u plus v comma w that is equals to summation u i plus v i into w i bar over i where we are taking w as w 1 up to w n. So, this is summation over i u i w i bar plus summation over i v i w i bar; and this is inner product of u and w plus inner product of v and w.

The fourth property is inner product of alpha u and w is equals to summation over i alpha u i into w i bar which is equals to alpha times summation over i u i w i bar, which is alpha times inner product of u and w. So, we have shown all the four properties over this definition that means, this defines an inner product over the over the complex over C n over the complex field.

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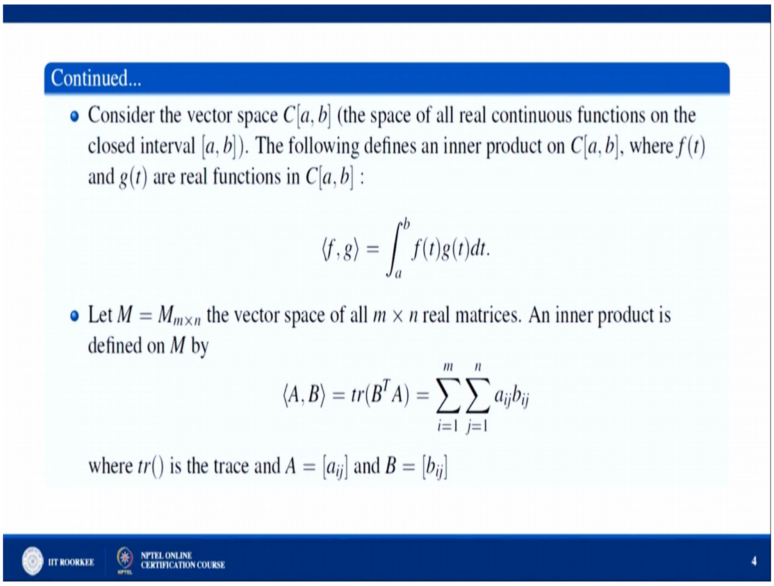

Now, similarly we can go for the third problem that if we define a vector space, if we define if we take a vector space of all real continuous functions on the interval on the close interval a to b, and we defined as the product as integral a to b f t g t d t then this defines an inner product very easy to show. Because you can simply see that if you take f and f inner product f and f then which is simply x square f square from a to b which is always greater than equal to 0 for every f. And if f inner product of f and f is equal to 0 that means, integral a to b f square equal to 0 and that is true only when f equal to 0. So, the first property holds.

Now, when you take inner product of f and g or g or f both are same because f t into g t is same as g t into f t. Now, inner product of f plus g comma h if you take f plus g into h so that property the third property can be directly obtained. And alpha times f comma g is you can simply take alpha outside the integration. And we can get the four property also. So, these definitions define inner product on the vector space C of close interval a b. Now, if you take this example, we considered all the matrixes all the real matrices of order m cross n over the real field, and this defines the inner product. How it is defined inner product let us see.

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$$V = M_{m \times n}, \quad F = \mathbb{R}$$

$$\langle A, B \rangle = \text{trace}\,(B^T A) = \sum_i \sum_j a_{ij} b_{ij}$$

- $\langle A, A \rangle = \text{trace}\,(A^T A) = \sum_i \sum_j a_{ij}^2 \geq 0$

  $\langle A, A \rangle = 0 \Rightarrow \sum_i \sum_j a_{ij}^2 = 0 \Rightarrow a_{ij} = 0 \ \forall\, i \,\&\, j \Rightarrow A = 0.$

  $A = 0, \quad \langle 0, 0 \rangle = 0$

- $\langle A, B \rangle = \text{trace}\,(B^T A) = \sum_i \sum_j a_{ij} b_{ij} = \sum_i \sum_j b_{ij} a_{ij}$

  $= \text{trace}\,(A^T B)$

  $= \langle B, A \rangle$

- $\langle A+B, C \rangle = \text{trace}\,(C^T (A+B))$

  $= \text{trace}\,(C^T A + C^T B)$

  $= \text{trace}\,(C^T A) + \text{trace}\,(C^T B)$

  $= \langle A, C \rangle + \langle B, C \rangle$

- $\langle \alpha A, B \rangle = \text{trace}\,(B^T (\alpha A)) = \sum_i \sum_j (\alpha a_{ij}) b_{ij} = \alpha \sum_i \sum_j a_{ij} b_{ij} = \alpha\, \text{trace}\,(B^T A)$

  $= \alpha \langle A, B \rangle$

So, here we are taking vector space as all matrix of order m cross real matrix of order m cross n and field as real. And inner product between matrices A and B we are defined as trace of trace of B transpose A which is simply summation over i summation over j a i j b i j. Now, the first property is that inner product of A and A inner product of A and A means trace of A transpose A which is which means summation over i summation over j a i j whole square that is always greater equal to 0 of course, because some of non negative quantities.

Now, inner product of A comma A is equal to 0 implies summation i summation j a i j whole square equal to 0 and that is true only when a i j equal to 0 for all i and j and that means matrix A itself is 0. And again if matrix A is 0 the definition trivially holds because it is inner product of 0 and 0 that means, trace of 0 transpose 0 which is of course, 0. So, the first property holds.

Now, if I take the second property that is A and B which is same as trace of B transpose A which is given by a summation i summation j a i j b i j if we this can be written as summation over i summation over j b i j a i j. So, we can easily write with a trace of A transpose B which is inner product of B and A.

Now, a third property is inner product of A plus B and C which is same as trace of trace of C transpose A plus B by the definition which is trace of C transpose A plus C transpose B. And this can be written as by the property of trace the trace of C transpose A plus trace of C transpose B, which is inner product of C and with inner product of a and c sorry a and c and the plus inner product of b and c. So, third property also holds.

Now, for the last property it is trace of inner product of alpha A comma B is equals to trace of B transpose alpha A which can be written as summation over i summation over j here a will we will replace a i j by alpha times a i j. So, alpha times a i j because all you are multiplying A with alpha. So, it is b i j. This can be written as alpha times summation over i summation over j a i j b i j. And this will be alpha times trace of B transpose A or alpha times inner product of A and B. So, hence we can say that all the four properties holds over this definition that means, this defines an inner product for this vector space over this field. So, these are the few examples of inner product.

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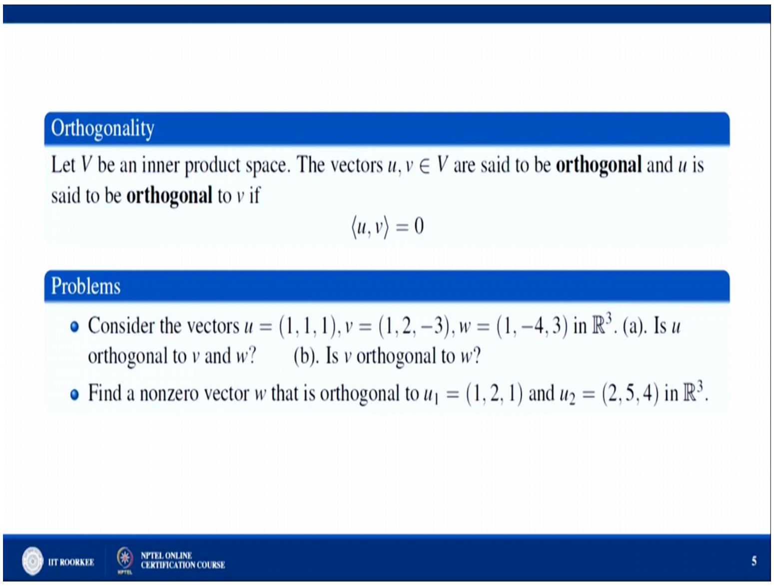

Now, when we can say that two vectors are orthogonal, so two vectors are said to be orthogonal if the inner product of between them is 0. Then we say that u and v are orthogonal. Now, let us see this problem consider vectors u equal to this v and w. So, the first problem is u orthogonal to v and w. So, you see here we are defining inner product as a standard inner product that may a dot product. If nothing is given that means we are taking the inner product as the dot product of the two vectors.

So, what a dot product of u and v it is 1 plus 2 is 3 minus 3 is 0 that means, u is orthogonal to v it is clear. Now, is u is orthogonal w also. So, let us see; 1 into 1 is 1; 1 minus 4 minus 4; 1 minus 4 is minus 3 plus 3 is 0. So, yes, it is orthogonal to w also. So, the first part the also the first part is yes.

Now, is v orthogonal w? So, to see that v is orthogonal to w try to find out the inner product of v and w; and if it is equal to 0; that means v is orthogonal to w. So, see here 1 into 1 is 1; 2 into minus 4 is minus 8; and 3 into minus 3 is minus 9, it is not equal to 0 clearly. So, v is not orthogonal to w. Now, find a nonzero vector w that is orthogonal to u 1 and also u 2.

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$u_1 = (1,2,1), \quad u_2 = (2,5,4) \qquad w \neq 0$

$w \perp u_1$ and $w \perp u_2$

$\langle w, u_1 \rangle = \langle w, u_2 \rangle = 0$

$$w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 2 & 5 & 4 \end{vmatrix}$$

$$= \hat{i}(3) - \hat{j}(2) + \hat{k}(1)$$

$$w = (3, -2, 1)$$

$$\langle w, u_1 \rangle = 3 - 4 + 1 = 0$$

$$\langle w, u_2 \rangle = 6 - 10 + 4 = 0$$

So, what is u 1 and what is u 2. Here u 1 is it is 1, 2, 1; and u 2 is it is 2, 5, 4. So, we have to find a nonzero vector w which is orthogonal to w is orthogonal to u 1 and w is orthogonal to u 2 also that means the inner product of w and u 1 and inner product of w and u 2 is equal to 0. So, so a vector which is orthogonal to these two vector is simply the cross product of these two. You simply find the cross product of these two that is will be a w which is orthogonal to these two vectors. So, how to find w, so w will be simply you find the cross product of these two cross product of these two will be i j k, it is 1 2 1 2 5 4. So, it is i cap times 8 minus 5 is 3 minus j cap time it is 4 minus 2 is 2 plus k cap times it is 1.

So, we can say that w is simply 3 minus 2 1 you can you can check also you see 3 into if you take the inner product of w and u 1. So, what are the inner product it is 3 minus 4 plus 1 which is 0. Inner product of w and u 2 is it is 6 minus 10 plus 4 which is again 0 of course, it will be because it is the cross product of these two vectors and hence w will be orthogonal to these two vectors. Or though other way out is in fact in fact any alpha time this vector 3 minus 2 1 will be orthogonal to u 1 and u 2 both. Now, how you define orthogonal complements.

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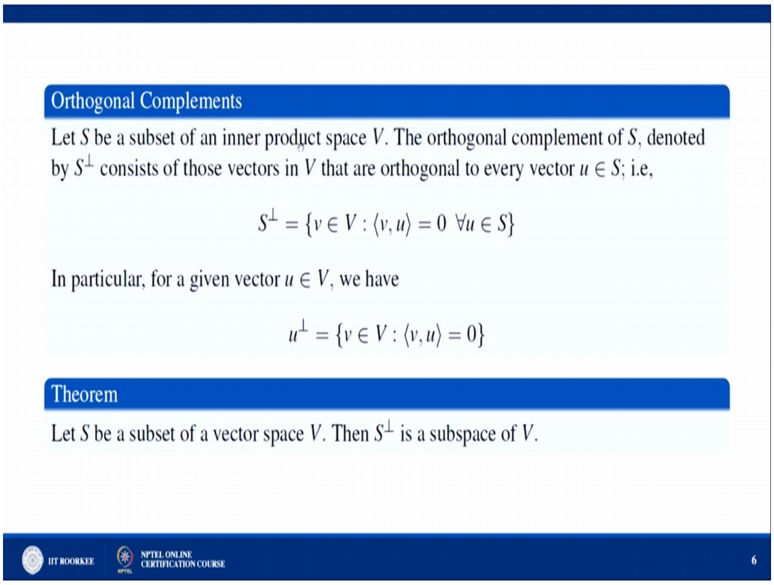

Let S be a subset of an inner product space V the orthogonal complement of S denoted by S perpendicular consists of all those vectors in V that are orthogonal to every vector of u belongs to S u is a subset of inner product space V. So, this is a orthogonal compliment of S which is given by all those v in V such that inner product of v and u is 0 for every u in S. In particular, for a given vector u belongs to V, we have orthogonal compliment of u as that V which is orthogonal to u collection of all those V which is orthogonal to u. Now, the first property of this orthogonal compliment is that it is a subspace of V.

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$S \subseteq V$ $\quad S^{\perp} \subseteq V,$ $\quad S^{\perp} = \{ v \in V \,/\, \langle v, u \rangle = 0, \ \forall u \in S \}$

let $v_1, v_2 \in S^{\perp}$, $\alpha \in F$ $\qquad \alpha v_1 + v_2 \in S^{\perp}$

To Prove

for any $u \in S$,

$$\langle \alpha v_1 + v_2, u \rangle$$
$$= \langle \alpha v_1, u \rangle + \langle v_2, u \rangle$$
$$= \alpha \langle v_1, u \rangle + \langle v_2, u \rangle$$
$$= 0$$
$$\Rightarrow \alpha v_1 + v_2 \in S^{\perp} \Rightarrow S^{\perp} \text{ is a subspace of } V.$$

So, how we define orthogonal complement orthogonal compliment is basically a subset of vector space V which is defined as all those v in V such that inner product of v and u is equal to 0 for all u in S S is a subset of V. Now, in order to show that this is a subspace of this vector space V let v 1 v 2 in S orthogonal compliment and alpha belongs to field. And we have to show that alpha v 1 plus v 2 also belongs to S complement. This to prove to show that it is a it is a subspace we have to show that this also belongs to orthogonal compliment of S.

So, how we will show this? So, let us try to find out the inner product of this with any u, any u in s for any u in s. Now, this is equals to by the definition of inner product, this is this u plus v 2 comma v 2 comma u. Again this is alpha times inner product of v 1 u plus inner product of v 2 u. Now, since u v 1 and v 2 are in orthogonal compliment of S, so this is 0 and this is 0 for any u. So, this is equal to 0. So, we have shown that alpha v 1 plus v 2 belongs to orthogonal complement of s and that means orthogonal complement of S is a subspace of S of V.

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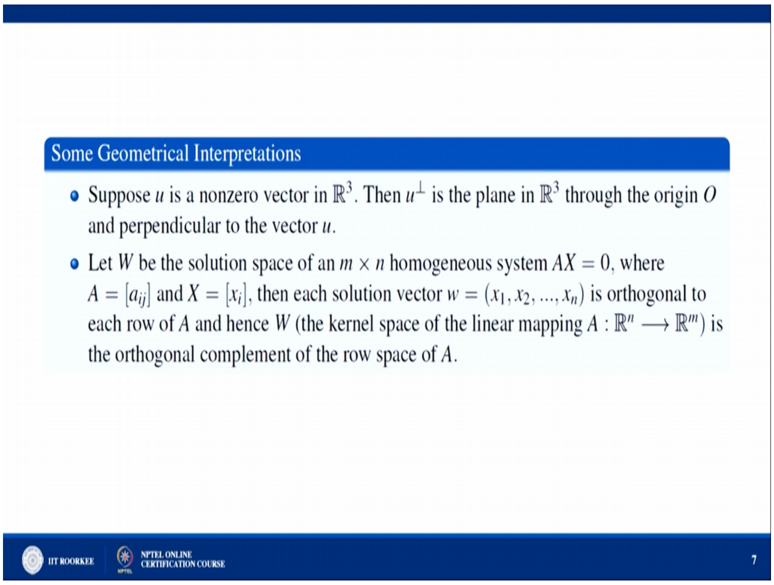

Now, we have few geometric interpretations of a complement orthogonal complement. Suppose that u is a nonzero vectors in R 3 then orthogonal complement of a vector u is a plane in R 3 passing through origin and is perpendicular to the vector u. You see what is what is an orthogonal compliment orthogonal compliment of element are all those V in V such that orthogonal.

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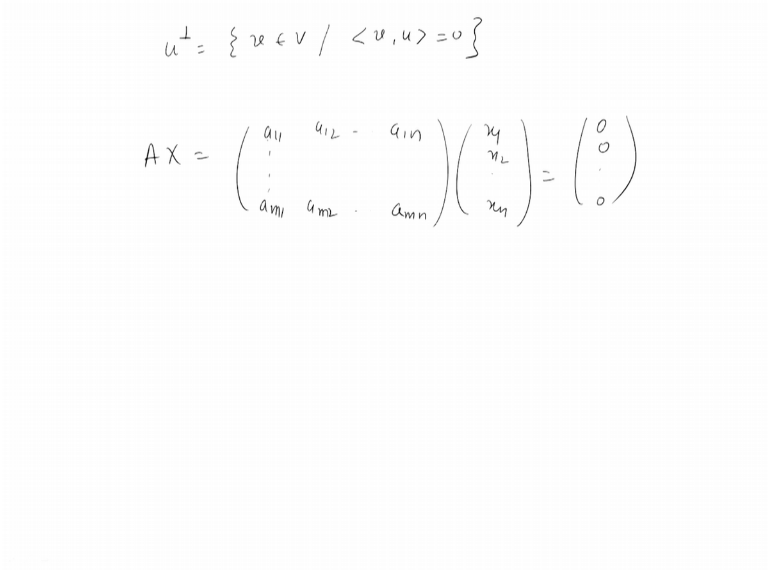

Ah inner product of v and u is equal to 0 is equal to 0. So, how we can how we can see geometry geometrically. So, geometrically it is simply it is simply all the planes it is something like plane you see we have to find out V. All the planes which are passing through origin because it is equal to 0 and perpendicular to u it is very clear from the definition itself.

Now, now let the second one is let W be a solution of a space of an m cross n homogeneous system of equations A X equal to 0. Suppose, for this equation for this system equations the solution space is W ok, where A is a matrix a i j and X is a column vector x i. Then each solution w which is given by x 1, x 2 up to x n is orthogonal to each row of a and hence w the kernel space of the linear mapping a from R n to R m is the orthogonal complement of the row space of A. You see here we have matrix of order m cross n and x is a column vector of order n cross 1.

You see here we are having A into X A is a matrix a 1 1 a 1 2 and so on up to a 1 n. Similarly, a n 1 a n 2 and a m 1 a m 2 a m n and here it is x 1 x 2 up to x n equal to it is 0 0 0. What does this mean this means that when you multiply this row with this column it is equal to 0. The second row with this column is equal to 0; the third row with this column is equal to 0. So, what does it mean this means that that every row every row is orthogonal to this that is that is what is it is here that each solution each solution is orthogonal to each row because the inner product of row and the x i is 0.

So, each solution is orthogonal to each row and hence W is orthogonal complement of the row space of A. So, this W which is a solution space of this is simply the orthogonal complement of row space of A, row space is a space generated by the rows of A. So, this is all about inner product space. So, in this lecture, we have seen that what inner product space is and what are the properties of inner product. We have also seen that if a set is known to you then how we can find out orthogonal complement of that S.

Thank you.