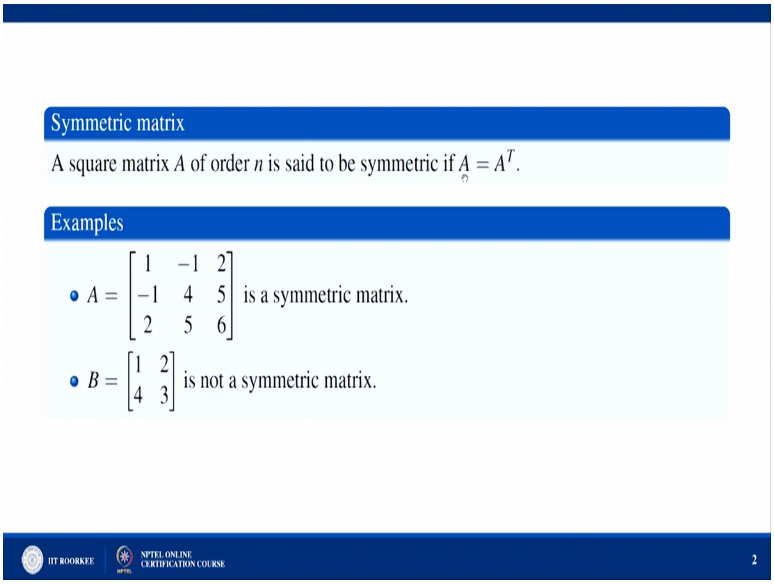
Matrix Analysis with Applications
Dr. S. K. Gupta
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 16
Special Matrices

Hello friends. So, welcome to lecture series on Matrix Analysis with applications. So, today we will discuss about a Special Matrices ok. What is special matrices are and how we can find out eigen values of special practices, what are the important characteristic properties they are having.

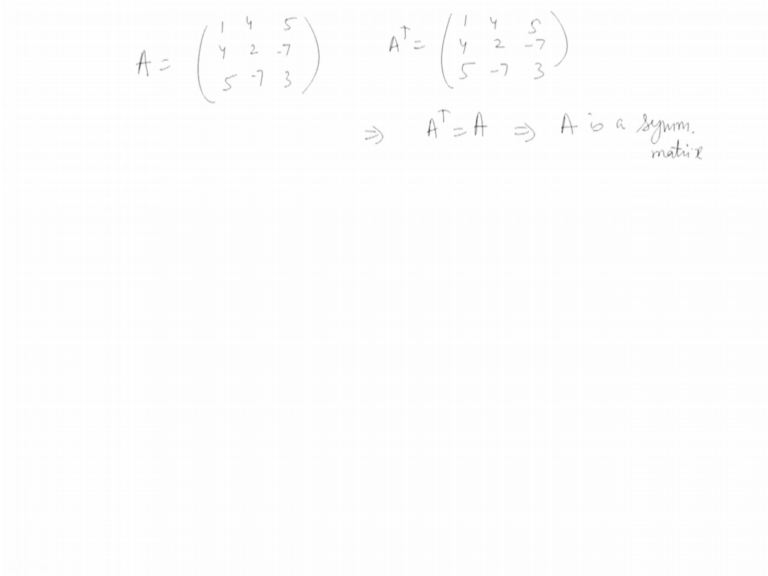
So, first matrix is symmetric matrix.

(Refer Slide Time: 00:46)



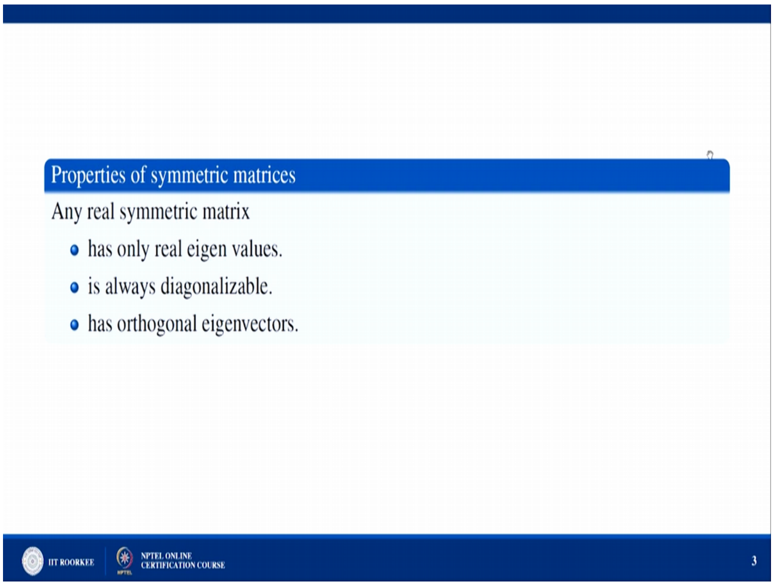
So, a square matrix A of order n is said to be symmetric if A is equal to A transpose, transpose means you interchange rows and columns and if interchanging rows and column the matrix will remain unchanged; that means, matrix is symmetric. Then we say the matrix is symmetric for example, you see this matrix, if you interchange rows and columns ok. So, this matrix is you can find that A transpose is equal to A, if you see this matrix. So, these two elements are not same, that is A is not equal to A transpose here, so this matrix is not symmetric. So, we can always construct a symmetric matrix like this.

(Refer Slide Time: 01:27)



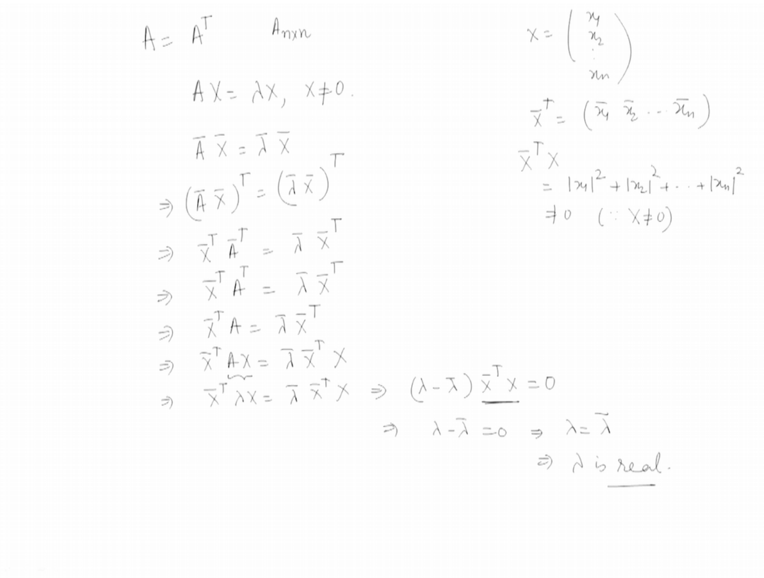
Suppose you are having A matrix you can arbitrary put any diagonal elements, if it is 4 here, 4 here 5 here, 5 here say 7 here minus 7 here minus 7 here. So, if you find its transpose so, it is always 1 4 5, you interchange these rows and columns 4 2 minus 7, 5 minus 7 3. So, you can see easily see that A transpose is equal to A. So, we can say that A is a symmetric matrix ok.

(Refer Slide Time: 02:03)



Now, what are the properties of symmetric matrices? Any real symmetric matrix, real means all entries are real ok, has only real eigenvalues.

(Refer Slide Time: 02:18)



Suppose matrix is symmetric that is A is equals to A transpose and matrix of order n cross n. Now, we know that many matrix A, matrix may have complex roots also. I mean complex eigen values also, but if it is a real symmetric matrix it always have only real eigen values, how we can prove it? So, the proof is simple you can see here.

Let us suppose A X equal to lambda X and X is not equal to 0 ok. So, that means, A as an eigen value lambda and the corresponding eigen vector is x. Now, you take bar both the sides, bar means conjugate, how you can find the conjugate of a matrix? You simply find conjugate of each element of the matrix that will be A bar and is equal to lambda bar X bar, you can take conjugate both the sides, now you take transpose both the sides. So, it is conjugate whole transpose is equal to lambda bar X bar whole transpose. So, this implies X bar transpose, A bar transpose is equal to lambda bar X bar transpose lambda is a, lambda bar is a scalar. So, it is transposes will be itself.

Now, since A is a real matrix so, A bar will be itself. So, we can write is equal to X bar transpose A is equals to lambda bar X bar transpose ok. A A transpose and A transpose is A because A is symmetric. So, this implies X bar transpose A is equals to lambda to the bar X bar transpose. Now, you have to multiply both side by a post multiply both sides by a vector X. So, this is X bar transpose A X is equals to lambda bar X bar transpose X.

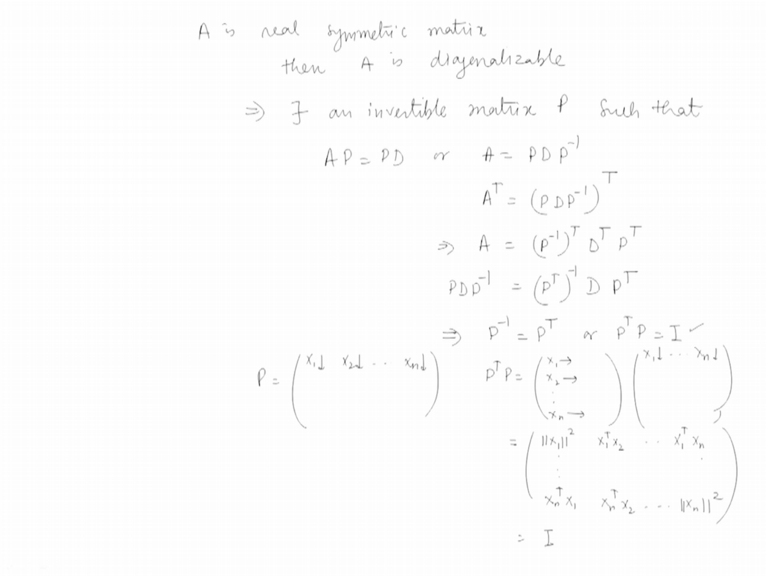
Now, this A X is lambda X this we have already taken. So, it is lambda X lambda bar transpose lambda X lambda bar X bar transpose X. So, this implies lambda can be taken out. So, lambda minus lambda bar times X bar transpose X is equal to 0.

Now, if X is this vector x 1, x 2 up to x n then X bar transpose will be x 1 bar, x 2 bar and so on up to x n bar. And what will be X bar transpose X? This will be mod X 1 square because X 1 bar into X 1 which is mod X 1 square plus mod X 2 square and so on up to mod x n square. And it is not equal to 0 since X is not equal to 0. It will be 0 only when all X i's are 0 and all X i equal to 0 means X equal to 0, but X is not equal to 0. So, that mean this is not equal to 0.

Now, this is not equal to 0 this means lambda minus lambda bar is equal to 0 because product of these two is equal to 0. So, this implies lambda equal to lambda bar and if number and is equal to its conjugate; that means, lambda is purity real, lambda is real. So, we have taken an arbitrary lambda for an symmetric matrix A and we have shown that lambda is real; that means, all eigen values are real for a symmetric matrix. So, this is a first property of symmetric matrix that eigen values are always real.

The next property is it is always diagonalizable; that means, if you have a any real symmetric matrix A then you can always find an invertible matrix p such that AP is equals to PD, if A is symmetric always. Next is it has orthogonal eigen vectors, by using the second property is it easy to show you see because we note that every symmetric matrix is always, every real symmetric matrix is always diagonalizable.

(Refer Slide Time: 06:37)



That means if A is a real symmetric matrix then A is diagonalizable, always diagonalizable ok. Now, that means, this implies there exist an invertible matrix P, invertible matrix P such that, such that A P equal to P D or A is equals to P D P inverse. Where D, D are the these is are the diagonal values of I mean eigen values of a which are all real this we have already seen because a is symmetric ok.

Now, you can take transpose both the sides because A is symmetric. So, it is P D P inverse whole transpose, this is equal to A A transpose is equal to a because A is real symmetric matrix and it is p inverse transpose D transpose, P transpose. It is P transpose whole inverse D transpose is D itself because it is a diagonal matrix and it is P transpose and A is nothing, but P D P inverse. So, it is P D P inverse is equal to this. So, from here we can conclude that one of the, one of the conclusion from here is P inverse is equals to P transpose or P transpose P is equal to identity ok.

Now, now what is P? P is simply a matrix of eigen vectors, we have already seen the construction of such P's. So, what is P? P is a simply X 1, X 2 up to X n these are eigen vector corresponding to lambda 1, lambda 2 up to lambda n. So, what is P transpose P, it will be it will be X 1 in this row, X 2 in this row up to X n in this row and this is X 1 in this column and so on up to X n in this column ok.

So, what is it is equal to, you see this row this column, when you multiply this row this column this is nothing, but norm of X 1 square ok. When you multiply this row with this column, this will be nothing, but X 1 transpose X 2 and similarly it is X one transpose X n X n transpose X 1 X n transpose X 2 and. So, on up to X n norm square and this is equal to identity, this we have shown this equal to identity means all norms of all the vectors I is equal to 1 and all other entries are 0; that means, that means, eigen vectors are orthogonal in fact, orthonormal ok.

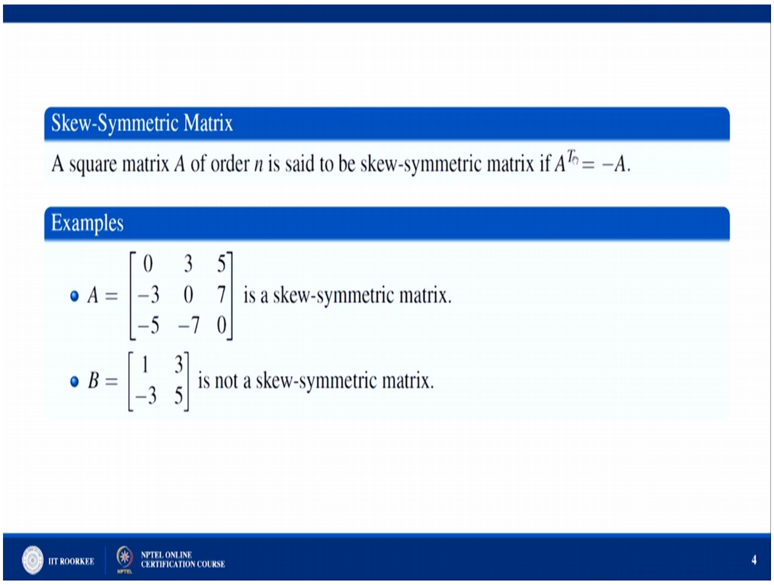
(Refer Slide Time: 09:59)

$$X_i^T X_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$\{X_1, X_2 \cdots X_n\} \rightarrow \text{orthonormal}$$

It simply means that X i transpose X j is equal to 0 if i not equal to j and it is 1 when i equal to j. So that means, this set X 1, X 2 up to X n this set is orthonormal. So, hence we can say that they has orthogonal eigen vectors or if it is orthonormal then it is orthogonal also ok. So, these are the properties of similar matrices. Now, next is and skew symmetric matrix.

(Refer Slide Time: 10:35)



A square matrix a of order n is said to be skew symmetric matrix if A transpose is equal to minus A ok. Suppose you see this matrix, if you take A transpose of this matrix it is simply equal to minus of A so, hence this matrix is a skew symmetric. However, this matrix is not a skew symmetric because when you take minus A transpose it is not equal to minus of A for this problem, minus of B ok. So, what are properties of skew symmetric matrix let us see.

(Refer Slide Time: 11:10)

$A^T = -A, \quad A_{n \times n} \rightarrow$ skew symm. matrix.

$A^T = -A \Rightarrow |A^T| = |-A|$

$\Rightarrow |A| = (-1)^n |A|$

if $n =$ odd then $(-1)^n = -1$

$|A| = -|A| \Rightarrow \underline{|A| = 0}$

Now, for a skew symmetric A transpose is equals to minus A, A is n cross n is skew symmetric ok.

Now, if A is skew symmetric the first property is, that if n is odd order of the matrix is odd that determinant of A is 0, if A is skew symmetric. Proof is simple you can see here A transpose is equal to minus A this implies determinant of A transpose is equal to determinant of minus A. This implies determinant of A transpose is same as determinant of A and determinant of minus 1 minus minus times A is minus 1 raise to power n into determinant of A. So, if n is odd then minus 1 raised to power n will be minus 1. So, determinant of a will be minus determinant of A and this imply determinant of A is equal to 0.

So, the first property is that a skew symmetric matrix of odd order has 0 determinant or any skew symmetric matrix of order odd, of odd order is always singular the first property. Now, second property is, a skew symmetric matrix all the eigen values of a skew symmetric matrix are either 0 or purely imaginary how we can prove this. So, let us see the proof.

(Refer Slide Time: 12:52)

$$A^T = -A, \quad A_{n\times n}$$

$$AX = \lambda X, \quad X \neq 0$$

$$\bar{A}\bar{X} = \bar{\lambda}\bar{X}$$

$$\Rightarrow \bar{X}^T \bar{A}^T = \bar{\lambda}\bar{X}^T$$

$$\Rightarrow \bar{X}^T A^T = \bar{\lambda}\bar{X}^T \Rightarrow -\bar{X}^T A = \bar{\lambda}\bar{X}^T$$

$$\Rightarrow -\bar{X}^T A X = \bar{\lambda}\bar{X}^T X$$

$$\Rightarrow -\bar{X}^T \lambda X = \bar{\lambda}\bar{X}^T X$$

$$\Rightarrow (\bar{\lambda}+\lambda)(\underbrace{\bar{X}^T X}) = 0$$

$$\text{As } \bar{X}^T X \neq 0 \Rightarrow \bar{\lambda}+\lambda = 0$$

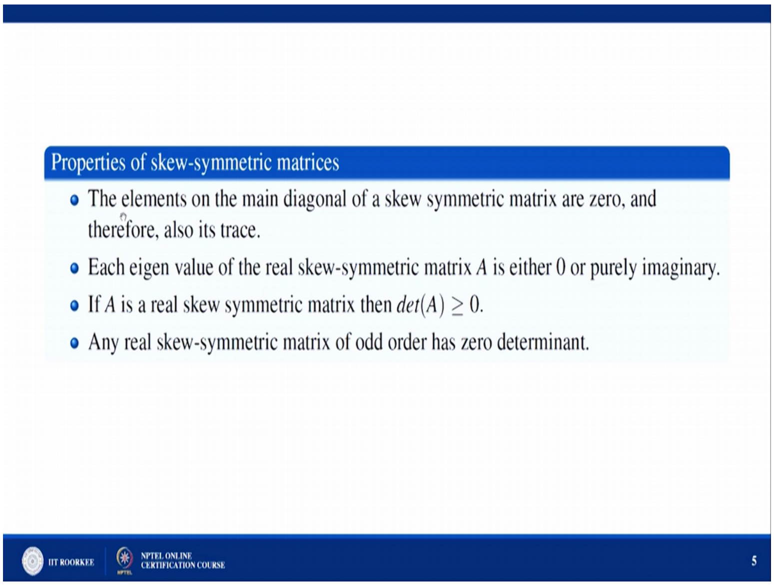
$$\text{or } \lambda = -\bar{\lambda}$$

$\Rightarrow$ either $\lambda = 0$ or purely imaginary.

Now, A transpose is equals to minus A, A is n cross n a skew symmetric matrix. So, A X equal to lambda X, take a X equal to lambda X X is not equal to 0, again take bar both the sides, take transpose both the sides. So, taking transpose we get this expression ok, now since A is a we are we are taking a as a real skew symmetric matrix for this case. So, X bar transpose it is A, A transpose lambda bar X bar transpose, this implies X bar transpose A transpose is minus A. So, it is minus of A is equals to lambda bar X bar transpose post multiply both side by vector X, this implies A X equal to lambda X you can substitute here and this implies lambda bar plus lambda times X bar transpose X is equal to 0 and X bar transpose X is not equal to 0 this we have already showed in the last proof. So, as X bar transpose X is not equal to 0. So, this implies lambda bar plus lambda is equal to 0 or lambda is equals to minus lambda bar and this implies either lambda equal to 0 or purely imaginary ok.

So, for a real skew symmetric matrix eigen values are either 0 or purely imaginary. So, these are second property for a skew symmetric matrix ok.

(Refer Slide Time: 14:44)



So, the property is here the elements of the main diagonal of a skew symmetric matrix are 0 and therefore, its trace is also 0. This is very easy to show, you see that if A matrix has skew symmetric matrix then a ii that a diagonal element diagonal element a 11, a 22, a 33 and so on.

(Refer Slide Time: 15:10)

$$A^T = -A, \qquad a_{ii} = -a_{ii} \quad \forall i$$

$$\Rightarrow a_{ii} = 0 \quad \forall i$$

$$A \to 0, \ \pm \alpha i, \ \pm \beta i, \ldots$$

$$\pm \underline{\alpha i}, \ \pm \underline{\beta i}, \ \pm \underline{\gamma i}, \qquad \alpha i, -\alpha i \to \alpha^2$$

$$\alpha^2 \beta^2 \gamma^2 \ldots \qquad \det(A) \geq 0.$$

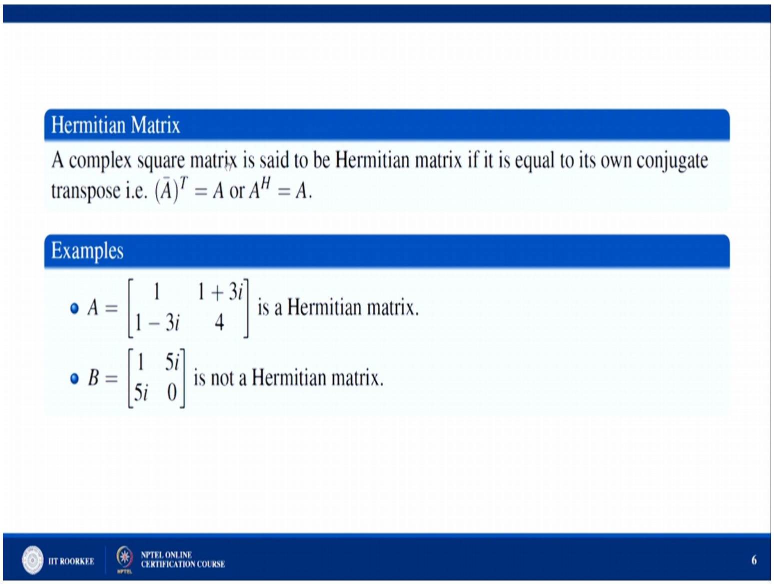
It will be equal to minus of a i i because A transpose equal to minus of a for all i and this implies a ii equal to 0 for all i. This means diagonal elements of a skew symmetric matrix are all are all 0 and hence the trace is also 0 because trace is nothing but sum of diagonal elements.

The next property we have shown that each eigen value of a real symmetric matrix A is either 0 or purely imaginary. Now, if a is a real symmetric matrix then determinant of A is always greater than or equal to 0, this is again easy to show because they have shown that eigen values of eigen values of skew symmetric matrix, real skew symmetric matrix either 0 or purely imaginary ok. If one of the eigen value is 0 so determinant will be 0 because determinant is nothing, but product of eigen values ok. So, A has eigen values like 0 plus minus alpha i plus minus beta i like this, the eigen values of a if a is a skew symmetric matrix.

So, if 0 is one of the eigen value then the product will be 0; that means, determinant is 0, if 0 is not an eigen value suppose eigen values are plus minus alpha i plus minus beta i types. So, if 0 is not the eigen value of a matrix skew symmetric matrix a then the eigen values are of the are of the types plus minus alpha i plus minus beta i plus minus gamma i or so on and determinant of a will be nothing, but product of eigen values.

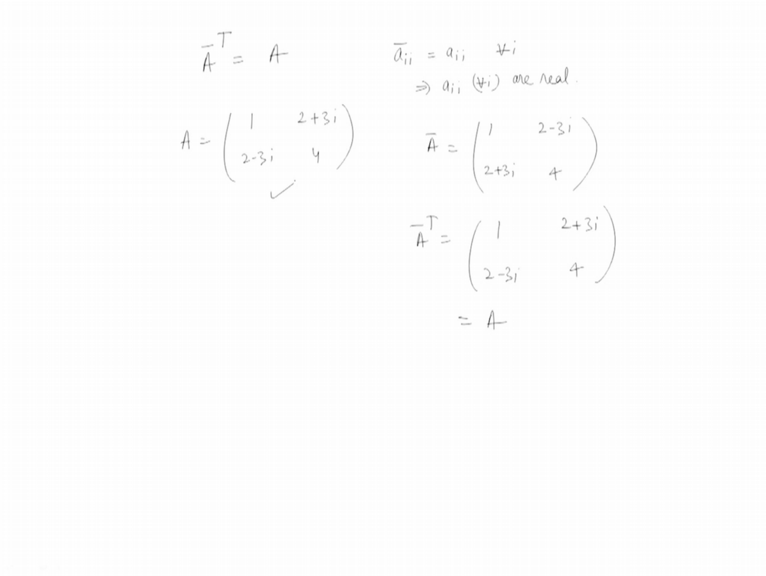
So, if you multiply these two it will be alpha i minus alpha i. So, product will be alpha square, similarly product of these will be beta square product of these will be gamma square. So, product of I mean eigen values will be something like this. So, we can see the determinant of A will be greater than or equal to 0 because complex roots always exist in pairs, any real skew symmetric matrix of order of odd order has determinant 0 that we have already seen.

(Refer Slide Time: 17:24)



Now, Hermitian matrix a complex square matrix A is said to be Hermitian if it is equal to its own conjugate transpose, that is A, A conjugate transpose is equal 2 A. So, let us see this example suppose it is. For the Hermitian matrix we have seen that a, we have seen that it is.

(Refer Slide Time: 17:48)

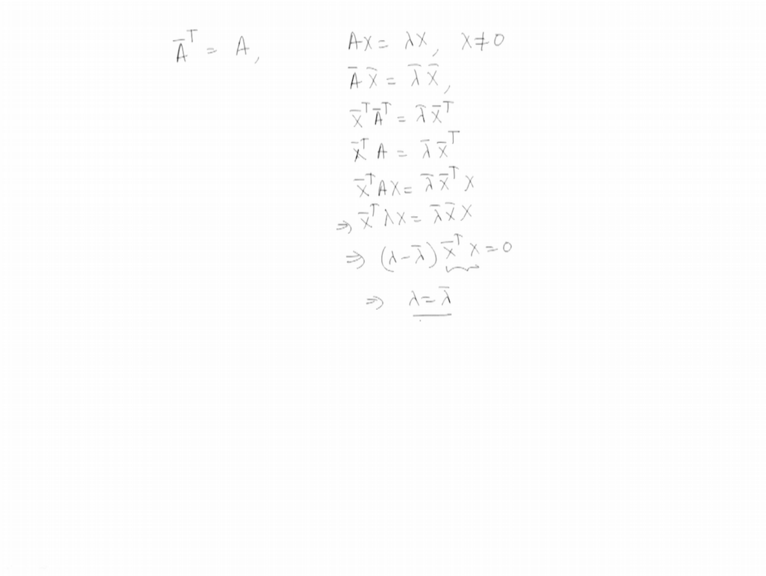


A conjugate transpose is equal to A, the first most property for this matrix is you see that a ii which are diagonal elements conjugate should be equals to a ii for all i. So, this implies a ii for all i's are real.

So, the first most property of Hermitian matrix is that the diagonal elements must be real, it is a complex matrix, but for Hermitian matrix all diagonal elements must be real. So, you take this example suppose it is 1 it is 2 plus 3 iota, it is 2 minus 3 iota and suppose 4. When you take the conjugate of this conjugate is 1 conjugate is 1, conjugate of this is 2 minus 3 iota conjugate of this 2, 3 iota and 4. Now, when you take transpose of this, it is 1 2 minus 3 iota row will go to column 2 plus 3 iota and 4. So, which is equal to A So, we can say that this matrix is a Hermitian matrix.

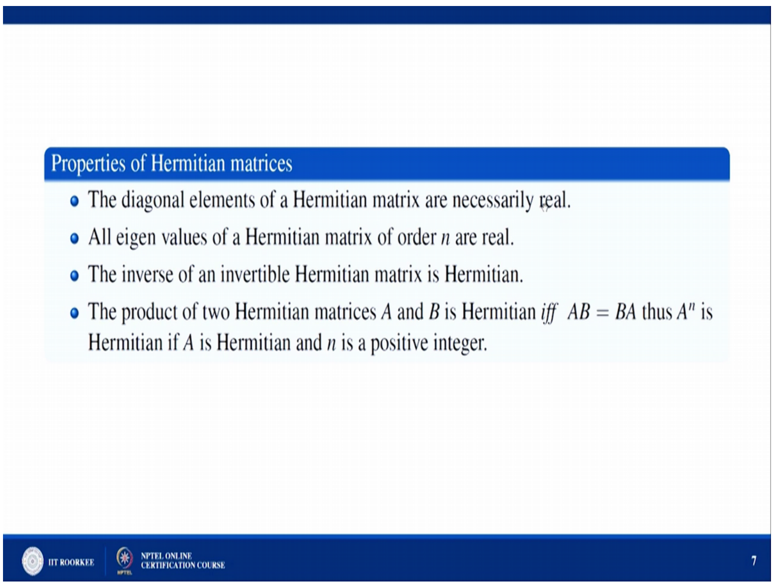
Now, the next property of Hermitian matrix is all eigen values of Hermitian matrix are real, the proof follows on the same lines as we did for symmetric matrix, you take a X equal to lambda X the proof is same, you can see.

(Refer Slide Time: 19:14)



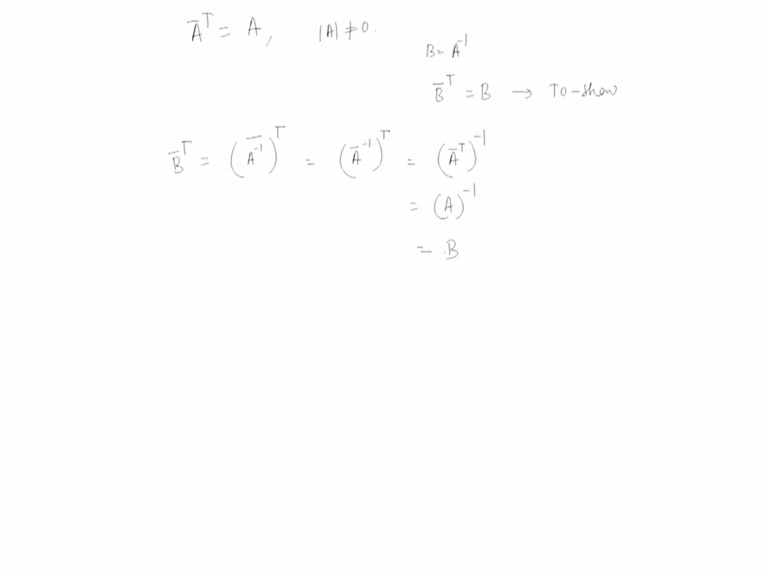
Here we are having a bar transpose equal to A ok. So, AX equal to lambda X you can take here, X a is not equal to 0 take bar both the sides. Take transpose both the sides and replace a bar transpose by A, X bar transpose a is equals to lambda bar, lambda bar transpose it is X bar transpose a X equals to lambda bar, X bar transpose X, X bar transpose it is lambda X and it is lambda bar X bar transpose x. So, this implies lambda minus lambda bar into X bar transpose X equal to 0 and say this is not equal to 0. So, this implies lambda equal to lambda bar. So, hence all eigen values of Hermitian matrix are real.

(Refer Slide Time: 20:12)



So, the properties are of Hermitian matrix is the diagonal element of Hermitian matrix are necessarily real, all eigen values of Hermitian matrix of order n are all real. Now, the inverse of a invertible Hermitain matrix is also Hermitian ok, you can see the proof is very easy you see here a trans a conjugate transpose is equal to a because A is Hermitian.

(Refer Slide Time: 20:35)



We have to show that if A, if this matrix is invertible if this matrix A is invertible, determinant of A is not equal to 0 then A inverse is also Hermitian. So, A inverse Hermitian means for A inverse also for if B is A inverse then B bar transpose is also equal to B this we have to show.

So, you can take B bar transpose, B bar transpose is A inverse whole bar and transpose which is same as A bar inverse whole tend to transpose which is same as A bar transpose whole inverse. This is by the properties of matrices and A bar transpose is A because A is Hermitian. So, this is A inverse and this is nothing, but B. So, we have shown that B is equal to b bar transport; that means, inverse of a Hermitian matrix if A is invertible is also Hermitian.

Now, the product of two Hermitian matrices A and B is Hermitian if and only if A B is equal to B A and thus A raise to power n is Hermitian, if A is Hermitian n is a positive integer. So, let us try to prove this, the second proof is follows on this. So, product of two Hermitian matrix is Hermitian this we have to show if and only if AB equal to BA.

(Refer Slide Time: 22:07)

A, B → two Hermitian matrices

$$\bar{A}^T = A, \quad \bar{B}^T = B.$$

$$\overline{(AB)}^T = (\bar{A}\,\bar{B})^T = \bar{B}^T \bar{A}^T = BA$$
$$= AB$$

$$(\overline{BA})^T = BA$$

$$(\overline{AB})^T = AB \Rightarrow (\bar{A}\,\bar{B})^T = AB$$
$$\Rightarrow \bar{B}^T \bar{A}^T = AB$$
$$\Rightarrow \underline{\underline{BA = AB}}$$

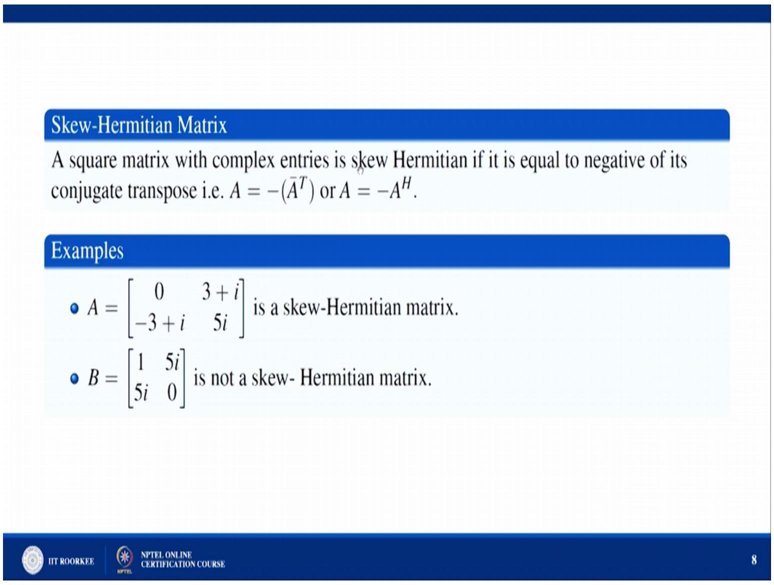
So, let us let us take A and B two Hermitian matrices ok, two Hermitian matrices, that means, A bar transpose equal to A and B bar transpose equal to B.

So, we have to show that A B, A B, A into B is also Hermitian if A B equal to B A. So, let us take this whole transpose and try to show that if A B equal to B A then this is equal to A B. So, this is equals to A bar, B bar whose transpose and this is equals to B bar transpose A bar transpose and this is equals to B into A because A and B are Hermitian matrices and A B is equals to B A. So, this is equals to A B. So, we have shown that product of two Hermitian matrices is also Hermitian if A and B, if A B equal to B A. Similarly, we can show that b a whole bar is transpose is equals to B A, same lines.

Now, the converse part, now let us suppose this is Hermitian that is this is equal to this and we have to show that A B equal to B A the second part. So, to show this part, so this implies A bar, B bar whole transpose is equals to A B and this implies bar transpose A bar transpose is equals to A B and B bar transpose is B and this is A. So, it is equals to A. So, we have shown that this is equal to this.

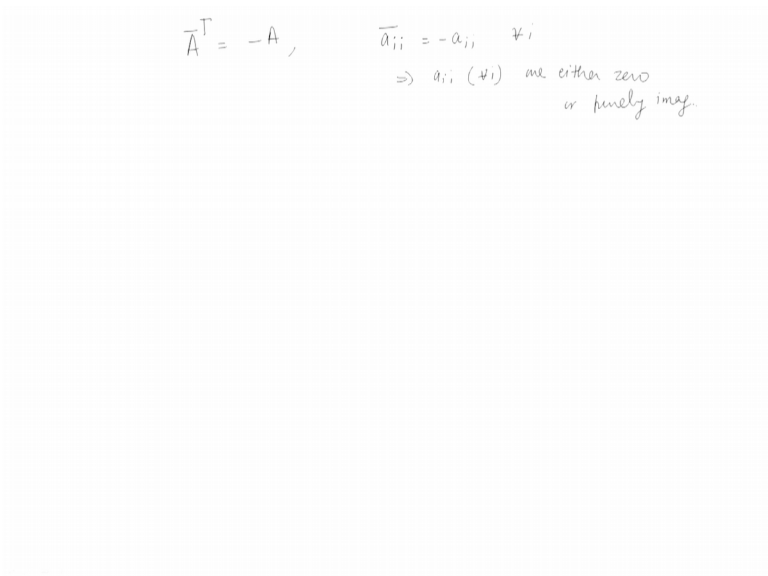
So, hence we can show, hence we can if you substitute B equal to A, then A square I mean A square is always equal to A square; that means, A square is also Hermitian if A is Hermitian. And similarly A cube, similarly A raise to power 4. So, this proof again is simply get so; that means, if A is Hermitian then any power any positive power of A is also Hermitian.

(Refer Slide Time: 24:10)



Now, for a skew Hermitian matrix a matrix with complex entries is called skew Hermitian if, A conjugate transpose is equal to minus of A. So, what are the properties of skew Hermitian matrices?.

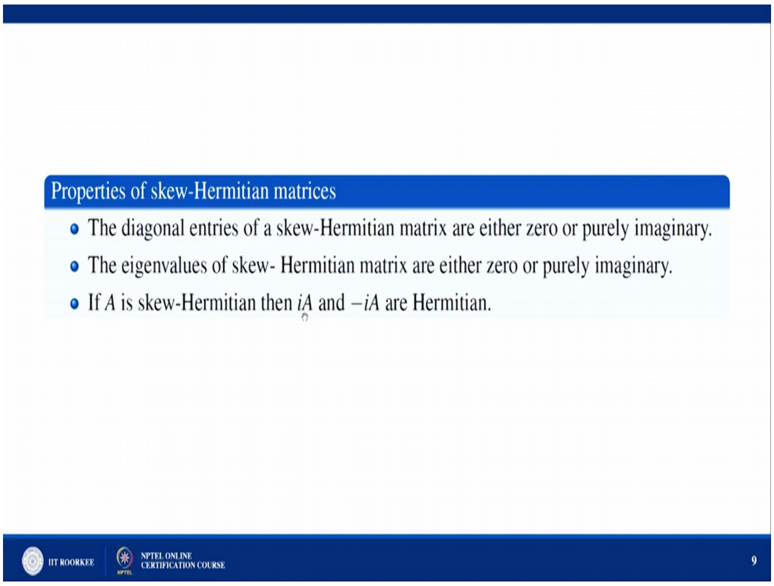
(Refer Slide Time: 24:25)



You see here A bar transpose is equals to minus A; that means, first of diagonal elements. So, this is equals to minus of a a ii for all i and this means a i i i other, a i i for all i are either 0 or purely imaginary ok. The first most point for a skew Hermitian matrix the first property.

The second property is all eigen values of the skew Hermitian matrices are either 0 or purely imaginary, the proof we can easily obtained ok, following the same as we did for a skew symmetric matrix ok.

(Refer Slide Time: 25:11)



Next is if A is a skew Hermitian matrix then iota A and minus iota A are Hermitian. So, this is used to show you see A is a skew Hermitian that means this is equals to minus A.

(Refer Slide Time: 25:21)

$$\bar{A}^T = -A,$$

$$B_1 = iA \qquad B_2 = -iA$$

$$\bar{B}_1^T = \left(\overline{iA}\right)^T = \left(\bar{i}\,\bar{A}\right)^T = -i\,\bar{A}^T = iA = B_1$$

$$\bar{B}_2^T = \left(\overline{-iA}\right)^T = -\bar{i}\,\bar{A}^T = -iA = +B_2$$

You take B 1 is equals to iota A and B 2 is equals to minus iota A. So, if it is B 1 conjugate transpose. So, it is minus iota whole conjugate transpose. So, it is iota conjugate and A conjugate whole transpose. So, conjugate of iota is minus iota and it is A bar transpose and A bar transpose is minus A because A is a skew Hermitian it is given to us. So, it is iota A which is B which is B 1. So, we have shown that B 1 is Hermitian matrix.

So, similarly we can proceed for B 2 if you take B 2 bar transpose so, it is minus iota A bar transpose. So, this is minus iota bar A bar transpose, minus iota bar is iota and this is A bar transpose is minus of A because A is skew hermitian. So, it is equals to minus of B 2. So, it is equal to B 2. So, we have shown that B 2 is Hermitian matrix. So, if A is skew A is I mean Hermitian, A is skew Hermitian then iota A and minus iota A's are Hermitian matrices.

So, here we have seen some of the important matrices like symmetric matrices, skew symmetric matrix, Hermitian matrix and skew Hermitian matrices and we have seen some of the important properties of these special matrices. In the next lecture we will see some more special matrices like orthogonal matrices, unity matrices etcetera and their properties.

Thank you.