Matrix Analysis with Applications

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Lecture – 15
Diagonalization

Hello friends, so welcome to lecture series on Matrix Analysis with Applications. So, today we will discuss about Diagonalization, what do you mean by diagonalization of the matrices?

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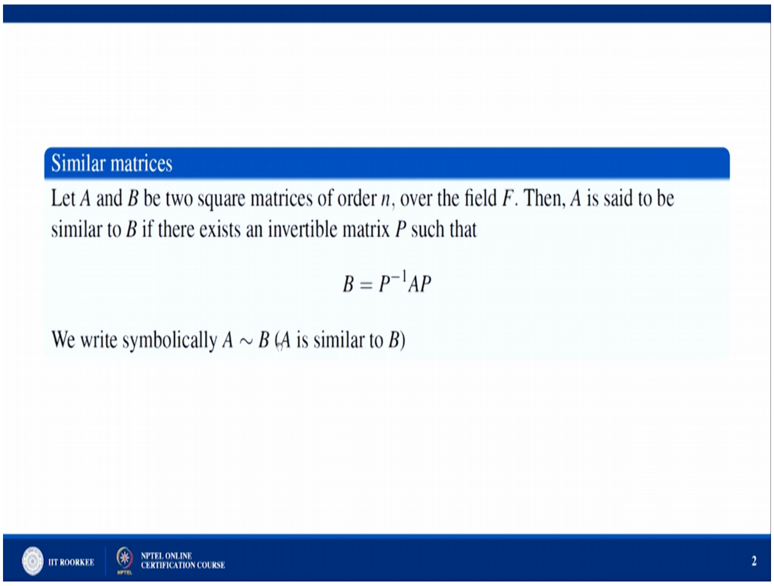

So, let us discuss now first of all before starting diagonalization let us discuss similar matrices. So, let A and B be two square matrix of order n over the field F, then A is said to be similar to B if there exists an invertible matrix P such that B equal P inverse AP and we symbolically write A is A similar to B like this ok. So, what do you mean by basically two similar matrices?

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$A \sim B$ if $\exists$ an invertible matrix P

st

$$AP = PB \quad \text{or} \quad \boxed{B = P^{-1}AP}$$

$$|B| = |P^{-1}AP| = |P^{-1}|\,|A|\,|P|$$
$$= \frac{1}{\cancel{|P|}} \cdot |A| \cdot \cancel{|P|}$$
$$= |A|$$

So, we say that A matrix A ok, A matrix A is similar to B if there exist an invertible matrix, invertible matrix P such that A P is equal to P B or B is equal to P inverse A P ok. So, one thing is very like obvious from this fact. So, what is that that determinant of B and determinant of A are equal. This is very clear from here you can see, the determinant of B will be determinant of P inverse A P, and that will be equal to determinant of P inverse determinant of A determinant of P. Since, determinant of A into B is equal to determinant of A into determinant of B.

Now, determinant of P inverse is 1 upon determinant of P into determinant of A into determinant of P. So, these two cancels out, so it is equal to determinant of A. So, what we obtain the first property of similar matrices is that if 2 matrices A and B are similar then their determinant are always equal, this is a first property of similar matrices. The second property is that if 2 matrices are similar then they have same eigenvalues ok.

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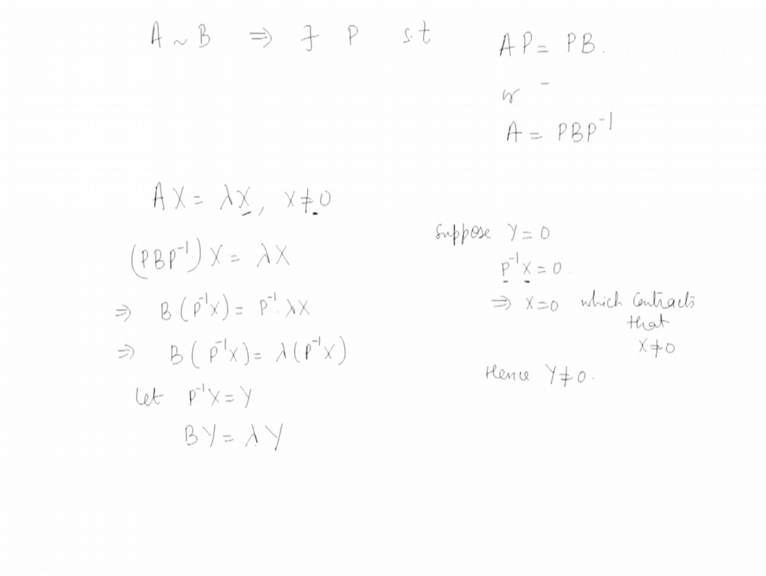

So, how can you prove this? So, if A is similar to B this means this means implies there exist P such that A P is equal to P B or A is equals to P B P inverse ok. If A is similar to B, now we have the second property we have to show that if 2 matrices similar then they have same eigenvalues.

So, how we can proceed for this supposes A x equal to lambda x where x is not equal to 0. So, what does it mean this means that A has an eigenvalue lambda and the corresponding eigenvector is x ok. Now since, A similar to B that means, A is equal to P B P inverse, so you can replace A by P B P inverse ok. Now this implies B into P inverse x this we can easily write you pre multiplied both the size by P inverse because P is invertible. So, it is P inverse of lambda x and this implies B into P inverse x is equal to lambda times P inverse x ok, because lambda is a scalar can be taken out. Now, this P inverse x you can suppose let this P inverse x is y suppose, then this is B y equal to lambda y.

So, what we have concluded we have concluded that B matrix as an eigenvalue lambda and the corresponding eigenvector is y. In order to proved at y is an eigenvector you have to showed at y is not equal to 0 of course. So, that is that is very easy to show suppose y is equal to 0, we can prove this by contradiction if y is equal to 0; that means, P inverse x equal to 0. Now x is not equal to 0 x is not equal to 0 from here because it’s an eigenvector and this matrices invertible this is a system of linear equations ok.

As system of linear equation with this matrices invertible; that means, only unique solution which is the x equal to 0, but x is not equal to 0. So, it is a contradiction because, from here it implies that x equal to 0 which contradicts that x is not equal to 0.

Hence, y is not equal to 0 because, because whenever we write such express that B y equal to lambda y, if y is not equal to 0 then only we can say that the eigenvalues lambda and the corresponding eigenvector is y ok. So, what we have concluded basically we have taken a matrix A with eigenvalue lambda and the corresponding eigenvector x, and using the property of similar matrices we have shown that a matrix B as a same eigenvalue lambda and the corresponding eigenvector is y ok. So, we can say that similar matrices have same eigenvalues of course, their eigenvector may not be same if x is an eigenvector corresponding to lambda here for A, then a corresponding to lambda for B the eigenvector is y which is P inverse x.

Now, if they have same eigenvalues this clearly shows that the characteristic polynomials of similar matrices are same, since roots are same. If roots are same eigenvalues are same; that means, characteristic polynomial of matrix A and B which are similar to each other are also same ok. Now since they have same eigenvalues; that means, that trace of the 2 metals are also same trace is nothing, but sum of eigenvalues. Now if the similar matrix are same eigenvalues; that means, sum of eigenvalues are also same and this implies that trace of 2 matrix are also equal ok.

So, what we have concluded, we have concluded that if 2 matrices A and B are similar then their determinant are also are equal. Then the tracers trace of the 2 matrices are equal trace of A is equal to trace of B, where A and B are similar matrices they have the same characteristic polynomial. And hence they have the same characters same characteristic roots or eigenvalues ok, eigenvectors may not be same ok.

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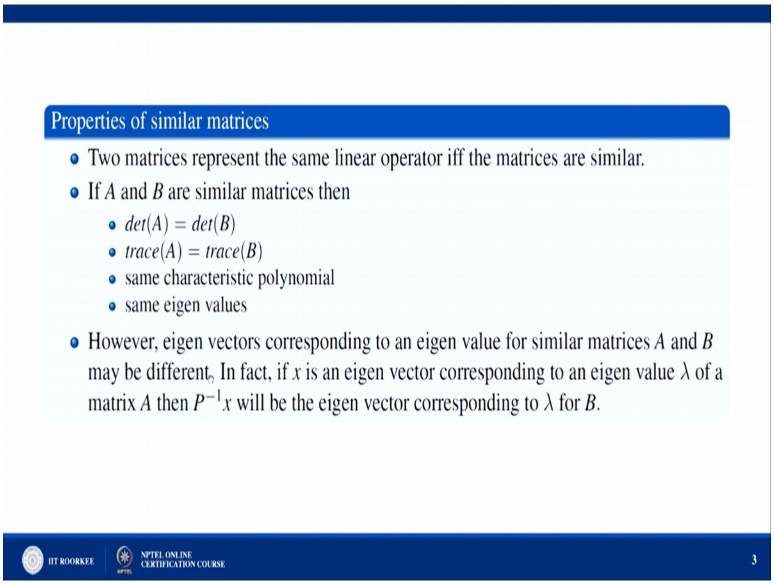

So, this these are what they are in this properties you can easily see, if 2 matrices represent the same linear operator if and only if the matrices are similar number 1. If matrices are similar than the linear operators are also same or they have if they have the linear operator then the matrices are similar. If A and B are similar matrices then determinant are equal, trace are equal same characteristic polynomial and same eigenvalues this way we have already discussed

However eigenvector corresponding to an eigenvalue for similar matrix A and B may be different. In fact, if x is an eigenvector corresponding to an eigenvalue lambda of matrix A, then P inverse x will be the eigenvector corresponding to lambda for the matrix B this we have already discussed.

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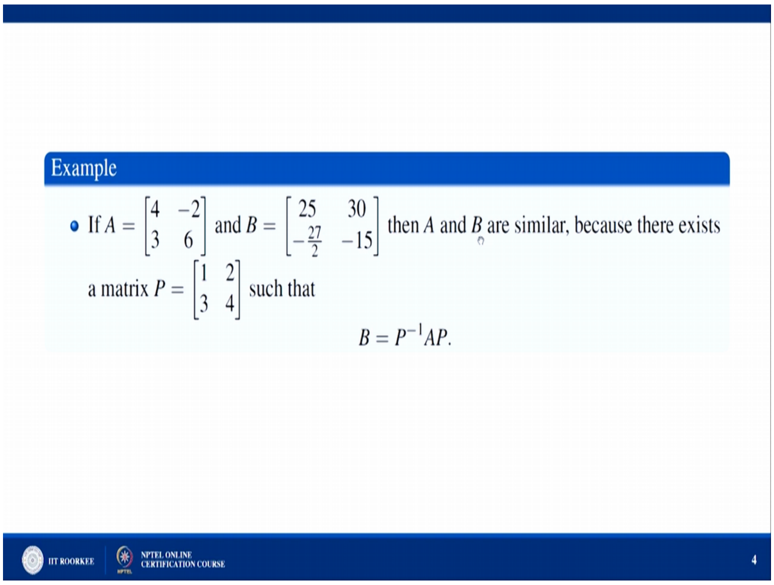

Now, let us discuss one example suppose A is this matrix and B is this matrix then A and B are similar, how we have concluded this is very easy to conclude basically if you have 2 matrices A and B any 2 matrices A and B ok.

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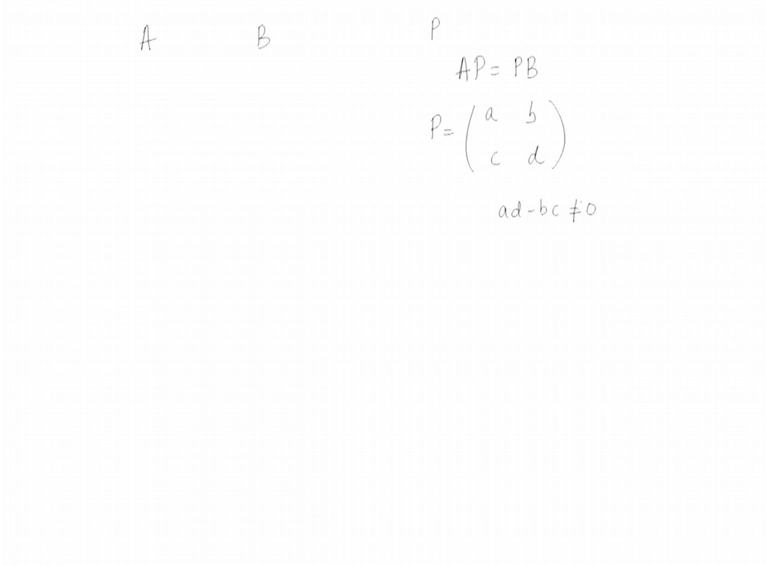

You have to check whether these 2 matrices are similar or not. So, you basically take arbitrary matrix P such that A P is equals to P B it here A and B are 2 cross 2 matrices. So, you can take P as a b c d, then you can solve the left hand side you can solve the right hand side ok. And you will get 4 equations in 4 unknowns, you can find a b c d, and if you find abcd such that this matrix P is invertible that is a d minus b c is not equal to 0 that clearly means matrix A and B are similar.

Because we have shown the existence of such P such that such invertible P, such that A P equal to PB; So, here also here also you can you can find such P which we are getting as this you can easily verify here that B equal to P inverse A B. So, we can say that these 2 matrices are similar, you can also verify here you can see that trace is 10 here the trace is also 10, here the determinant is 24 minus 6, 24 plus 6 30. And here also if you simplify you will get 30 the determinant of P because they are similar. Now, in terms of linear operator what do you mean by diagonalizability.

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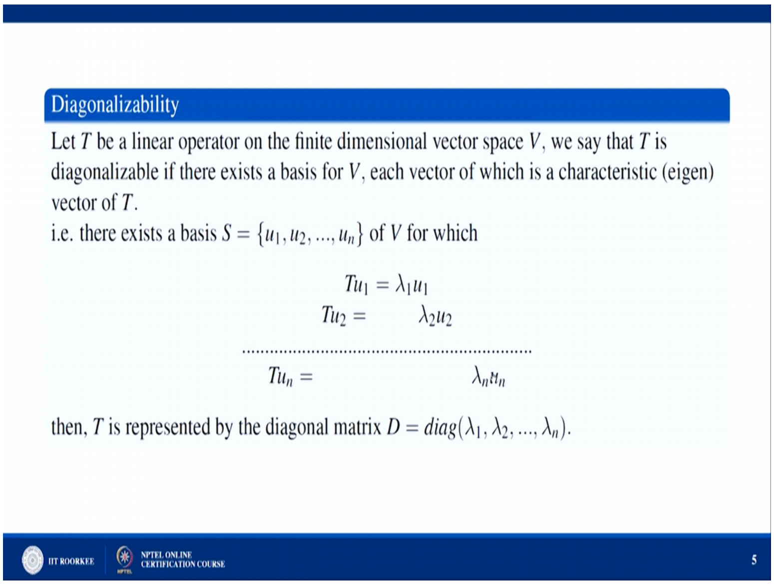

So, if T is A linear operator on a finite dimensional vector space V then we say that T is diagonalizable if there exist a basis for, each vector of which is a characteristic vector of T. So, what does it mean? It means that there exist a basis S which is given by u 1, u 2, up to u n of V, for which T of u 1 equal to lambda 1 u 1, it is something like A x equals to lambda x. You we already know that every linear transformation correspond to a matrix respect to some basis.

So, if this is a basis and, so T of u 1 will be lambda 1 u 1 T of u 2 will be lambda 2 u 2 and so on T n u n will be lambda n u n, then T is represented by a diagonal matrix D which is given by this. So, what does this mean we will discuss later on by taking any example of matrices?

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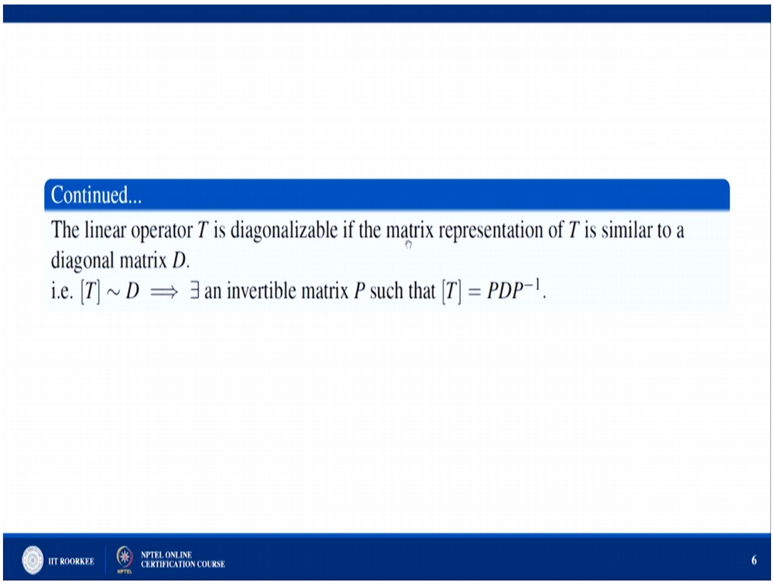

So, the linear operator T is diagonalizable if the matrix representation of T that we already know that every linear transformation represent a matrix ok. So, if a linear transformation if a matrix representation of a linear operator T, is similar to a diagonal matrix D then we say that T is diagonalizable that clearly means that.

The matrix representation of T similar to D; that means, there exists an invertible matrix P again such that such that matrix representation of T is equal to PDP inverse.

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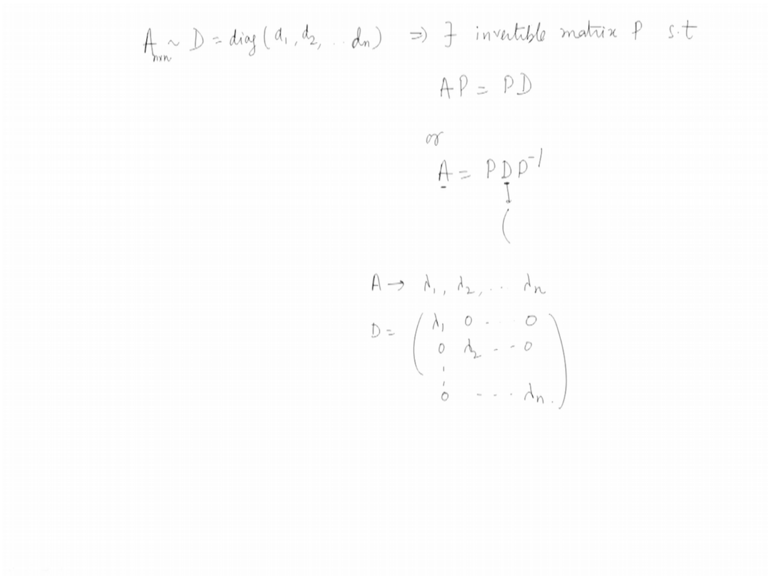

So, what we have concluded we have concluded suppose we have a matrix A which is a matrix correspond to a linear operator matrix representation of a linear operator T. Then we say that this matrix is diagonalizable if it is similar to a diagonal matrix D, D is a diagonal matrix ok. Diagonal d 1 d 2 up to d n if A is n cross n matrix ok. It is similar to a diagonal matrix means they are exist an invertible matrix P ok. Invertible matrix P such that A P is equal to a P D or A is equals to P D P inverse.

Now, we know that eigenvalues of A and eigenvalues of its similar matrices are eigenvalues of similar matrices are same this we have already proved ok. So, an it is a diagonal matrix and the eigenvalues of diagonal matrix are nothing, but the diagonal element itself. So, we can say that that here the diagonal elements of this D are nothing, but eigenvalues of A; that means, if A has an eigenvalues lambda 1 lambda 2 up to lambda n then this D will be nothing, but lambda 1 0 0 0 0 lambda 2 0 0 0 and 0 up to lambda n, because similar matrices have same eigenvalues now if characteristic polynomial.

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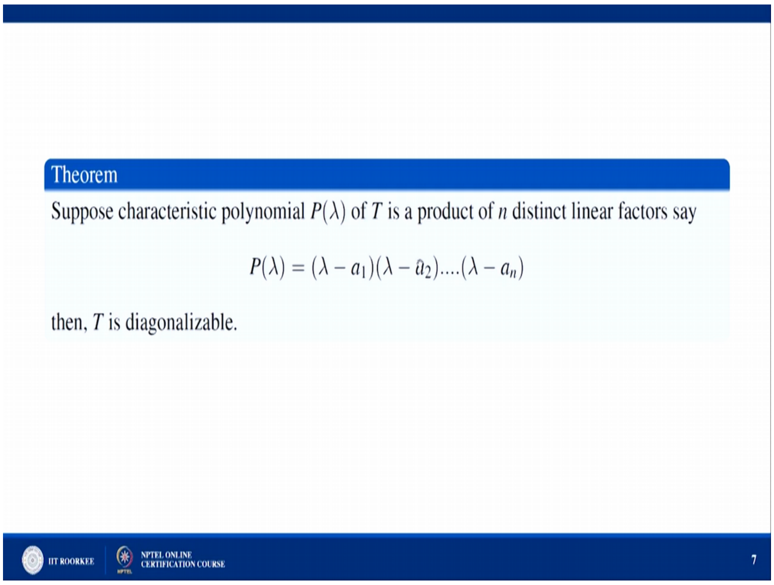

P lambda of T or of A is a product of n distinct linear factors like this lambda n minus A 1, all have a power 1, all have a power 1, then T is always diagonalizable. If it is a distinct linear factors are all have a power 1 in the characteristic polynomial then T is a matrix is always diagonalizable, now let us discuss algebraic and geometric multiplicity of a eigenvalue and eigenvector. So, what is it mean let us discuss ok.

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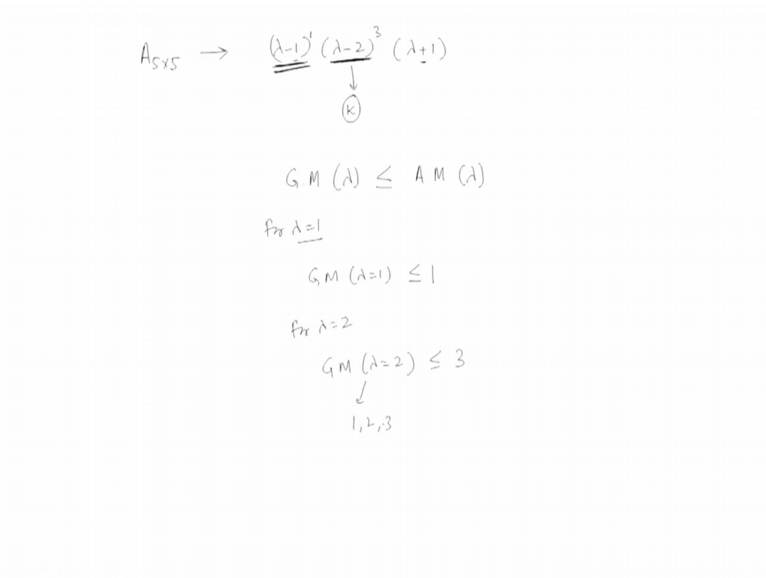

Suppose A is a 5 cross 5 matrix and it is it is characteristic polynomial is suppose lambda minus 1 lambda minus square cube and lambda minus lambda plus 1 say ok. So, it is 3 plus 2 is 5 now corresponding to lambda equal to 1 there is only 1 factor. So, we can we say that algebraic multiplicity for lambda equal to 1 is 1. Algebraic multiplicity means numbers of times that lambda repeats this lambda equal to 1 is repeating only 1 time.

So, algebraic multiplicity corresponding to lambda equal to 1 is 1, now this lambda equal to 2 is repeating 3 times. So, we say that algebraic multiplicity corresponding to lambda equal to 2 is 3 again this lambda lambda equal to minus 1 is repeating 1 time. So, we can say that algebraic multiplicity for lambda equal to minus 1 is 1. So, algebraic multiplicity corresponding to a lambda is nothing, but number of times that lambda repeats ok. Now corresponding to this lambda, lambda equal to 1 or lambda equal to 2, suppose lambda equal to 2 correspond lambda equal to 2 number of linearly, linearly independent eigenvectors are suppose k of course, k will be less than equal to 3 ok.

Now, suppose number of linearly independent eigenvectors corresponding to this lambda equal to 2, are k so this k is called geometric multiplicity for lambda equal to 2, geometric multiplicity corresponding to lambda equal to 2. So, geometric multiplicity is nothing, but number of linearly independent eigenvector corresponding to a lambda ok. Now, geometric multiplicity which is G M correspond to a lambda is always less than equal to algebraic multiplicity correspond to that lambda, it cannot be more than this.

Now, since lambda equal to 1 is repeating 1 time, so what is algebraic multiplicity for lambda equal to 1 is 1 ok, for lambda equal to 1 for this particular example. So, G M for lambda equal to 1 will be less than equal to 1. So, G M will be 1 only it cannot be 0, there is at least one linearly independent eigenvector correspond to a lambda ok.

So, that means, if a lambda is repeating only 1 time, so number of linearly independent eigenvector correspond to that lambda will be 1 only; now here for lambda equal to 2 algebraic multiplicity is 3. So, what should be the geometric multiplicity, geometric multiplicity will be less than equal to 3 because by this property we can say that for lambda equal to 2, geometric multiplicity for lambda equal to 2 will be less than equal to algebraic multiplicity which is 3.

So, geometric multiplicity of correspond lambda equal to 2 either is 1 2 or 3 and for lambda equal to minus 1 geometric multiplicity is 1 ok. Now, what is a sufficient by which we can surely say that matrix is diagonalizable or not, because findings such P always for a bigger matrix say 10 cross 10 or 5 cross 5 is not an easy task. So, what should be the sufficient condition by which we can surely say seeing a matrix the whether it is diagonalizable or not. So, what is that condition let us discuss here these are this is a example you see.

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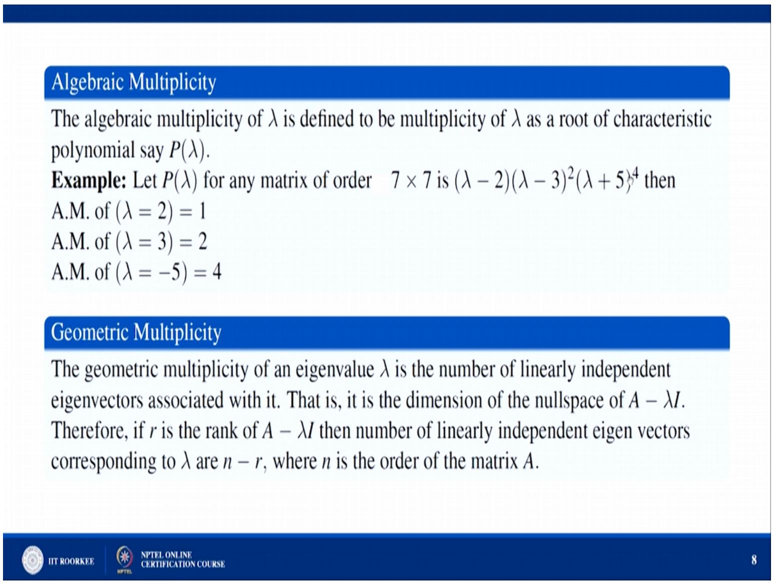

Here is a 7 cross 7 matrix here it is lambda minus 2 lambda minus 3 whole square lambda plus 5 whole is of 4. So, for lambda equal to 1 algebraic multiplicity is 1 for lambda equal to 3 algebraic multiplicity is 2 for lambda equal to minus 5 algebraic multiplicity is 4 geometric multiplicity is 4.

Geometric multiplicity is simply number of linear independent eigenvector associated with the with it, that is it is the dimension of the null space of A minus lambda i, how it is we will discuss by an example.

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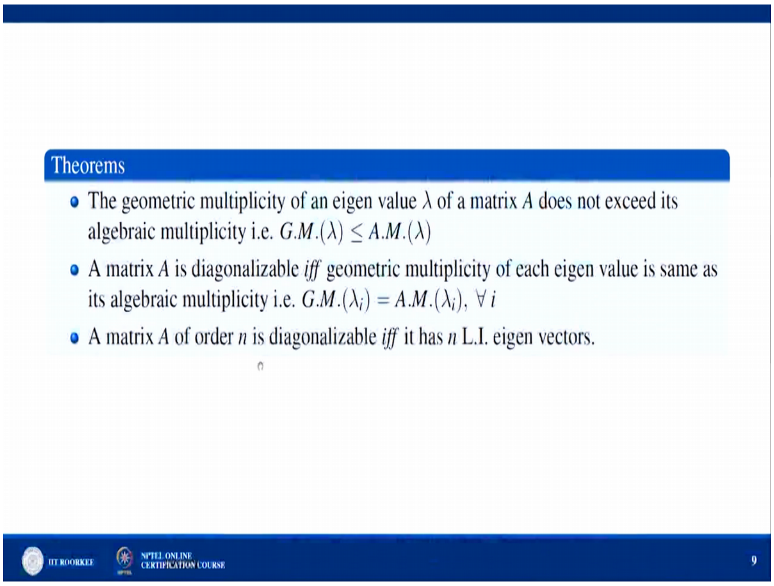

So, this we have already seen that geometric multiplicity of an eigenvalue lambda does not exceed its algebraic multiplicity number 1. Now, a matrix is diagonalizable if and only if geometric multiplicity corresponding to each lambda is same as its algebraic multiplicity ok; that means, if geometric multiplicity corresponding to each lambda i is equal to algebraic multiplicity then only matrix will be diagonalizable.

So, this is a sufficient condition, or we can say that if a matrix as an order n then it will be diagonalizable if and only if it has n linearly dependent eigenvectors because, corresponding to each lambda geometric multiplicity equal to algebraic multiplicity number of number of number of linearly number of eigenvalues are total n. So, it must have n linearly independent eigenvector if matrix diagonalizable otherwise it is it will not be diagonalizable. So, let us discuss these two example quickly and from here we can analyze each and everything, so what is the first example.

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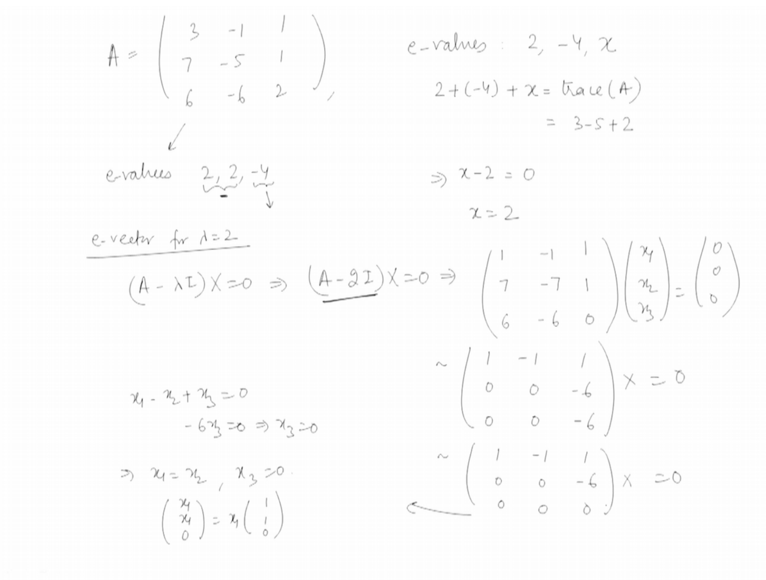

The first example is you see a is given to us as 3 minus 1 1 7 minus 5 1 6 minus 6 2, eigenvalues which is given to us r 2 minus 4 and x ok. It is given to us or we can find out the eigenvalues by finding the characteristic polynomial of this matrix. Now, how you can find out x? We can find out because the sum of eigenvalues is equal to trace of A and trace of A is simply 3 minus 5 plus 2 ok. So, it will be x minus 2 will be equal to it is 5 minus 5 0 that is x equal to 2.

So, what are the eigenvalues of this A eigenvalues of A will be 2 2 minus 4 because x is 2. So, what is algebraic multiplicity of 2 is 2 and what is the algebraic multiplicity of minus 4 is 1, corresponding to lambda equal to minus 4 geometric multiplicity will be 1 only, because geometric multiplicity is always less than equals to algebraic multiplicity. Now correspond to lambda equal to 2 if geometric multiplicity is equal to 2, then only this matrix will be diagonalizable. If it is less than 2 that is 1 then this matrix is not diagonalizable, now diagonalizable means they are weak they are does not exist any D such that a similar to S, that does not exist any P such that A P equal to P D.

So, let us find eigenvector correspond to lambda equal to 2, so eigenvector for lambda equal to 2. So, how you will find it a minus lambda i x equal to 0 this implies lambda is 2. So, put it 2 this implies what is a minus 2 i it is 1 minus 1 q it is a 7 minus 7 1 ok. It is 6 minus 6 and 0 times x is means x 1 x 2 x 3. So, 0 means 0 0 0 now you can it is 1 minus 1 1 you can take, 1 operation that is this minus 7 time this is 0, this 1 is 7 time this is 0 this 1 is 7 time this is minus 6. This minus 6 time this is 0 this minus 6 time this is 0 this minus this time minus 6, and this minus this minus this row will give 0 here, we are trying to make echelon form of this.

So, what we have obtained from here it is x 1 minus x 2 plus x 3 equal to 0 and minus 6 x 3 equal to 0. So, this implies x 3 equal to 0 if you substitute x equal to here, so we obtain x 1 equal to x 2. So, how many linearly independent eigenvector correspond to lambda equal to 2 it is only 1, which is x 1 x 1 0 or 1 1 0 you taken any 1 linearly independent eigenvector which is you say 1 1 0.

So, so it is a only 1 1 0 only 1 that is geometric multiplicity corresponding to lambda equal to 2 is 1 it should be 2 for diagonalizibility, so we can say that this matrix A is not diagonalizable. So, how we can how we can check you see you see we have find the null space of this basically A minus lambda i, we have basically finding all this things means we have find null space of this matrix A minus 2 i ok, or we can say that if r is the rank of the use rank of this matrix is 2 ok, and number of unknowns are 3 or order of matrices 3.

So, 3 minus 2 is 1 and 1 is the number of linear independent eigenvectors so; that means, if r is the rank of A minus lambda i and n is the n is the order of A then n minus r will be the geometric multiplicity corresponding to that lambda. And if it is equal to geometric if it is equal to algebraic multiplicity for each lambda then only matrix will be diagnosable ok, now you can see here.

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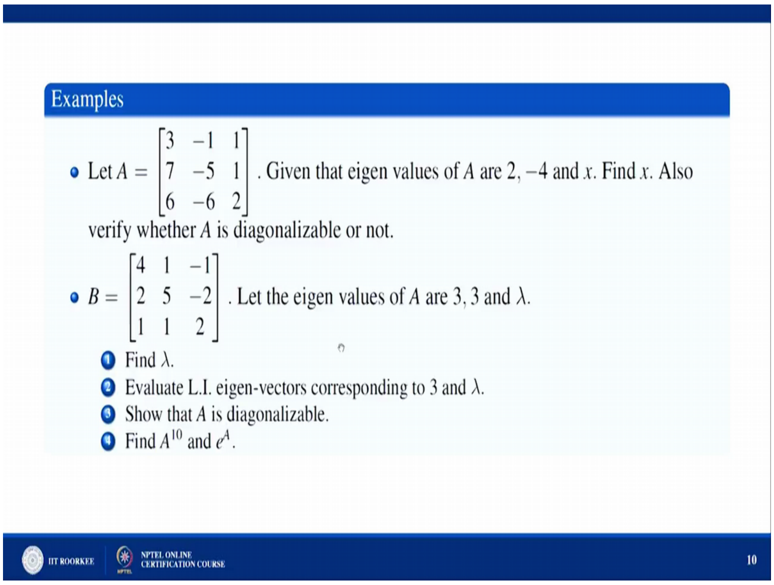

You can see a second example here.

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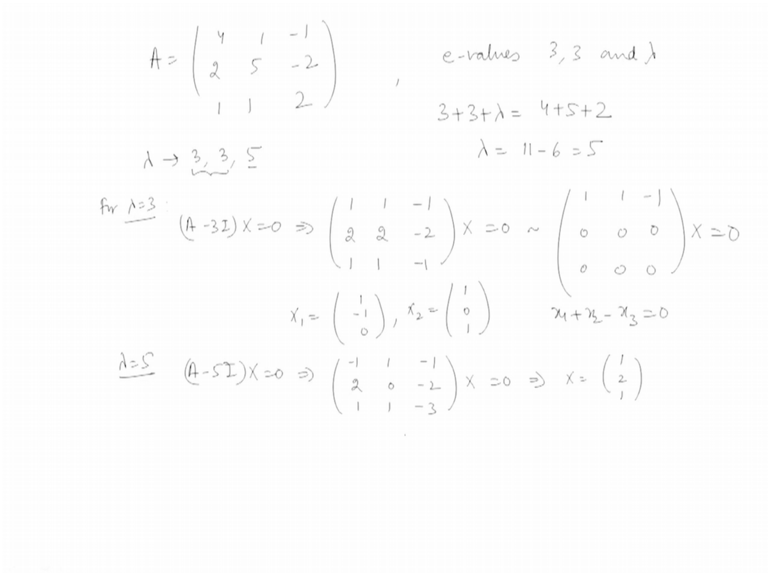

Here A is 4 1 minus 1 2 5 minus 2 1 1 2 and here eigenvalues are 3 3 and lambda, let us see whether it is diagonalizable or not and if yes what will be P and how we can find the other expressions in this problem ok.

So, this is a problem let us let us try to find it quickly, so the sum of eigenvalues again is equal to trace of A. So, lambda will be it is 9 plus 2 11 11 minus 6 that is 5, so we can say that eigenvalues of this A are 3 3 and 5. So, we will find number of linearly dependent eigenvector correspond lambda equal to 3 first we will check ok. So, for lambda equal to 3, so for lambda equal to 3 it is A minus 3 I times x equal to 0 and this implies 1 1 minus 1 2 2 minus 2 and it is 1 1 minus 1 x equal to 0.

So, when we convert into basically echelon forms we can easily see that it is 2 times this and it is 1 time this. So, it is 0 0 0 0 0 0 x equal to 0, so what is the rank of this matrix rank is 1 and what is the order of the matrix is 3 3 minus 1 that is 2; that means, 2 is the geometric multiplicity corresponding to lambda equal to 3. So, yes because algebraic multiplicity is 2 and geometric multiplicity is also 2 for lambda equal to 5 it is 1 only so; that means, this matrix is diagonalizable ok. So, what are what are those 2 vectors x 1 plus x 2 minus x equal to 0. So, we can to pick any 2 arbitrary linear depend eigenvector satisfying this equation.

So, we can take x 1 as 1 minus 1 0 because 1 minus 1 0 is satisfying this and x 2 as 1 0 1 and these two are linearly independent is it ok, because 1 minus 1 0 satisfying this and 1 0 1 also satisfying this and these two are linearly independent. So, yes they will they will form the entire eigen space. Now, corresponding to lambda equal to 5, for lambda equal to again we can find out A minus A minus I times x equal to 0 and when we simplify it is minus 1 1 minus 1 it is 2 0 minus 2 and it is 1 1 minus 3 x equal to 0. And when you simplify this you will get x as x as 1 2 1 you can find its echelon form and then vector you find it this vector.

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$$P = \begin{array}{c} \begin{array}{ccc} X_1\downarrow & X_2\downarrow & X_3\downarrow \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \end{array}, \qquad P^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 0 & -2 \\ -1 & -1 & 3 \\ 1 & 1 & -1 \end{pmatrix}$$

$$AP = PD \Rightarrow A = PDP^{-1} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$A^2 = A \cdot A = (PDP^{-1})(PDP^{-1})$$
$$= PD(P^{-1}P)DP^{-1} = PD^2P^{-1}$$

$$A^k = PD^kP^{-1}, \qquad A^{10} = PD^{10}P^{-1} \qquad D^{10} = \begin{pmatrix} 3^{10} & 0 & 0 \\ 0 & 3^{10} & 0 \\ 0 & 0 & 5^{10} \end{pmatrix}$$

$$A^{50} = PD^{50}P^{-1}$$

So, how you can find out P because it is diagnosable now we have ensured at that this matrix diagnosable. So how you can find out that P? So, do find P, to find P how you can find out P we simply write first x 1 vector here x 2 vector here x 3 eigenvector here. So, what is x 1 eigenvector; x 1 eigenvector we have already find 1 minus 1 0 x 2 is 1 0 1 and 1 2 1 this will be that P and it is always invertible, because eigenvectors are linearly independent ok. So, it is always invertible ok. So, what will be P this inverse, so P inverse you can easily find out and the P inverse which you can find will be 2 0 minus 2 minus 1 minus 1 3 1 1 minus 1 this you can find out the P inverse.

Now, A P, so AP will be equal to PD PD and this implies A is equals to PDP inverse. So, and what is D here D, will be nothing, but what is the first eigenvector first eigenvector is this x and this is corresponding to lambda equal to this corresponding lambda equal to 3 ok. So, here you can write first is 0 3 0 0 0 the second eigenvector is corresponding to again lambda equal to 3 see ok.

So, it is 3 0 and the third eigenvector is called you know 5. So, it is 5 now suppose if you want to find out A square say A square is nothing, but A into A which is equals to PDP inverse into PDP inverse and this is nothing, but PDP inverse P into D into P inverse. So, it is a P D square P inverse; that means, that means A square is also diagonalizable ok.

Similarly, if you proceed like this, so what is A raised to power k it is PD raised to power k into P inverse, suppose you want to find out A raised to the power 10. So, it is PD raised to power 10 into P inverse, and what is d raised to power 10 it is 3 raised to power 10 0 0 0 3 raised to the power 10 0 0 0 3 raised to power 10 0 0 0 5 raised to power 10 P you know P inverse you know D D raised to power 10 you know. So, simply multiplication of these three matrices will give you A raised to the power 10 which is otherwise we have to find out A raised to power 10 by 10 matrix multiplication suppose you want to find A raised to power 50.

Similarly, you can find out A raised to power 50 also P D raised to power 50 into P inverse, similarly by the matrix multiplication of these three now suppose, suppose you want to find out e raised to power A.

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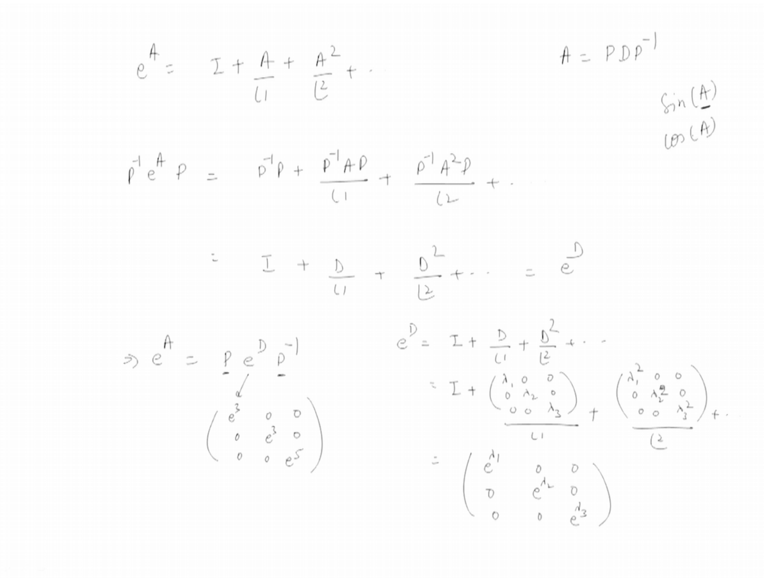

It is I plus A plus A square by 2 factorial and so on ok. So, if you multiply P you see here it is A A equal to PDP inverse ok. So, here A equal to A is equal to PDP inverse, so if you if you take P inverse e raised to power P into P. So, this will be P inverse P plus P inverse A P upon factorial 1 plus P inverse A square P upon factorial 2 and so on. And this is identity plus it is D because A is invertible, I mean A is diagonalizable and it is D square upon factorial 2 as we have already discussed

So, it is e raised to power D, so what we have concluded that e raised to power A is nothing, but P e raised to power D into P inverse and how P is P be know P inverse be know how we can find e raised to power D, e raised to power D is nothing, but I plus D upon factorial 1 plus D square upon factorial 2 and so on. D is nothing, but here it is lambda 1 0 0 lambda 2 0 0 0 lambda 3 yeah here lambda 2 and lambda 2 are 3 and lambda 3 is 5 for this particular problem ok.

Now, square will be lambda 1 square 0 0 0 lambda 2 square 0 0 0 lambda 3 cube square upon factorial 2 and so on. So, when you club all these terms, so it is the first term is 1 plus lambda 1 upon factorial 1 lambda 1 square upon factorial 2 lambda 1 cube upon factorial 3 and so on, which is nothing, but e raised to power lambda 1 second term is 0 throughout this is 0 this again e raised to power lambda 2 0 0 0 e raised to the power lambda 3.

So, e raised to power D we can simply find out e raised to power lambda 1 0 0 0 e raised to power lambda 2 0 0 0 e raised to power lambda 3. So, in this case e raised to power lambda e raised to power D will be what e raised to power 3 0 0 0 e raised to power 3 0 0 0 e raised to the power 5. So, hence we can find out e raised to power A also e raised to power of a matrix by simply multiplying these three matrices. Similarly, if you want to find out sign A sign of A matrix this is A matrix not an angle that also we can find out doing the same derivation same lines following the same lines similarly cos of A matrix this also we can find out.

So, these are some of the applications of diagonalizibility. If a matrix is diagonalizable we can easily find out higher powers of A by the multiplication of only three matrices e raised to power A or sin A cos A or other expressions of the similar types. So, in this lecture we have seen that when a matrix is diagonalizable what are important properties of the diagonalizability and, what are the some, some applications of a diagonal matrices.

Thank you.