Matrix Analysis with Applications
Dr. S. K. Gupta
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 14
Cayley-Hamilton Theorem and Minimal Polynomials

Hello friends so, welcome to lecture series on Matrix Analysis with Applications. So, in today's lecture we will focus on Cayley-Hamilton theorem and minimal polynomial that, what Cayley-Hamilton theorem is and how can you find the minimal polynomial using characteristic polynomial. So, what Cayley-Hamilton theorem state let us see.

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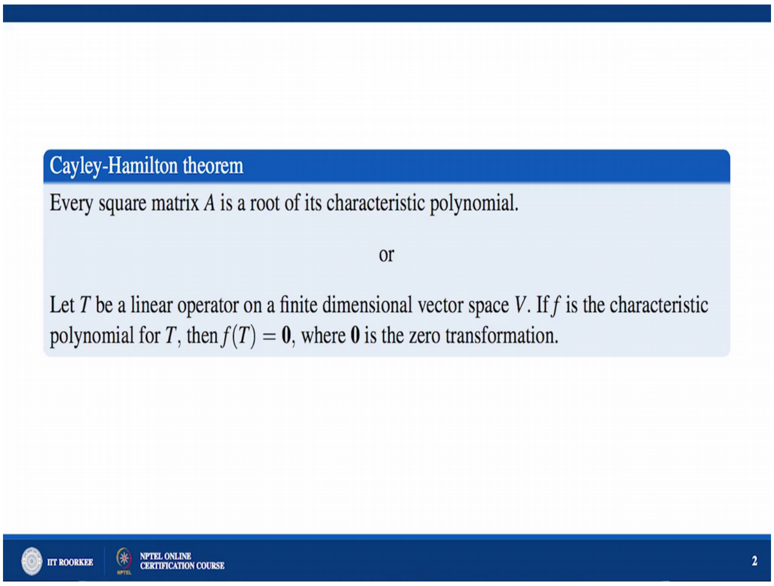

Cayley-Hamilton theorem states that every matrix A, every square matrix A is a root of its characteristic polynomial ok. So, what does it mean let us see.

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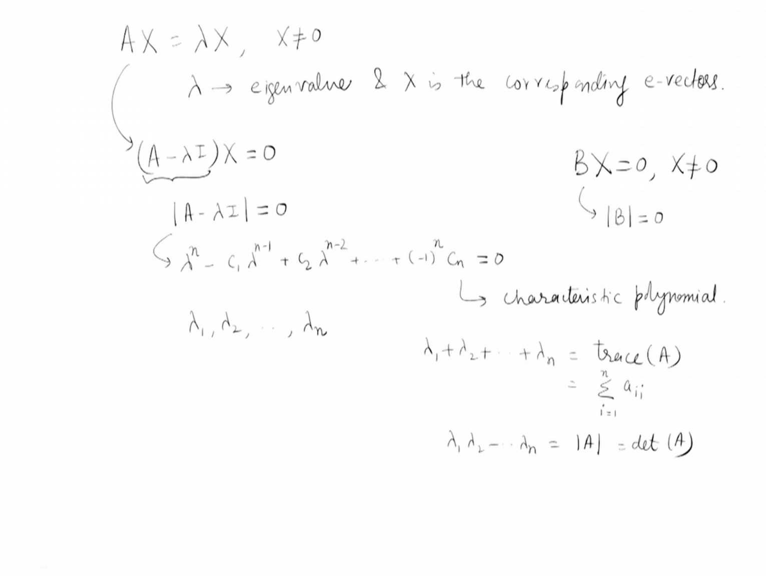

We know that A X is equal to lambda X, we already discuss this thing that where X is not equal to 0 and lambda is a corresponding eigenvalue, I mean if lambda is a corresponding Eigen is an eigenvalue, then the corresponding eigenvector is X ok.

So, lambda is eigenvalue and X is a corresponding eigenvector, this we have already discussed when we are discussing eigenvalues and eigenvectors. Now, from this we can easily see that A minus lambda I times X equal to 0 ok. Now, since X is not equal to 0. So, this system of linear equations has infinitely many solutions, it is it is something like B X equal to 0, where X is not equal to 0.

Now, you see for this system for this system of linear equations, where B is a matrix of order n cross n, this system equation if X is not equal to 0 means have infinitely many solutions. And this will be having infinitely many solution only when determinant of B is equal to 0. So, here instead of B we are having a minus lambda I. So, this means since X is not equal to 0 this means determinant of A minus lambda I is equal to 0. So, this means this expression determinant of A minus lambda I equal to 0, gives a polynomial in lambda and that polynomial in lambda is called characteristic polynomial. And what is the degree of that polynomial? The degree of that polynomial is order of the matrix.

Here, here I am taking out the matrixes n cross n. So, the degree of the polynomial of this will be simply of order n. So, suppose a polynomial which we are getting after opening this determinant is a lambda raised to the power n minus C 1 lambda raised to power n minus 1 plus C 2 lambda raised to power minus 2 and so, on plus minus 1 raised to power n C n is equal to 0 ok. So, this polynomial lambda this polynomial is called characteristic polynomial ok.

So, how many how many roots this equation is having, this equation is having n number of roots. Suppose a roots are lambda 1, lambda 2 up to lambda n, roots maybe distinct roots maybe real roots maybe complex ok. Some roots are equal some are distinct anything.

Now, we have already discussed at some of the eigenvalues is equal to trace of the matrix, trace means sum of the diagonal elements, that is sum of over i sum of a i i, a 11 plus a 22 plus a 33 upto a nn and i is varying from 1 to n. And this also we have discussed at product of eigenvalues is nothing, but determinant of A ok.

Now, Cayley-Hamilton theorem is states that for if this is characteristic polynomial of matrix A ok, then matrix always satisfies characteristic polynomial.

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$$A_{n\times n} \longrightarrow |A-\lambda I| = 0$$
$$\Rightarrow \lambda^n - C_1\lambda^{n-1} + C_2\lambda^{n-2} + \cdots + (-1)^n C_n = 0$$

Then,

$$A^n - C_1 A^{n-1} + C_2 A^{n-2} + \cdots + (-1)^n C_n I = 0.$$

$$\operatorname{adj}(A-\lambda I) = B_1\lambda^{n-1} + B_2\lambda^{n-2} + \cdots + B_n$$

Here $B_1, B_2 \ldots B_n$ are the matrices of order $n\times n$.

$$(A-\lambda I)\operatorname{adj}(A-\lambda I) = |A-\lambda I|\, I$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix}_{n\times n}$$

$$(A-\lambda I) = \begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix}$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$A \operatorname{adj} A = |A|\, I$$

That means, that means if for a matrix a of order n cross n, the characteristic polynomial is determinant A minus lambda I is equal to 0, which is equal to suppose lambda raised to power n minus C 1 lambda raised to power n minus 1, C 2 lambda raised to power n minus 2 and so, on plus minus 1 raised to power n C n is equal to 0.

Then by Cayley-Hamilton theorem a raised to power n minus C 1 a raised to power n minus 1 plus C 2 a raised to power n minus 2 and so, on plus minus 1 raised to the power n, C n into identity should be 0 should be a 0 matrix, or a null matrix; that means, if you have a characteristic polynomial corresponding to matrix A. It always be satisfied by the matrix itself, that is a main statement of Cayley Hamilton theorem.

Now, how can we prove it, how can we prove that a matrix whose characteristic polynomial is given by suppose this expression, it will be satisfied by the matrix also. So, let us try to prove this result the proof of the Cayley-Hamilton theorem you see, for in order to prove let us find adjoint of A minus lambda I. Let us find adjoint of A minus lambda I. You see what is what is a matrix suppose a matrix is a 11, a 12 and so, on up to a 1n a 21, a 22 and so on up to a 2n a n1 and so on up to a nn.

This is n cross n matrix A and, what is a minus lambda I, A minus lambda I will be a 11 minus lambda a 12 and so on a 1n, a 21, a 22 minus lambda and so on a 2n and here a n1, a n2 and so on up to a nn minus lambda, this is a minus lambda i.

Now, if you want to find out adjoint of this matrix A minus lambda i. So, how you will procedure you first find cofactors of each element, take the transpose of the matrix found by the cofactors, that will be the adjoint of that matrix. If suppose you want to find out the cofactor of this matrix this element, the first element you want to find out cofactor of this element will leave, this column will leave this row and the determinant of n n minus 1 cross n minus 1 matrix will be the co factor corresponding to first element.

Similarly, if you want to find out cofactor of a 1 2 elements say for example, then you leave this column and you leave this row and, you find out the cofactor of determinant of the remaining matrix will be the cofactor of that particular element with of course, cofactor of C i j is minus 1 raised to power i plus j minor of i j that we already know.

So, so what is it mean it mean that we if you want to find out the cofactor of any element of this matrix say first element. So, it will be the highest power of lambda will be lambda raised to power n minus 1, you see you want if you see cofactor of this element. Then the determinant of this will contain lambda raised to the power n minus 1. Now, lambda raise to power n, if you want to find cofactor of this element, then the highest power of lambda will be lambda raised to power n minus 2 ok.

So, if you find cofactors of each and every element of this matrix. So, this will be something like, you can say that it is B 1 lambda raised to power n minus 1 plus B 2 lambda raised to power minus 2 and so, on up to B n ok. This B 1, B 2 up to B n they are the matrices itself ok, where itself the matrices there here B 1, B 2 and so, on up to B n are the matrices ok. Because, because when you write the determinant of this matrix you open it will contain lambda raised to power n minus 1 and the other terms.

Similarly, when you take the cofactors of other element and so on so, you will get lambda raised to the power n minus 1 A matrix B 1. Similarity lambda raised to power n minus 2 A matrix be 2 and so on up to B n. You can easily verify this result by taking a 3 cross 3 matrix so, you will find that B 1, B 2 up to B n in that case is are the matrixes of order n cross n of order n cross n.

Now, we know that we also know that A minus lambda I into adjoint of A minus lambda I that means, A into adjoint of A we already know it is equal to determinant of a times identity ok. So, this is a matrix into adjoint of the matrix is equal to determinant of A minus lambda I into I, this we already know. So, so you substitute this expression who are here. So, what you will get what you will get you see.

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$$(A-\lambda I)\,\mathrm{adj}(A-\lambda I) = |A-\lambda I|\, I$$

$$\Rightarrow \quad |A-\lambda I|\left(B_1\lambda^{n-1} + B_2\lambda^{n-2} + \cdots + B_n\right) = \left(\lambda^n - C_1\lambda^{n-1} + \cdots + (-1)^n C_n\right) I$$

$$-B_1 = I \qquad \text{—}\ A^n$$

$$AB_1 - B_2 = -C_1 I \qquad \longrightarrow A^{n-1}$$

$$B_n A = (-1)^n C_n I \qquad \longrightarrow I$$

$$A^n - C_1 A^{n-1} + C_2 A^{n-2} + \cdots + (-1)^n C_n I = 0.$$

We are having A minus lambda I into adjoint of A minus lambda I, we are taking is equal to determinant of A minus lambda I times identity.

This implies a minus lambda I, this is we are resuming it is equals to B 1 lambda raised to power n minus 1 B 2 raised to power n minus 2 and so, on up to B n is equal to determinant of. Now, determinant of A minus lambda I is we already know that it is lambda raised to power n minus C 1 raised to power n minus 1 and so, on up to minus 1 raised to power n into C n times identity.

This we have already assumed at the determinant of this matrixes lambda raised to the power n minus C 1 lambda raised to power minus 1 and so, on minus 1 raised to power and C n times identity.

Now, let us compare the coefficient from both the sides, you see what is the coefficient of lambda raised to the power n, from here and here when you multiply these two brackets, then lambda raised to power n will be minus B 1, from this into this itself ok. And that must be equal to I from here.

Now, lambda raised to power n minus 1 lambda raised to the power n minus 1, when you multiply this with this element. So, it is A B 1 and this with this I mean this element the second element of this bracket, that is minus of B 2 that will be equal to lambda raised to power n minus 2 from here C 1 times identity.

Similarly, if you similarly if you take the lambda raised to the power 0 from here. So, lambda raised to power 0 will be this into this that is B n into A from here and no other term and, that will be equal to minus 1 raised to the power n C n times identity. Now, now you multiply the first equation by A raised to power n, the second equation by A raised to power n minus 1. The last equation with the identity and you add them, when you multiply this by A raised to power n multiply this by a raised to power n minus 1. Then it become A into A raised to the power n minus 1 will become A raised to power n into B 1.

So, these two will cancel out. Now, similarly when you multiply B 2 with this element this will be minus A raised to power n minus 1 into B 2 which will be cancel from the next expression, term in the next expression. And similarly last expression will be cancels from the second last one ok. So, when you add them when you add them it we obtain A raised to the power n minus C 1, A raised to power n minus 1 plus C 2, A raised to power n minus 2 and so on plus minus 1 raised to power n C n times identity will be equal to 0.

So, hence we have obtained hence we obtained this result, which states that I mean which tells us that characteristic polynomial will be satisfied by the matrix itself. So, hence we got the proof of the Cayley-Hamilton theorem.

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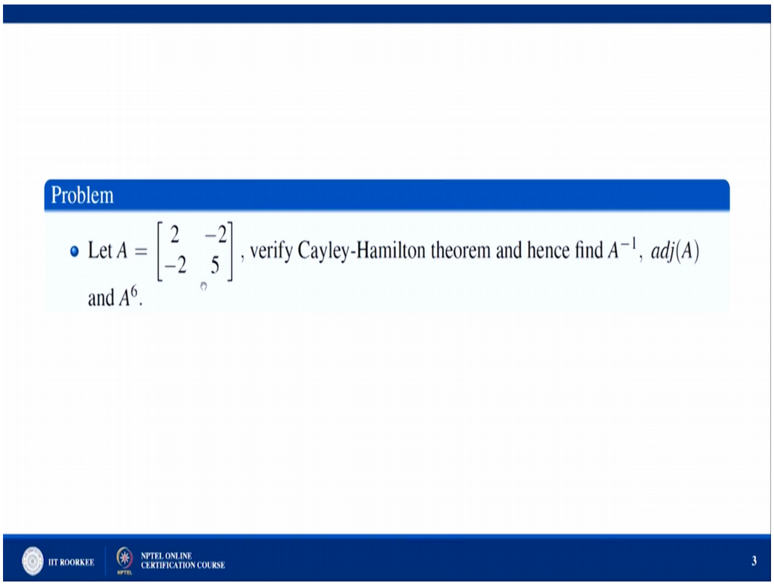

So, suppose we are having this problem now, matrix A for simplicity we have taken example of 2 cross 2. Similarly we can go for 3 cross 3, or higher orders. Now, let us suppose a is this matrix ok, we have to verify Cayley-Hamilton theorem and hence find A inverse adjoint of A and A raise to power 6.

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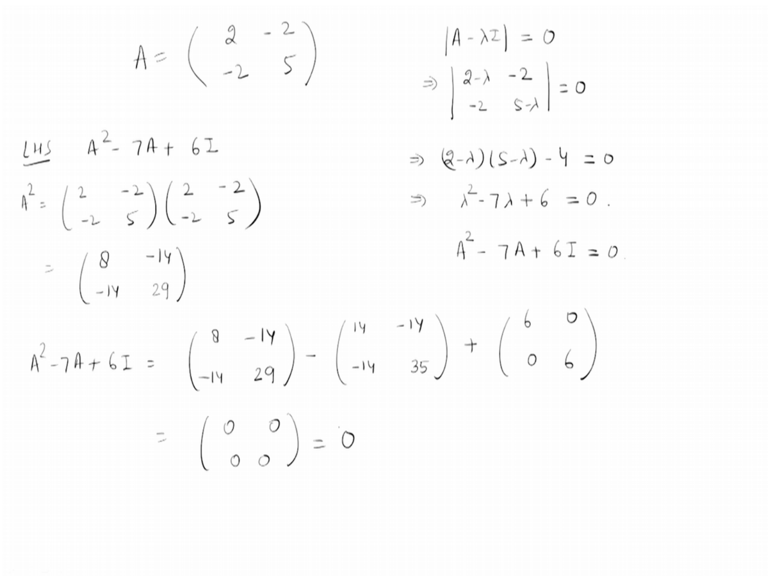

Now, what is A here? A is you see A is 2 minus 2 minus 2 and 5. First we will find out the characteristic polynomial of this matrix A and, from there we will try to verify Cayley-Hamilton theorem. So, what is the characteristic polynomial determinant of A minus lambda I is equal to 0 is a characteristic polynomial. So, this implies determinant of 2 minus lambda minus 2 minus 2 5 minus lambda determinant should be 0.

So, this implies 2 minus lambda into 5 minus lambda minus 4 should be 0. This implies lambda square, this is minus 7 lambda plus 10 minus 4 is plus and minus 4 is plus 6 equal to 0. So, this is a characteristic polynomial of this matrix A. Now, what from Cayley-Hamilton theorem, from Cayley-Hamilton theorem we must have A square minus 7 A plus 6 I should be 0, this we have to verify in order to verify you simply take left hand side, left hand side is a square minus 7 A plus 6 I.

And we have to show that it is equal to null matrix. So, first find A square, what is a square A square is 2 minus 2 minus 2 5 into A itself 2 minus 2 minus 2 5, when you multiply these two this row this column, it is 4 plus 4 is 8. This row this column minus 4 minus 10 minus 14. This row this column minus 4 minus 10 minus 14, this row this column is 4 plus 10 is 14.

Now, you now let us try to find this expression A square minus 7 A plus 6 I, which is 8 minus 14 minus 14 14 minus 7 times A you multiply this with 7. So, it is it will be 14 minus 14 minus 14 7 5's at it is 35 7 into 5 is 35 plus 6 I 3 6 0 0 6.

Now, 8 plus 6 is 14 and 14 minus 14 is 0 similarly, minus 14 and plus 14 is 0 minus 14 plus 14 is 0 and, it is 14 14 plus 6, 14 plus 6 is 20. So, let us let us again verify this thing, it is this row this column that is 4 plus 25 ok.

So, here we have a doubt here is a correction, you see this element. It is 4 this row this column that is 4 plus 25 is 29 ok. So, this element is 29 now 29 plus 6 is 35 and 35 minus 35 is 0. So, it is 0 0 0 0 so, it A an unmatrix 0 ok. So, hence we have hence this Caylen Hamilton theorem is verified. Now, we have to find out inverse of a using Cayley-Hamilton theorem so, how we will find that.

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$$A^2 - 7A + 6I = 0$$

$$A^{-1}(A^2 - 7A + 6I) = 0$$

$$A - 7I + 6A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{6}(7I - A) = \frac{1}{6}\begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{\text{adj}\,A}{|A|} \Rightarrow \text{adj}\,A = A^{-1}\cdot|A| = 6A^{-1} = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\lambda_1 \lambda_2 = 6 = |A| \neq 0$$

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$$

$$7I - A = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} - \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

So, for this matrix we have seen that A square minus 7 A plus 6 I is equal to 0, this we have obtained by Cayley-Hamilton theorem. Now, now what is what is the determinant of determinant of the matrix, determinant of the matrix is simply product of eigenvalues and, product of eigenvalues is simply given by the last term upon first term that is 6 that is the determinant.

So, determinant is not equal to 0 this means inverse exist. Now, first we have to get the ensurity that inverse exist and, for inverse to be to exist matrix must be invertible, I mean non singular for that determinant must be non 0 and, from here the product of eigenvalue is 6. So, we can say that determinant is not equal to 0 so, A inverse exist.

Now, how to find A inverse. Now, since A inverse exist and A satisfied this equation by Cayley-Hamilton theorem. So, we can multiply both the sides, or we can operate both the sides by A inverse. So, let us operate both the sides by A inverse, it is 0 A inverse into A square is A minus 7 times identity plus 6 times A inverse should be 0.

Now, this implies A inverse will be you can put all the other terms on left hand side, it is 7 I minus A ok. Now, what is A? A is given to us as it is 2 minus 2 minus 2 5, so what is 7 I minus A 7 I minus A will be 7 0 0 7 minus A A is 2 minus 2 minus 2 5 and, this will be 5 0 plus 2 is 2, it is again 2 a 7 minus 2 is 2.

So, this A inverse will be 1 6th of this method, it is 5 2 2 2. So, this should be the inverse of this matrix. Now, next is we have to find out adjoint of A. Now, we know that A inverse is adjoint of A upon determinant of A. So, this implies adjoint of A will be A inverse times determinant of A. Now, determinant of A is 6 that we already shown. So, it is 6 into A inverse and A inverse is this expression so, 6 is cancel out. So, adjoint of this matrix recently this matrix now, next we have to find out A raise to power 6 for the same problem ok.

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$$A^2 - 7A + 6I = 0.$$
$$A^2 = 7A - 6I \checkmark$$
$$A^3 = 7A^2 - 6A = 7(7A - 6I) - 6A$$
$$= 43A - 42I$$
$$A^4 = 43A^2 - 42A = 43(7A - 6I) - 42A$$
$$= \_\_\_$$
$$A^5 =$$
$$A^6 = \_\_\_$$
$$A^6 = \alpha \underline{A} + \beta I$$

Now, by Cayley-Hamilton theorem the characteristic polynomial is this ok, A square minus 7 I plus 6 I equal to 0 and, we have to find A raised to the power 6.

So, it is very easy to find out using this expression you see, A square is 7 A minus 6 I, you can find a cube by multiplying both sides by A, it is 7 A square minus 6 A 7 times. Now, A square again you substitute this value it is 7 A minus 6 I minus 6 A, it is equal to now 42 minus 6, 42 minus 6 is a 36 A minus 42 I ok, it is 49, it is 49 minus 6 so, it is 49 minus 6 is 43 it is 43 A minus 42 I.

Similarly, you can now multiply again you can multiply by A both the sides. So, it is a raised to power 4 43 A square minus 42 A. So, 43 A square A square is again 7 A minus 6 I from this expression, minus 42 A you can simplify this. And again you can multiply by a raised to the power 5, I mean a both the sides and substitute A square from this expression and finally, A raised to the power 6 ok.

So, finally, you will get a expression of a raised to the power 6 of this form some alpha raised to power A plus beta raised to power I, some expression of this form where alpha and beta can be computed by the by successive computation, then you can substitute A because you know the matrix A and identity you already know you can easily find out A raised to power 6 ok.

And other way out is using Cayley-Hamilton theorem, it is like you are you might be seeing that for 2 cross 2 matrix, either you can multiply A by 6 times A square, then A cube then A raised to the power 4 A raised to the power 5 and, then A raised to power 6, but if it is a larger matrix of order say 10 cross 10, then finding a raised to the power 6 is the difficult task.

So, but by the Cayley-Hamilton theorem using Cayley-Hamilton theorem, it is easy to find out. So, this is the main importants of Cayley-Hamilton theorem to find out thehigher powers of A to find out A inverse adjoint of A and other things about the matrix A.

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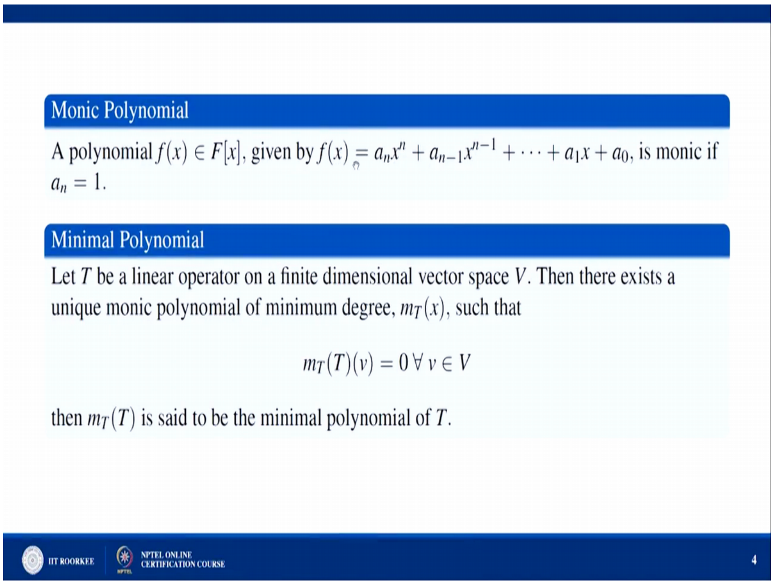

Now, what is monic polynomial a polynomial f x given by f x equal to a raised to a and x raised to power n plus a n minus 1 x raised to power n minus 1 and so, on up to a naught, is said to be monic. If the leading coefficient is 1, if this leading coefficient which is a coefficient of highest power of this polynomial is 1, then this polynomial is called monic polynomial ok.

Now, what do you mean by minimal polynomial. Let T be a linear operator on a finite dimensional vector space V, then there exist a unique monic polynomial of minimum degree m T x such that m T T v equal to 0 for all v in V, then this m T T is called minimal polynomial of T.

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$$A_{4\times4} \longrightarrow (\lambda-1)^2(\lambda-2)(\lambda-3)=0$$

$$(A-I)^2(A-2I)(A-3I)=0.$$

$$(\lambda-1)(\lambda-2)(\lambda-3) \longrightarrow (A-I)(A-2I)(A-3I)=0$$

or

$$(\lambda-1)^2(\lambda-2)(\lambda-3) \longrightarrow (A-I)^2(A-2I)(A-3I)=0.$$

So, what does it mean basically suppose, suppose A is a matrix of order say 4 cross 4 ok. And it is characteristic polynomial is suppose lambda minus 1 whole lambda minus 2 lambda minus 3 of course, if it is a order 4 cross 4.

So, the degree of the characteristic polynomial will be 4. So, it will be having 4 roots, suppose all the 4 roots are real. So, roots are roots of I mean characteristic roots are or the eigenvalues of this matrix are suppose 1 1 2 3. So, what is the characteristic polynomial these are characteristic polynomial of A, this matrix A and by the Cayley-Hamilton theorem we can easily say that A minus I whole square into A minus 2 I into A minus 3 I this should be 0, because by the Cayley-Hamilton theorem matrix satisfies characteristic polynomial.

So, hence this will be also this will be also be equal to 0. Now, minimal polynomial is the lowest degree polynomial, for which is to be satisfied by the matrix itself, you see what is the degree of this polynomial degree of polynomial is 4 ok. The important property of minimal polynomial is it contains all the different roots ok, it contains all the different roots that different roots are 1 2 and 3. So, minimal polynomial will always contain lambda minus 1, lambda minus 3 at least ok.

All the irreducible roots it will be having the minimal polynomial. So, so either it is either it is this polynomial, which is the lowest degree polynomial, which maybe the lowest degree polynomial. And A is be satisfied in this expression or it will be characteristic polynomial itself, I mean I want to say that matrix A will be satisfied by either this, or this for this it is a for this it is a obviously, true because by the Cayley-Hamilton theorem.

But we need a polynomial which is lowest degree. So, it may be this polynomial also. So, for this polynomial we have to check whether A is as is satisfying this expression or not. If A satisfying this expression so, this will be the minimal polynomial. Otherwise this is the minimum polynomial which is of course, be satisfied by A by the Cayley-Hamilton theorem ok.

So, minimal polynomial has important property number 1 it is a of lowest degree polynomial, which is satisfied by the matrix itself, it contains all the irreducible factors, all the linear factors which is the characteristic polynomial is having ok. Now, suppose you want to find out minimum polynomial correspond to a matrix A how you will proceed.

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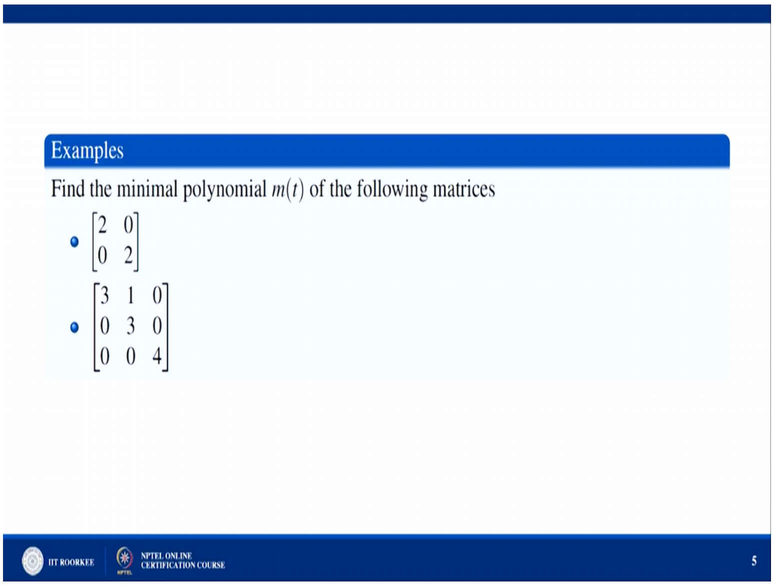

So, let us discuss it by an example, suppose you are having the first problem 2 0 0 2 2 0 0 2.

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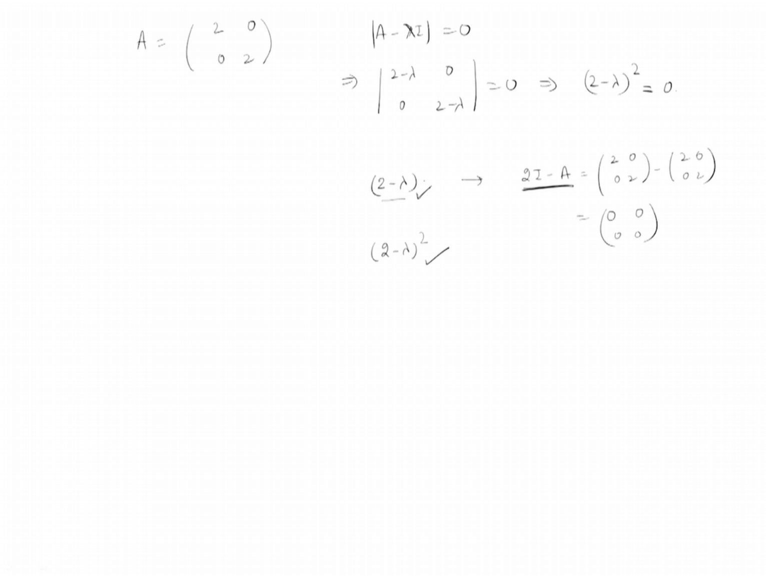

So, what is the characteristic polynomial of this matrix this, which means determinant 2 minus lambda 0 0 2 minus lambda should be 0, or it implies 2 minus lambda whole square is equal to 0. So, this is a characteristic polynomial of this matrix A, how many roots it is having only 2 roots both are equal lambda equal to 2 2 ok.

Now, the minimal polynomial is the lowest degree polynomial contains all the irreducible roots, all the linear roots all the different roots distinct roots ok. So, that means, the minimal polynomial will be either 2 minus lambda, or 2 minus lambda square ok. It is either having 2 minus lambda, or itself. This is will be always satisfied by matrix itself by the Cayley-Hamilton theorem.

So, we have to check whether this is the satisfied by the matrix, or not if it is satisfied by the matrix; that means it is a minimal polynomial. So, this means we have to check that 2 I minus A should be 0, or it is not 0 let us see 2. So, 2 I is 2 0 0 2 and A is simply of course, the same thing.

So, it is comes out to be a null matrix; that means, that means this expression, this expression satisfying by matrix A; that means a minimal polynomial of correspond to this matrix is 2 minus lambda ok. It is of degree 1 not 2. So, the minimal polynomial of this matrix is 2 minus lambda.

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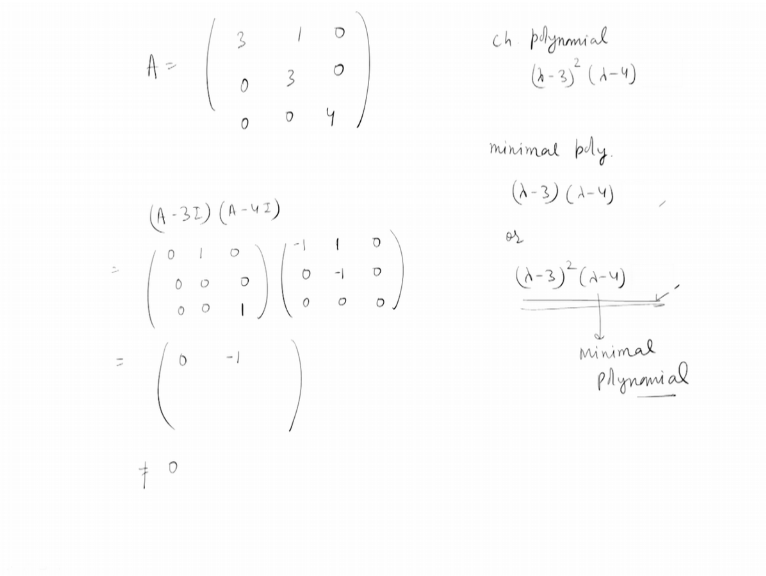

Now, let us take the second example to see whether it is what is the meaning of polynomial of this. The second example is if you see A is A here is 3 1 0 0 3 0 0 0 4. It is the upper triangular matrix you can easily see upper triangular matrix and, the in case of upper triangular matrix eigenvalues are simply the diagonal elements.

So, what are eighenvalues of this matrix these are 3 3 and 4. And if you know the eigenvalues you can simply find out characteristic polynomial, which is lambda minus 3 whole square into lambda minus 4 that is that we can easily see. So, characteristic polynomial of this matrix A will be nothing, but lambda minus 3 whole square into lambda minus 4. Now, you have to see what is the minimal polynomial of this matrix A. So, to see the minimal polynomial of this matrix A minimal polynomial is lowest degree polynomial which is satisfied by the matrix itself.

So, either it is so, minimal polynomial of this matrix, is either lambda minus 3 into lambda minus 4, or lambda minus 3 whole square into lambda minus 4, because it contain all the different roots, or the distinct roots all the factors having distinct roots.

So, this is obviously, satisfied because by the Cayley-Hamilton theorem. Let us see whether it is satisfied or not if it is satisfied, then this will be the minimal polynomial of this matrix A. So, we have to see basically that A minus 3 I into A minus 4 I, what is expression this let us see what is A minus 3 I, it is 0 1 0 it is 0 0 0 0 0 4 and, what is a minus 4 I, it is A minus 3 I is 0 1 0 0 0 1 0 0 0 and 0 0 0 1 ok.

Because you are multiplying this we are subtracting 3 with each diagonal elements. Now, A minus 4 I will be minus 1 minus 1 1 0 0 minus 1 0 0 0 0. Now, what is the port of these two you multiplied this with this 0, you multiplied this with this is minus. If one element is non-zero, if one element comes out to be non 0, this means it is not equal to a null matrix. And if it is not equal to null matrix; that means, this cannot be the minimal polynomial

So, what is the minimal polynomial then the minimum polynomial will be the this expression this will be the minimal polynomial this. So, in this way we can find out the minimal polynomial of a matrix A ok.

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$$A_{5\times 5} \rightarrow (\lambda-1)^3(\lambda-2)(\lambda+3)=0$$

Minimal poly

$$(\lambda-1)(\lambda-2)(\lambda+3)$$

$$(\lambda-1)^2(\lambda-2)(\lambda+3)$$

$$(\lambda-1)^3(\lambda-2)(\lambda+3) \checkmark$$

So, basically if I have I am having a matrix A of order say 5 cross 5 and the characteristic polynomial is suppose lambda minus 1, whole raised to power 3 into lambda minus 2 into lambda plus 3. Suppose is equal to 0, then the minimal polynomial of this I mean for this matrix A corresponding characteristic polynomial may be so, what about cases we have to check.

For minimal polynomial we have to check this product, whether it is 0 or not this product, whether it is 0 or not and this product this is of course, 0 by the Cayley-Hamilton theorem. So, if these two are not equal to 0. So, the matrix characteristic polynomial itself is a minimal polynomial ok, but we have to check for these two also whether these are 0 or not ok.

So, first you will check for this if it is equal to 0. So, this the least lowest degree polynomial, which is satisfied by the matrix. Otherwise we check for this, if these two are not equal to 0, then the characteristic polynomial itself will be the minimal polynomial ok.

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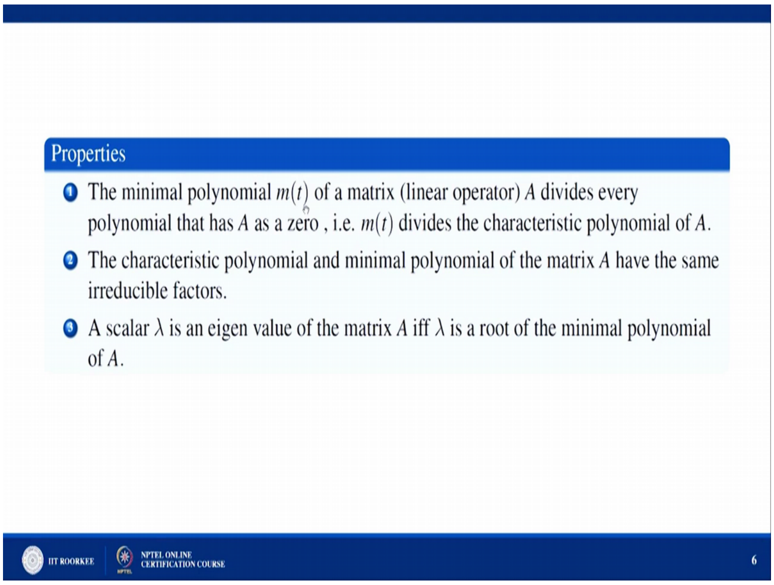

So, what are property minimal polynomial, the minimal polynomial m t of matrix a divides every polynomial that has A as a zero, that is m t divide the characteristic polynomial of A.

So, always minimal polynomial divides it is characteristic polynomial ok. Number 2 it has the characteristic polynomial and minimal polynomial of matrix A has the same irreducible factors, which we have discussed. And number 3, A scalar lambda is eigenvalue of matrix A if and only if lambda is a root of a minimal polynomial of A also. So, if lambda is eigenvalue it is also it is always a root of minimal polynomial also.

So, these are some of the important properties of minimal polynomial. So, in this lecture we have seen that that, how we can find out characteristic polynomial, I mean how we can see and see the advantage, or the applications of Cayley-Hamilton theorem, how can you find out minimal polynomial corresponding to a matrix A ok. In the next few lectures, we will see some of the important properties of important advantage of minimal polynomial so.

Thank you.