Matrix Analysis with Applications
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Lecture – 13
Eigenvalues and Eigenvectors

Hello friends, welcome to lecture series on Matrix Analysis with Applications. Now, today's lectures based on Eigen values and Eigenvectors. That if we have a given matrix A, then how can you find this eigen values in the corresponding eigenvectors.

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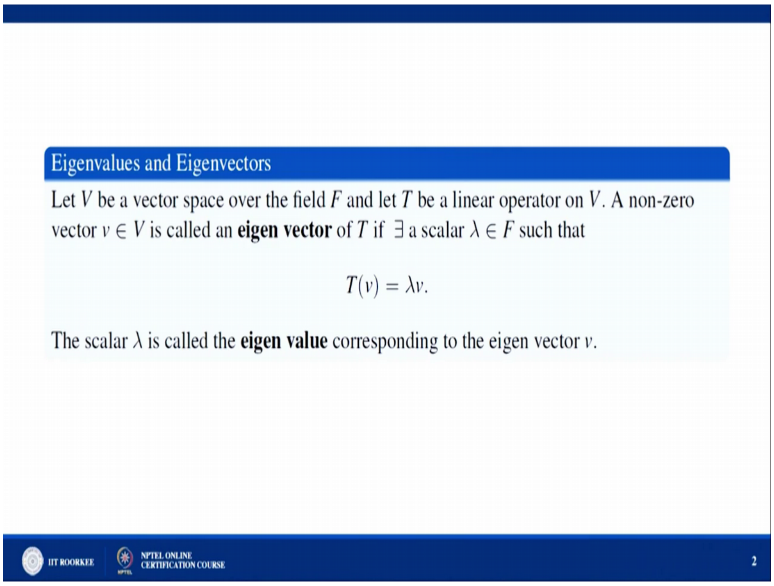

So, first of all what eigen values and eigenvectors are. So, let V be a vector space over the field F, and T be a linear operator on V that is T is a linear transformation from V to V. A non-zero vector v belongs to V is called an eigen vector of T, if there exists a scalar lambda belongs to F such that T v equal to lambda v. So, if we have a linear operator T, and we have a non-zero vector v in V, and there exist a lambda belongs to field such that T v equals to lambda v, then we say that lambda is eigen value, and the corresponding eigen vector is v.

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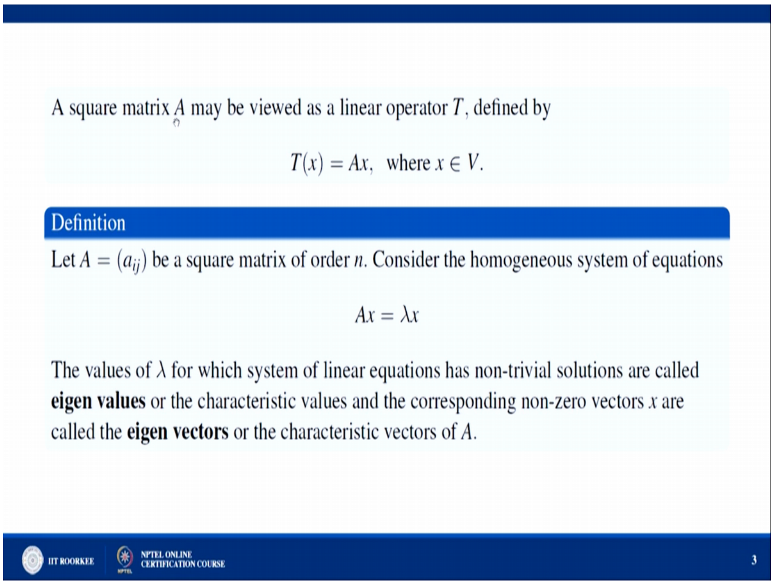

Now, a square matrix A may be viewed as a linear operator T, and is defined as T x equals to A x, where x belongs to V ok. So, if you have a linear operator the linear operator, then A matrix can be viewed as a linear operator also. Now, now let us define a matrix A, which is a square matrix of order n. Then if we consider a system of homogenous equation A x equals to lambda x.

Then the value of lambda for which the system of linear equations has non-trivial solution are called eigen values or the characteristic values of A, and the corresponding non-zero vectors x are called eigen vectors or the characteristic vectors of A. Since, since every matrix can be viewed as a linear operator T, so in the previous definition we have replace this T by a matrix A here. And this x is in V, this is a scalar lambda in field ok.

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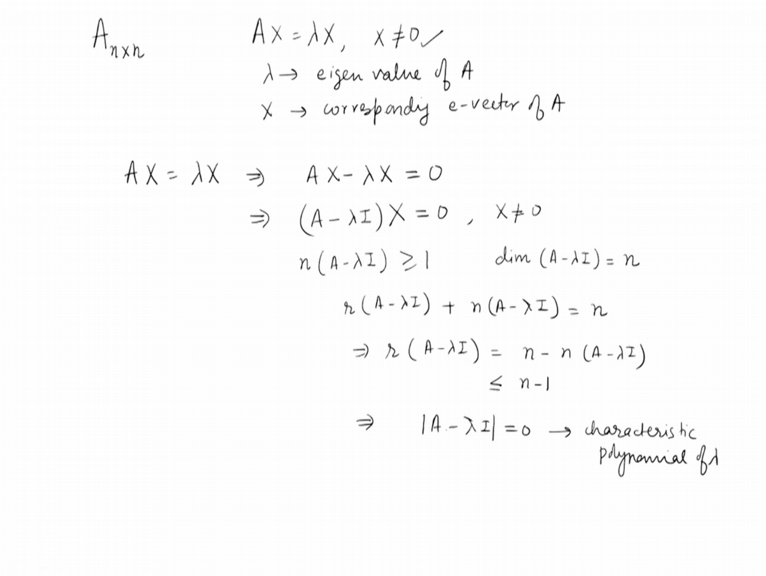

So, what I want to say basically, suppose you have a A matrix A, which is a order of n cross n. Then if A x is equals to lambda X, where X is not equal to 0. Then this lambda is called this lambda is called eigen value of a and X is called corresponding eigen vector corresponding corresponding eigen vector of A, or eigen vector corresponding to lambda.

Now, we are having a A X equal to lambda X, so this can be written as A X minus lambda X equal to 0 ok. Now, this can be written as A minus lambda into I times X equal to 0, where I is an identity matrix. Now, what we are having basically, we are having we are having A minus lambda I, X equal to 0 means, we are having system of homogeneous linear equations you know, because iterate side is 0.

Now, now since X is not equal to 0 as we already defined a definition X is not equal to 0. So, this means if we are taking this as a linear operator T, if we take A minus lambda I as a linear operator T, then T X equal to 0, that means, that means and X is not equal to 0, that means, the nullity the nulls space. The nullity of this matrix or this operator is greater than or equal to 1 ok, because X is not equal to 0 means this means nullity is not 0, and nullity is not 0 means, it is greater than or equal to 1.

Now, what is what is the dimension of dimension of A minus lambda I of course, dimension of A minus lambda I is n. So, rank by rank nullity theorem, the rank of this A minus lambda I plus nullity of A minus lambda I must be equal to dimension of matrix or operator, which is n So, this implies rank of A minus lambda I must be equals to n minus nullity of A minus lambda I.

And since, nullity is greater than equal to 1, this may this is less than equal to n minus 1. And since, rank is less than equal to n minus 1 less than equal to n minus 1, this means in the echelon form of this matrix at least 1 0, 1 row is there, which is having all 0 elements, and that means, this implies determinant of A minus lambda I must be 0. Now, this equation, determinant of a minus lambda equal to 0 is called a characteristic polynomial, this will give a characteristic polynomial in lambda characteristic polynomial of lambda [FL]. Now, now this is a matrix of order and cross n. So, determinant, so this determinant will give a polynomial of degree n in lambda.

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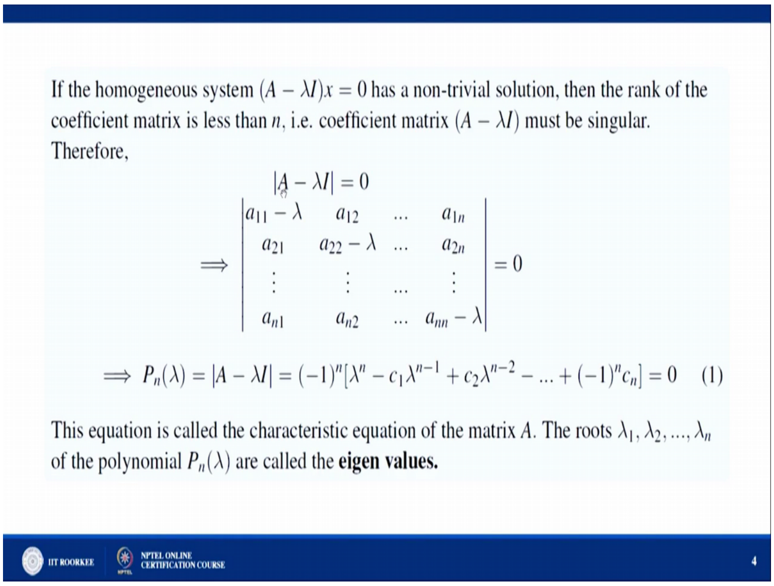

So, so what we are having now? So, now if we if we visualize the determinant of A minus lambda I equal to 0, that means, that means this determinant is equal to 0 you see.

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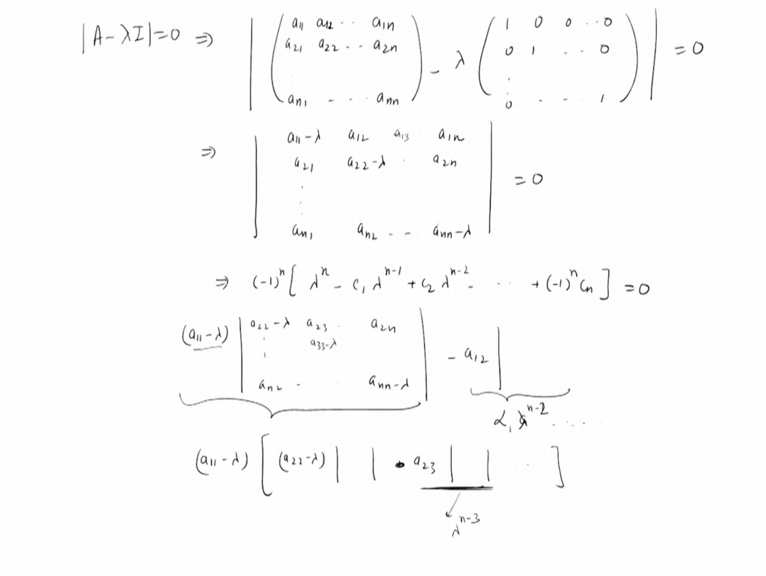

If we are taking determinant of A minus lambda I equal to 0, so it implies that determinant of A is this matrix you see, a 1 1, a 1 2 if you are taking this matrix as a and a and 1 up to a n n minus lambda times I is identity matrix. So, when you take determinant put it equal to 0, so this is nothing but determinant of you see, only the only this lambda will be subtracted from the diagonal elements, because all other entries are 0. So, this will be determinant of a 1 1 minus lambda a 1 2 and so on up to a 1 n, then a 2 1, a 2 2 minus lambda, then a 2 n and so on a n 1, a n 2, and so on up to a n n minus lambda, this determinant is equal to 0.

So, this will give us a polynomial in lambda, and that polynomial is called characteristic polynomial ok. So, this is basically gives minus 1 raised to power n lambda raised to power n minus say c 1 lambda raised to power n minus 1 plus c 2 lambda raised to power n minus 2 minus and so on plus minus 1 raised to power n into c n is equal to 0 ok. Because when you when you expect this in terms of lambda, so you will get a polynomial in of degree n in lambda.

Now, now if you carefully observe this determinant you see, when you open from the first element you see, if you take a 1 1 minus lambda, then you take then you delete this row and this row and this column. So, this will be the remaining I mean determinant minus a 1 2.

Now, when you take, when you expand from this element, you delete this column and this row, that means, the two values of two linear factors of lambda are not coming in this in this expansion, so that means, this expansion will contain ah maximum lambda raised to power n minus 2, because when you take, when you expand from this, you are deleting this row and this column. So, two factors of lambda are not coming in are not coming in that expansion, that means, the maximum power of lambda, which is coming from this term is maximum power of lambda, which is coming from this term is lambda raised to power n minus 2.

Similarly, if you take a 1 3, so here will be something a 3 3 minus lambda, and when you delete this column and this row, then again two factors of lambda are not coming, so that means, maximum power of lambda, which are coming from this term is lambda raised to power n minus 2. Similarly from the other terms so, what we can conclude that the remaining terms will contain say alpha 1 lambda raised to power n minus 1, and so lambda raised to power n minus 2 and so on, so that means, the power of lambda raised to power n, and power of lambda raised to power n minus 1 are coming only from the first determinant.

Now, if you see here, again if you see here, now if you open from this is ok, if you open from this, here is something a 3 3 minus lambda is again here, if you open from this element, again you will delete this column and this row. So, two powers of lambda are not are still not here I mean not coming, so that means, you see if you take a 1 1 minus lambda, if you expend, this determinant then it is a 2 2 minus lambda, and again some determinant plus and minus a 2 3, and determinant of some other thing, which in which 2 power 2 powers of lambda are not there ok.

So, that means, that means, it will contain maximum lambda raised to power n minus 2, and minus 3, because this determinant is having lambda raise to power n minus 1 1 1 factor is already outside. So, this determinant is having lambda raised to power n minus 1. And two factors are not coming, that means, the maximum power, which this is this element is having is lambda raised to power n minus 3.

And when you multiply this with lambda, so it will be having lambda raised to power n minus 2, so that means, what I want to say basically that when you expand similarly, the entire determinant. So, lambda raised to power n and lambda raised to power n minus 1 are only coming in the product of the diagonal elements. When you similarly try to expend this entire determinant, so lambda raised to power n, and lambda raised to power minus 1 is coming only from product of the determinant diagonal elements ok.

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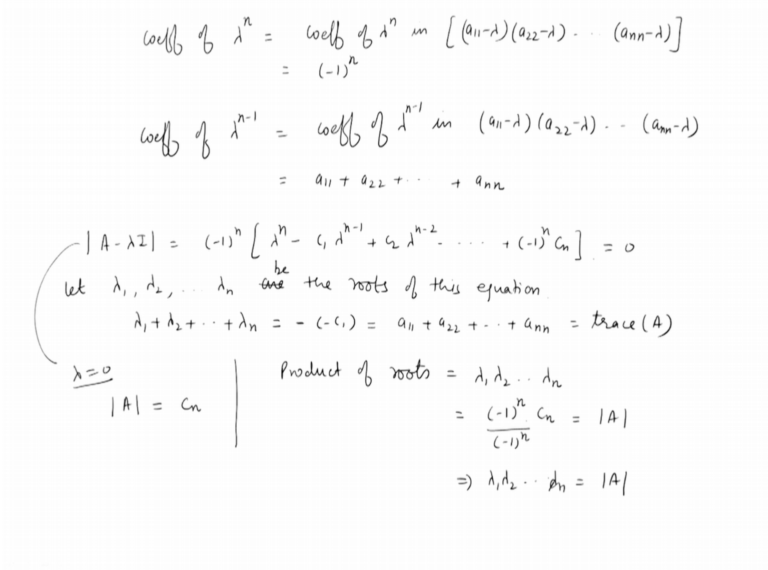

So, and all other and the lambda raised to power n minus 2 lambda raised to power n minus 3 are from the all from all the components, may be from all the components, all the elements, so that means, that means coefficient of lambda raised to power n is simply coefficient of lambda raise to power n in simply a 1 1 minus lambda a 2 2 minus lambda and so on a n n minus lambda. And it is simply minus 1 raised to power n, so carefully see it is simply minus 1 raised to power n.

And similarly, if you want to see coefficient of lambda raised to power n minus 1, it is simplt coefficient of lambda raise to power n minus 1 in again this product a 1 1 minus lambda a 2 2 minus lambda and so on up to a n n minus lambda. So, so what is the coefficient of lambda minus lambda raised to power n minus 1 in this, it is simply a 1 1 plus a 2 2 plus and so on a n n. You can you can simply see, the sum of these elements a 1 1 plus a 2 2 and so on up to a n n will be the coefficient of lambda raised to power n minus 1.

So, what basically now we are having, we are having determinant of A minus lambda I is something like minus 1 raised to power n, and it is lambda raised to power n minus c 1 lambda raised to power n minus 2 and minus 1 plus c 2 lambda raised to power n minus 2 minus and so on minus 1 raised to power n c n, so this is equal to this.

Now, let so how many roots this equation will be having, this equation is will be having n roots, because the it is of n degree polynomial. So, let lambda 1 lambda 2 and so on up to lambda n are the roots be the roots of this equation.

So, if these are the roots, then what is the sum of the roots, you can simply see, the sum of the roots is simply this is the equation. So, some of the roots of this will be simply negative of minus c 1. It is minus negative of negative of minus c 1, and this is equal to basically this is equal to you see the coefficient of lambda raised to power n minus 1 is simply this, so this is a 1 1 plus a 2 2 and so on up to a n n. And this is called a trace of a matrix, trace of a matrix A.

So, what we have concluded, we have concluded that sum of eigen values is nothing but the trace of the metrics, that means, some of the diagonal elements, the first property. The second property is now this result that this is equal to this whole for every lambda for any lambda ok, so this whole for lambda equal to 0 also. And when you substitute lambda equal to 0 we will get determinant of A as equal to c n as c n.

Now, if you find product of roots for this equation, product of roots. So, product of roots are lambda 1, lambda 2, up to lambda n, and that is simply equal to you see it is minus 1 raised to power degree of equation last term upon first term, which is nothing but c n ok, c n upon minus 1 raised to power n, and that is simply equal to determinant of A. So, this implies product of eigen values is nothing but determinant of A. So, we have noted here, two important properties of eigenvalues, the one property is that the sum of the eigen values is equal to trace of the matrix that is the sum of the diagonal elements, and the second property is the product of the eigen values is equal to the determinant of the matrix A.

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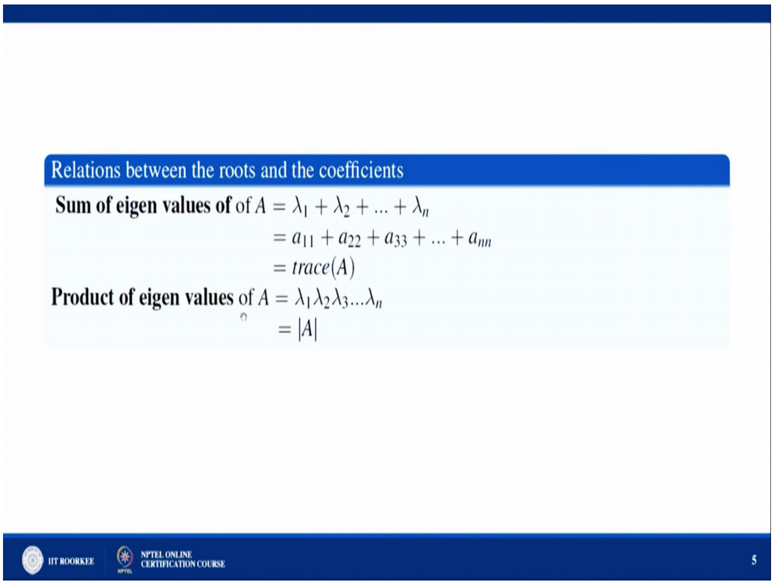

So, the sum of the eigen values simply equal the trace of the matrix. And the product of the eigen values simply the determinant of A.

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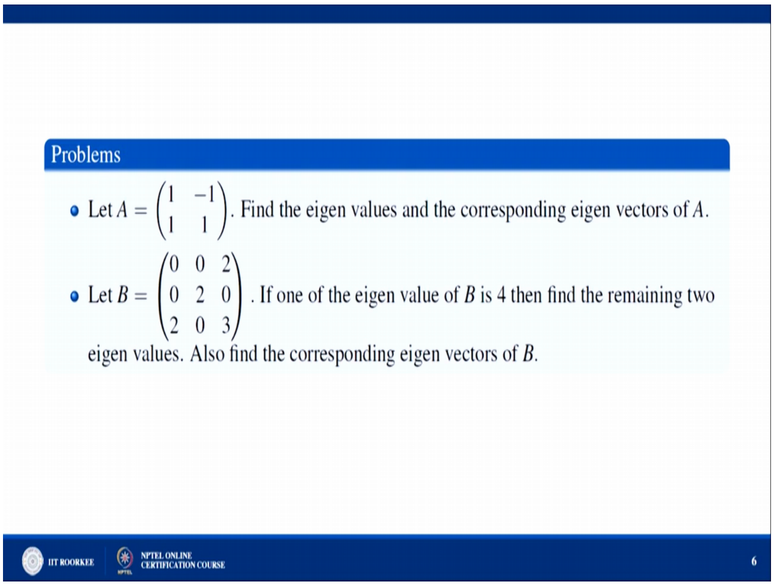

Now, let us let us discuss this problem. The first problem is let us consider this matrix A, which is 1 minus 1 1 1, let us find this eigen values in the corresponding eigenvectors.

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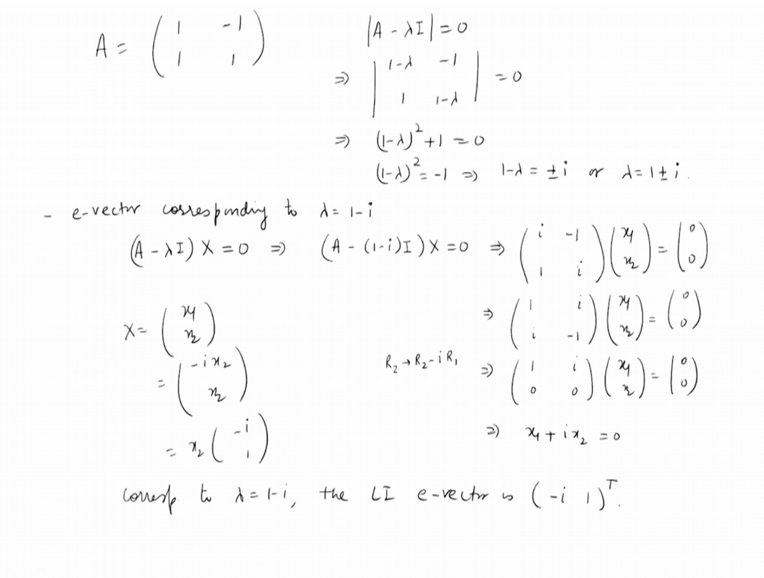

So, the matrix A is simply 1 minus 1 1 1. So, you can simply write A minus lambda I determinant this must be 0 for the characteristic polynomial, or finding the roots of the equation, roots of the latent roots or characteristic roots of this matrix A, so this is nothing but 1 minus lambda minus 1 1 1 minus lambda determinant equal to 0. And this implies on minus lambda whole square plus 1 should be 0, or 1 minus lambda square is equal to minus 1 that implies on minus lambda is equal to plus minus iota, or lambda equal to 1 plus minus iota so that two roots of two eigen values of this equations R 1 plus iota and 1 minus iota, so these are the eigen values.

So, let us find corresponding eigenvectors. So, first you find eigen vector eigen vector corresponding to lambda equals to say 1 minus iota. So, how you will find that you can simply see that A minus lambda I times X must be equal to 0, this we have already seen that this X is a is the eigen vector correspond to this lambda.

So, this implies, now you substitute lambda as lambda as A 1 minus iota, so this is A minus 1 minus iota times iota times identity matrix into x equal to 0. So, this implies, now when you when you take 1 minus lambda here, so it is you can simply take lambda here as 1 minus iota, so it is iota 1, and it is 1 and it is again iota, it is minus 1 it is iota, and x is x 1, x 2 equal to 0 0.

So, this implies, now you can you can apply some elementary row operation this in this matrix. You simply first interchange these two rows, it is 1 iota. Iota minus 1 x 1 x 2 remain as it is, interchange these two also 0 0. Now, this implies you can make 0 here, with the help of this by applying the elementary row operation R 2, this means R 2 minus iota times R 1.

When you apply this elementary row operations here, it is 1 iota it is 0 it is again 0, and it is x 1 x 2 0 0. So, this implies x 1 plus iota x 2 equal to 0. So, this give infinitely many solutions of x 1 and x 2 ok. You can substitute any value of x 1, you can find corresponding value of x 2 such that x 1, x 2 is not equal to 0 of course,.

So, 1 such value is 1 such x 1 is you can simply take you see x is what x is x 1, x 2. You can take x 1 as minus iota x 2 from here, and it is x 2, so it is x 2 times minus iota and 1, where x 2 is any ah real number.

So, so if you are talking about number of linearly independent eigen vector corresponding to lambda equal to 1 minus iota, so it is 1. So, you can pick any 1 linearly independent eigen vector from this, because because it gives infinitely many eigenvectors. You can pick out any one eigenvector, so we can said that vector is linearly independent eigen vector corresponding to lambda equals to 1 minus iota. So, we can say that corresponding to corresponding to lambda equal to 1 minus iota, the linearly independent eigen vector is say minus iota 1 transpose.

Now, similarly you will take ah lambda is equal to 1 plus iota, and we can you find out the corresponding linearly independent eigen vector on the same lines. Now, let us consider second problem. Here the matrix B is 0 0 2 0 2 0 and this matrix. It is given towards at the one of the eigen value of this matrix B is 4, then we have to find the remaining to eigenvalues, and also the corresponding eigenvectors of B.

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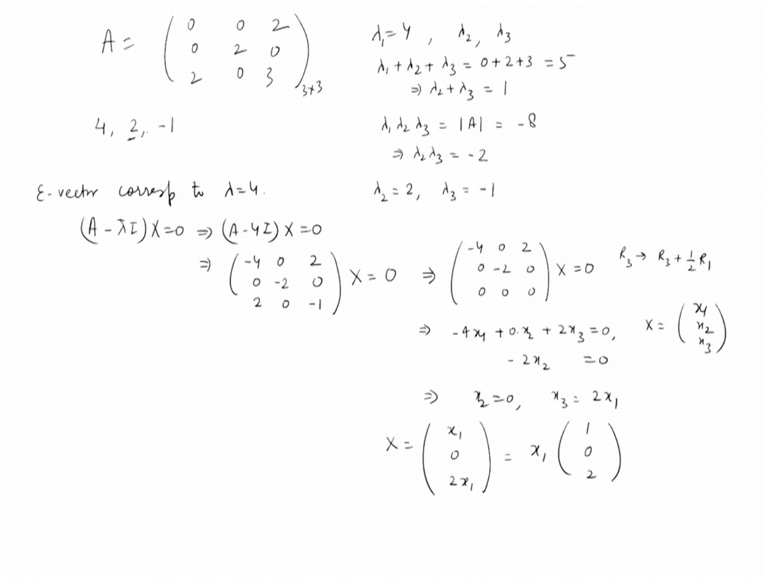

So, let us discuss this problem now. Here the matrix A is simply 0 0 2 0 2 0 and 2 0 3. Now, one lambda is 4, it is given to us say lambda 1 is 4. Now, we know that the trace is equal to the sum of the eigen values. So, let us suppose the other two eigen values are lambda 2, lambda 3, it is of order 3 cross 3, so it will be having three eigen values. So, some of the eigen values is equal to trace of the matrix that is 5 0 plus 2 plus 3. Now, lambda 1 is 4, so this implies lambda 2 plus lambda 3 is 1.

Now, the product of the eigenvalues, we know that product of eigen value is simply determinant of A. Now, the determinant of A, simply when you when you simply open this matrix, I mean determinant, then you simply get it minus 8. Now it is 4, so this implies lambda 2, lambda 3 will be minus 2. So, solving these two equations, we can easily find lambda 2 and lambda 3. So, it is clearly as we as we are seeing it is lambda 2 is 2, and lambda 3 is minus 1. So, sum is 1 and the product is minus 2. So, the other two eigen values are 2 n minus 1.

So, now we can say that this matrix are the eigen values 4, 2, and minus 1. Now, we have to find out the corresponding eigenvectors. Let us suppose, we have to find out the eigen vector corresponding to lambda equal to 4. Similarly, we can find eigen vector cross point lambda equal to 2, and lambda equal to minus 1.

So, let us find eigen vector corresponding to lambda equal to 4. How you will find that this is A minus lambda I X equal to 0 again, so this implies A minus 4 I into X equal to 0.

So, this implies minus 4 0 2, this is 0 minus 2 0, this is 2 0 minus 1, and X is X, which is equal to 0, this implies.

Now, we will try to convert this into its echelon form. So, we will make 0 here with the help of this, this is already 0, we will make 0 here with the help of this. So, which operation we will apply R 3 2 R 3 plus half of R 1. So, this is minus 4 0 2, this is 0 minus 2 0, this is 0 0, this plus half of this is against 0. Now, this implies, this implies minus 4 x 1 plus 0 into x 2 plus 2 into x 3 equal to 0. If you are taking X as x 1, x 2, x 3, it substitute it here, and we multiplied these two matrices, then we simply get these equations, and minus 2 x 2 equal to 0. So, this implies x 2 equal to 0, and x 3 as 2 x 1.

So, what is the corresponding eigen vector, corresponding eigen vector will be you can say, you can see, it is x 1 0 and 2 x 1, so it is x 1 times 1 0 2. So, again there are infinitely many eigenvectors corresponding to lambda equal to 4, but if you are talking about number of linearly independent eigen vector correspond to lambda equal to 4, so that is 1. You can pick any one eigen vector from this space, so we can say that is a linearly independent eigen vector correspond to lambda equal to 4. Similarly, we can find out eigen vector corresponding to lambda equal to 2, and lambda equal to minus 1.

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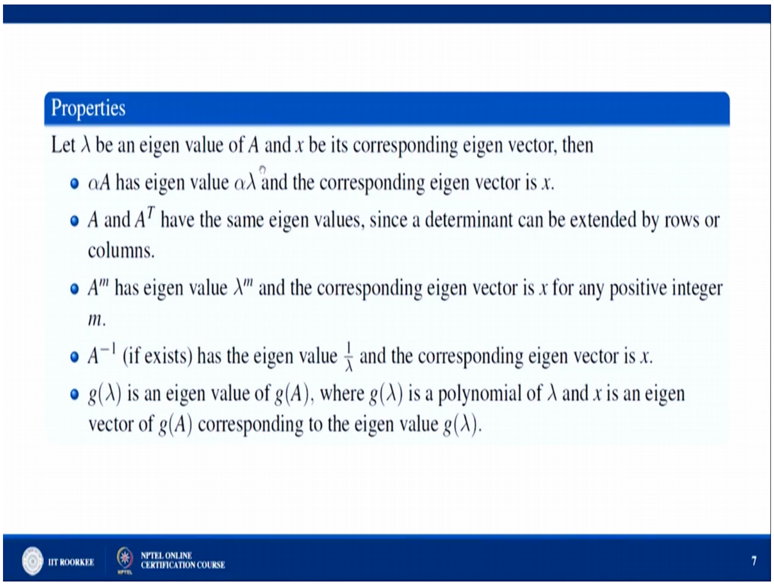

Now, if lambda is an eigen value of matrix A, and x is a corresponding eigen vector, then we have the following properties. Now, it is very easy to see these properties, you see here.

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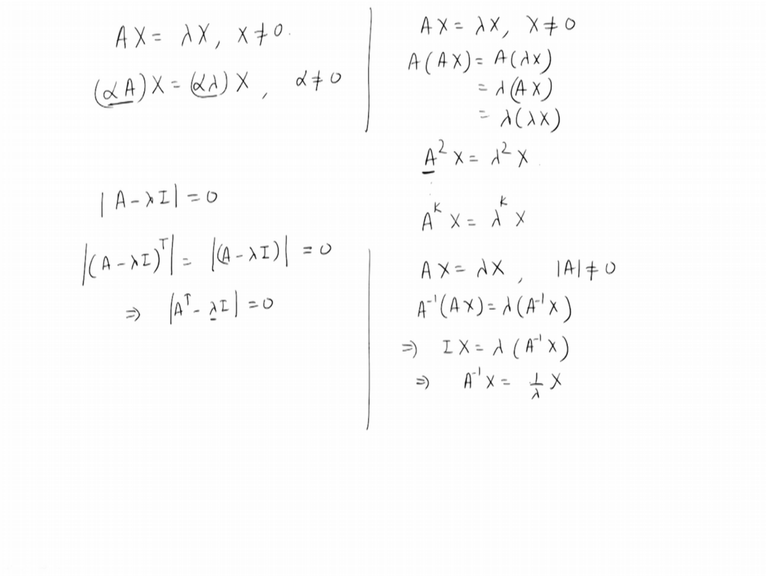

See if A has a eigen vector eigen value lambda and corresponding eigen vector is X, that means, A X equal to lambda X. Now, if you want to calculate the eigen, eigen values of alpha A, where alpha is a non-zero scalar. Then how we can find this, you simply multiplied these two by alpha, both sides by alpha. So, we can simply see that we can simply see that this alpha A has an eigen value alpha lambda, and the corresponding eigen vector is A corresponding eigen vector is X.

Now, if you want for A square, you see A X equal to lambda X, X not equal to 0. If you multiply both sides by A, it is A into X equals to A into lambda X, which is lambda times A X. And A X is lambda X, so it is lambda X. So, we can say that A square X is equals to lambda square X. So, what we can say that the if A as an eigen value lambda and the corresponding eigen vector is X, then A square as an eigen value lambda square and the corresponding eigen vector is X.

Similarly, similarly we can say that A raised to power K is equals to lambda raised to power K into X, you can simply find similarly find A cube, A raised to power 4 and so on. So, what we have concluded, we have concluded that if A as an eigen value lambda and the corresponding eigen vector is X, then A raised to power K has eigen value lambda raised to power K, and the corresponding eigen vector is X.

Now, next property is A and A transpose, both have same eigen values that is very easy to show, you can see that eigen values of or given by this expression. Now, A minus A this matrix, the transpose of this matrix is equals to A minus the determinant of this is same as determinant of this, because by changing interchanging rows and columns will not change the value of the determinant. This is equal to 0, it is given to us, and this implies A transpose minus lambda I whole determinant is equal to 0 so that lambda is not changing here, whatever lambda we are we are having here, same lambda we are having here, that means, the eigen values of A and A transpose are same.

The next is the next is if you say that A X equal to lambda X, and suppose determinant of A is not equal to 0, that means, A is invertible. Then you can take A inverse both the sides, then a inverse of A X will be equals to lambda times A inverse of X. And this implies, identity into x equals to lambda into A inverse of X. And this implies, A inverse of X will be 1 by lambda time X, this is this is this is one or identity here. So, we can say that if A as an eigen value lambda and the corresponding eigen vector is X, then A inverse as an eigen value 1 by lambda and the corresponding eigen vector is X.

So, these are few of the properties of eigenvalues, which is stated here. And if g lambda is an eigen value of g A, where g lambda is a polynomial of lambda, where x and x is an eigen vector of g A corresponding to eigen value g lambda. That that is very easy to show again, because the third property itself is states that if g lambda is a polynomial of lambda, then g lambda is an eigen value of g A.

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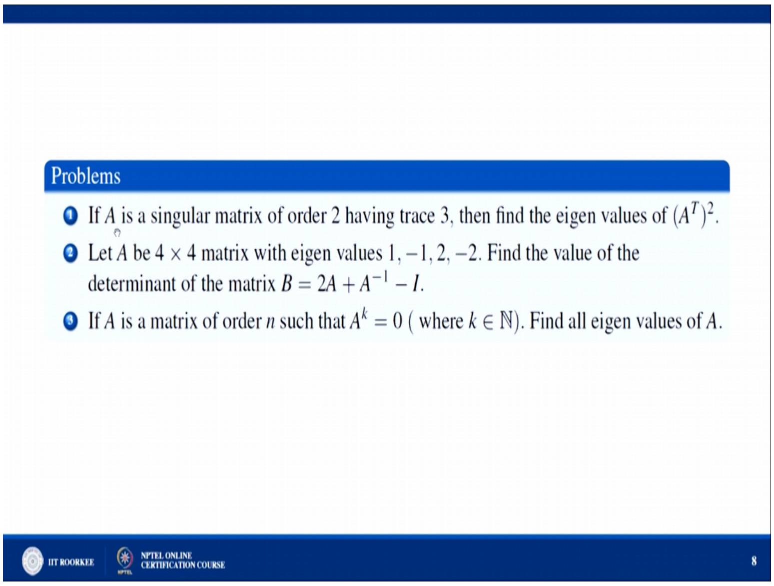

Now, let us let us discuss these few problems quickly. You see, if A is a singular matrix of order 2 and having trace 3. If it is singular matrix means, determinant is equal to 0, determinant is nothing but product of eigenvalues, product of eigen values equal to 0 means at least one of the eigen value is 0. Now, it is order to, that means, it has it is having only two eigenvalues, say lambda 1, and lambda 2.

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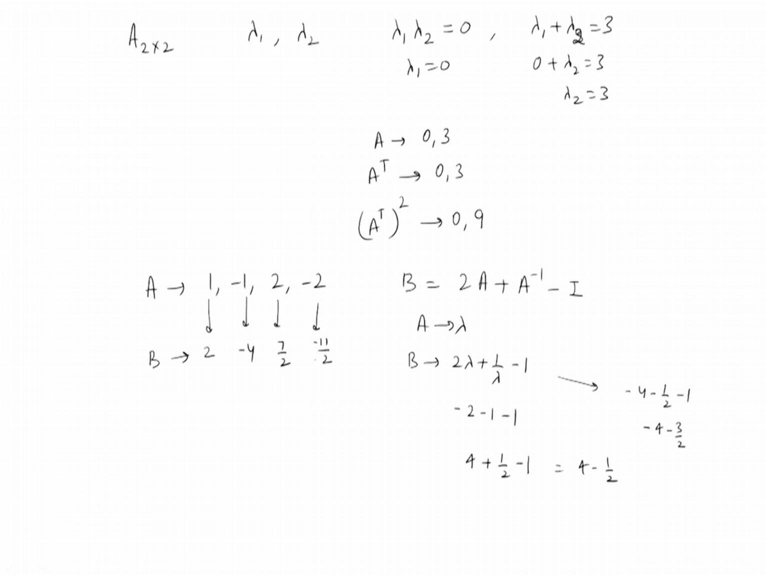

So, so we can say A is a order 2 cross 2 having two eigenvalues, lambda 1, and lambda 2. Then product of eigen values is equal to 0, because matrices singular, and trace is 3, trace is 3. So, here from here, we can say that one of the eigen value is 0 say lambda 1 is 0. If lambda 1 is 0, so lambda 2 will be 3. So, the eigen values are 0 and 3 of A; eigen value of A transpose will be again 0 and 3, and eigen value of A transpose square will be 0 and 9 ok.

The next one is if A is a 4 cross 4 cross 4 matrix of eigen values 1 minus 1 2 minus 2. Then the eigen values of this how we can find, you see eigen values of A are 1 minus 1 2 minus 2, and B is given as it is 2 A it is 2 A plus A inverse minus I. So, it is some polynomial in I mean, it is some polynomial in I mean B, I mean A. So, the eigen value of B will be nothing but if A as an eigen value lambda, then the eigen value of B will be nothing but 2 lambda plus 1 by lambda minus 1 2 has 2 lambda eigen value A inverse as 1 by lambda eigen value and this eigen value is 1.

So, simply, simply you can substitute here, cross point to 1, where having 2 plus 1 plus 1 minus 1 is 2 for B. Cross point to minus 1 if it substitute minus 1, it is minus 1 minus 1 that is minus 4. Cross point to 2, it is 1 plus 1 by 2 minus 1 that is that is 4 minus 1 by 2 that is 7 by 2. Cross point to minus 2, if I talking about minus 2, because minus 2, it is minus 4 minus 1 by 2 minus 1 that is minus 4 minus 3 by 2 that is minus 11 by 2. So, these are the eigen values corresponding to B.

So, the third problem is simple, if A is A matrix of order n, and such that A raised to power k is 0, where k belongs to a natural number. Then all this eigen values are 0, because if one of because if any of the eigen value is not equal to 0 of A, then A raised to power k will not be 0 ok. So, therefore, all the eigen values of a must be 0.

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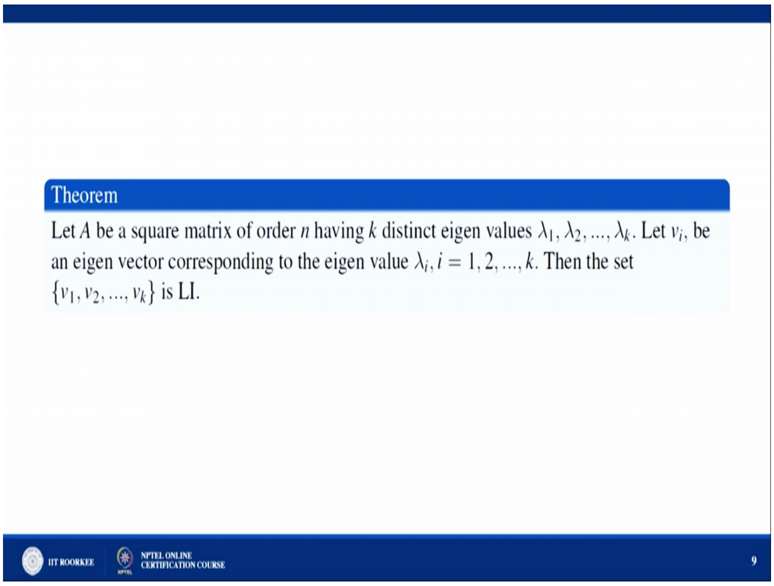

So, this is a result that if A is A square matrix of order n having k distinct eigen values lambda 1 up to lambda k. Let v i be the eigen vector corresponding to the eigen value lambda i, i from 1 to k. Then the set this is linearly independent.

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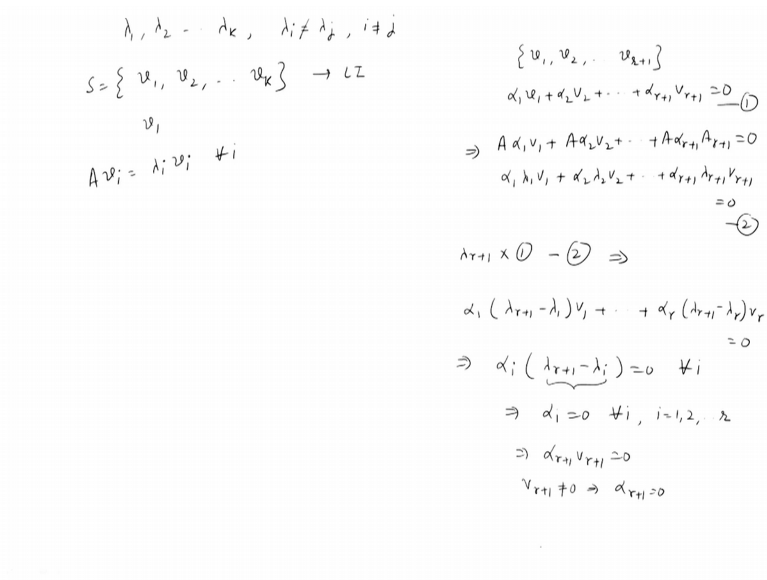

So, let us try to prove this result. You see, lambda 1, lambda 2 up to lambda k are distinct eigen values, that means, that means, lambda i is not equal to lambda j for i not equal to j number 1. Now, the set we are considering this v 1, v 2 up to v k, we have to showed at this set is linearly independent. v 1 is a eigen vector linearly independent eigen vector correspond to lambda 1, v 2 is a linearly independent eigen vector correspond to lambda 2 and so on. So, we will prove this by the method of induction ok.

Let us for n equal to 1 for k equal to 1, we are having v 1, and we v 1 a single set is I mean, it is it is linearly independent. I mean singleton set is always linearly independent, singleton non-zero vector, and v 1 is a non-zero vector is always linearly independent. Now, we will assume that it is true for k equal to r I mean this result hold for k equal to r, that means, the set of k r vectors are linearly independent. And we will try to showed that this also holds for k equal to r plus 1.

So, take the linear combination of these vectors now, to in order to showed that these are linearly independent. Now, you multiply both sides by matrix A, so we can simply take A into alpha 1 v 1 plus A into alpha 2 v 2 and so on A into alpha r plus 1 v r plus 1 equal to 0.

Now, A into v i is lambda i into v i for all i, because lambda i is the eigen value and the corresponding eigen vector is v i. So, we can simply write lambda 1 A v 1 is a lambda 1 v 1, similarly alpha 2 lambda 2 v 2 and so on up to alpha r plus 1 lambda r plus 1 v r plus

1. Now, say this equation 1, and this is equation 2. You can multiply 1 by lambda r plus 1 into 1 and subtract with 2, so what we will obtain, we will obtain lambda alpha 1 into lambda r plus 1 minus lambda 1 v 1 and so on up to alpha r lambda r plus 1 minus lambda r times v r equal to 0.

Now, we have already assumed that a set of r vectors are linearly independent. So, this is some linear combination of this linearly independent vector, which is equal to 0. So, this implies, alpha i is lambda r plus 1 minus lambda i equal to 0 for all i. But, eigen values are distinct, so this is not equal to 0. So, this implies, alpha equal to 0 for all i for all i means for i from 1 to r.

And when you substitute alpha from 1 to r here with all 0, then you will get you will get alpha r plus 1 v r plus 1 equal to 0. And since, v r plus 1 is not equal to 0 is eigenvector, so this implies, alpha r plus 1 equal to 0. So, we have shown that all alpha i's from i from 1 to r plus 1 equal to 0 that mean, this set of vectors are linearly independent. So, in this lecture, we have seen that that what eigen values and eigenvectors are, and some of the important properties of eigen values and eigenvectors.

So, thank you.