Matrix Analysis with Applications
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Lecture – 12
Matrix Associated with a LT

Hello friends, welcome to lecture series on matrix analysis with applications. In the last lecture, we have seen that how we can find out inverse of a linear transformations. We have seen that if T is invertible that is T is 1 1 and onto then T inverse is also linear and invertible. In this lecture, we will discuss that how we can find out a matrix associated with a linear approximation.

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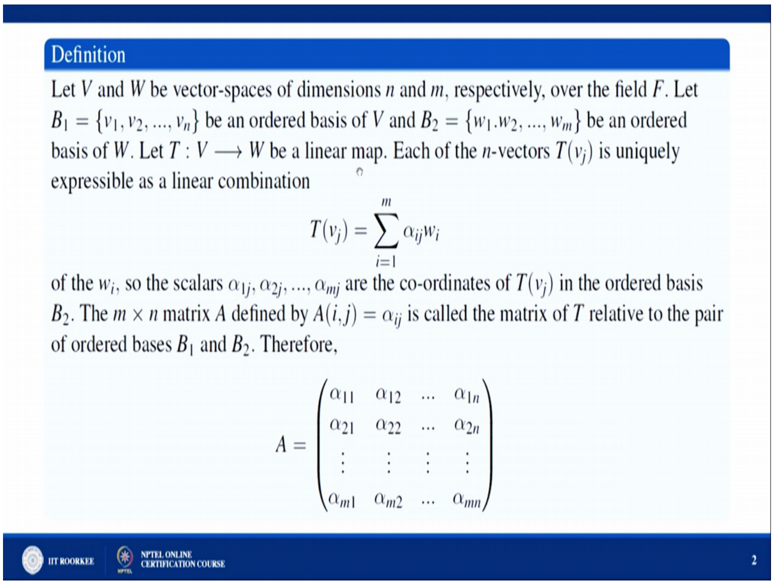

Let us suppose that V and W be vector spaces of dimensions n and m, respectively ok. Let us suppose these 2 are finite dimension vector spaces of dimension n and m over the field F. Let B 1 which is given by v 1, v 2 up to v n be an ordered basis of V and B 2 which is w 1, w 2 up to w m be an ordered basis of W. Let T from V to W be a linear map, be a linear transformation ok.

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$$T: V \to W, \quad B_1 = \{v_1, v_2, \ldots, v_n\} \to \text{ordered basis of } V$$
$$B_2 = \{w_1, w_2, \ldots w_m\} \to \text{'' '' '' } W$$

$$T(v_1) = \alpha_{11} w_1 + \alpha_{21} w_2 + \cdots + \alpha_{m1} w_m$$
$$T(v_j) = \alpha_{1j} w_1 + \alpha_{2j} w_2 + \cdots + \alpha_{mj} w_m$$
$$T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i, \quad j = 1, 2, \ldots n$$

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & & \alpha_{mn} \end{pmatrix}_{\underline{\underline{m \times n}}}$$

Now, what we are discussing basically here, you see. We have a linear transformation T from V to W B 1 which is given by v 1, v 2 up to v n is an ordered basis of V and B 2 which is given by w 1, w 2 and so on up to w m is an ordered basis of W ok; dimension of V is n, dimension of W is m. Now if you take T of say v i or v 1 suppose, first you take T of v 1; T of first element.

Now T of first element will be some element in w and that element in w can be expressed as the linear combination of element of the basis of w. So, so; that means, that means this T of v 1 can be written as alpha 1 1 w 1 plus alpha 1 2 w 2 plus and so on alpha 1 m w m or you can write like this alpha 2 1 alpha m 1 ok. These are these are the scalars, you see.

Similarly, if you want to write out say T of v j where j is running from 1 to n. So, it will be a alpha it will be alpha 1 j w 1 plus alpha 2 j w 2 and so on up to w alpha m j into w m or it will be written as summation i running from 1 to m alpha i j w i and j is running from 1 to n so; that means, each of the each of the element of this b 1 can be expressed as linear combination of elements of w i's or w yeah w i's.

Now if you take the first element v 1 and T of v 1 is given by this and these are called these scalars are called coordinates of this v 1, I mean T v 1 sorry. The coordinates of T v 1 are alpha 1 1, alpha 2 1 up to alpha m 1. The coordinates of v 2 will be similarly alpha 1 2, alpha 2 2 alpha and so on up to alpha m 2. So, if you represent matrix corresponding to this like this, the first coordinate the first coordinate of T v 1 as a column vector alpha

1 1 alpha 2 1 and so on up to alpha m 1. The second vector is alpha 1 2 alpha 2 2 and so on up to alpha m 2 and the last 1 is alpha m 1 n alpha 2 n and so on up to alpha m n. So, this m cross and matrix, this matrix correspond to this linear transformation respect to the basis b 1 and b 2 is the matrix associated with T respect to the ordered basis b 1 and b 2 ok.

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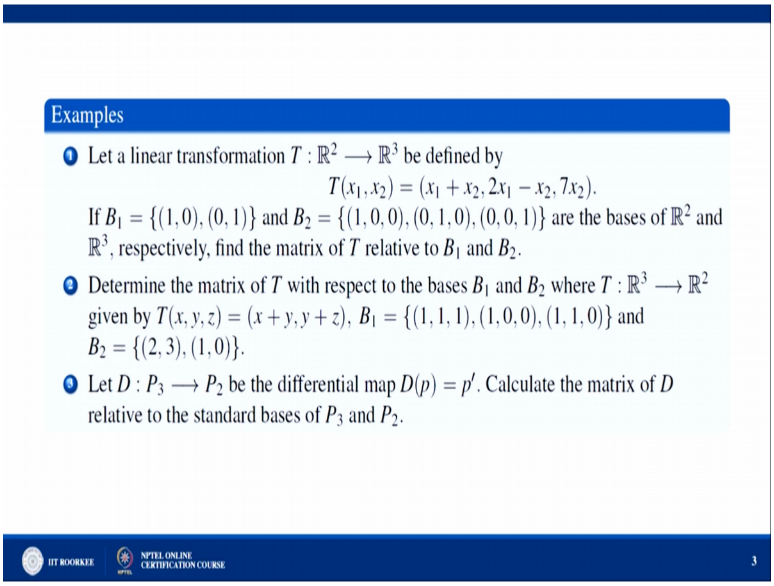

So, this is same A which we have discussed and this is called the matrix of T related to the pair of ordered basis B 1 and B 2 ok. Now, to understand this let us discuss few examples or few problems based on this.

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$T: \mathbb{R}^2 \to \mathbb{R}^3, \quad T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2, 7x_2)$

$B_1 = \{(1,0), (0,1)\}$

$B_2 = \{(1,0,0), (0,1,0), (0,0,1)\}$

$T(1,0) = (1,2,0)$

$= 1(1,0,0) + 2(0,1,0) + 0(0,0,1)$

$T(0,1) = (1,-1,7)$

$= 1(1,0,0) - 1(0,1,0) + 7(0,0,1)$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}_{3 \times 2}$$

The first problem is let us consider T from R 2 R 3 ok. The simple example and T is given by T of x 1 x 2 is given by x 1 plus x 2 2 x 1 minus x 2 and 7 x 2 and the basis of the first basis of R 2 is the ordered basis is 1 0 and 0 1, the standard basis and the basis of R 3 is 1 0 0, 0 1 0 or 0 1, 0 0 1 the standard basis of part 3.

Now, we have to find out the matrix of T respect to with respect to the basis B ordered basis B 1 and B 2. So, how we can find out that matrix? So, let us see here, first we find T of 1 0 ; this T of 1 0, T of 1 0 from here is it is 1, it is 2, it is 0. Now what will be and the and this is nothing, but this is nothing, but 1 times 1 0 0 2 times 0 1 0 and of course, 0 times 0 0 1. So, what are the coordinates of this vector T 1 0 ? The coordinate of this vector are 1 2 and 0 and you write this as a column of the matrix of T.

Again T of 0, 1 is nothing, but you simply take 1, it is minus 1 and it is 7 which can be written as 1 time 1, 0, 0 minus 1 time 0, 1, 0 and plus 7 time 0, 0, 1. So, the coordinate of this vector is 1 minus 1 and 7. So, the matrix associated with this linear transformation corresponding to the basis B 1 and B 2 is given by the first coordinate 1 2 0 as a column vector, 1 2 0. the the second 1 is 1 minus 1 seven as a column vector 1 minus 1 7

So, this this will be the matrix associated with associated with this linear transformation T corresponding to these standard basis of R 2 and R 3. Say now we have a second example, you see the second example is determine the matrix of T with respect to the basis B 1 and B 2 where T is given by this expression B 1 is this and B 2 is this. So, how we can find this ?

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$T : \mathbb{R}^3 \to \mathbb{R}^2 \ , \quad T(x, y, z) = (x+y,\ y+z)$

$B_1 = \{(1,1,1),\ (1,0,0),\ (1,1,0)\}$

$B_2 = \{(2,3),\ (1,0)\}$

$$T(1,1,1) = (2,2) = \tfrac{2}{3}(2,3) + \tfrac{2}{3}(1,0)$$

$$T(1,0,0) = (1,0) = 0\,(2,3) + 1\,(1,0)$$

$$T(1,1,0) = (2,1) = \tfrac{1}{3}(2,3) + \tfrac{4}{3}(1,0)$$

$$A = \begin{bmatrix} 2/3 & 0 & 1/3 \\ 2/3 & 1 & 4/3 \end{bmatrix}_{2\times 3}$$

So, now T is from R 3 to R 2 and T of x, y, z is given by x plus y and y plus z. So, B 1 is the basis of R 3 is given as 1, 1, 1 is given to us 1, 0, 0 and 1, 1, 0 and bases of R 2 is as 2, 3 and 1, 0.

So, of course, corresponding to different basis for the same linear transformation, the matrix will be different if you have this linear transformation, but we have a difference bases of R 3 and R 2. So, there will be a different matrices of this linear transformation corresponding to the basis B 1 and B 2 ok. Now suppose so, of course, the order of the matrix will be 2 cross 3. It is clear from here ok. So, what is T of 1, 1, 1? First you find this T of 1, 1 from here is 2 coma 2 and now this can written as you have to write this is as a linear combination of 2, 3 and 1, 0 ok. So, so it is 3 and you want 2. So, it is it is the 2 by 3 times, then only it will become 2 because no no no second component here. Now it is 4 by 3 and you to want to make it 2. So, it will be 2 by 3, then only it is 6 by 3 means 2.

So, the coordinates of vector with respect to with respect to this basis is 2 by 3 and 2 by 3. Now if you take T of 1, 0, 0 T of 1, 0, 0 here is you take 1 plus 0 is 1 and 0 plus 0 is 0 which is 0 times 2, 3 plus 1 time 1, 1, 0. So, the coordinate of this vector will be 0, 1.

Now T of 1, 1, 0 will be you see 1 plus 1 is 2 and 1 plus 0 is 1 and this can be written as you see you want to write it the linear combination of these two vectors. So, you want 1 here, so it is 1 by 3 and you want 2 here; a 2 by 3 means 4 by 3 ok.

So, the matrix corresponding to this linear transformation with respect to these 2 bases will be the first coordinate 2 3 2 by 3 2 by 3 as a column vector, the second vector; the second coordinate 0, 1. The third vector is 1 by 3 and 4 by 3. So, it has it is a order 2 cross 3. So, this will be the corresponding linear transformation ok. The third problem is you consider a linear map, I mean linear transformation d from p 3 to p 2 which is a differential map ok.

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$D: P_3 \to P_2, \qquad D(f) = f'$

$B_1 = \{1, x, x^2, x^3\}$

$B_2 = \{1, x, x^2\}$

$D(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$D(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$D(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$

$D(x^3) = 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3 \times 4}$$

Now, we have to calculate the matrix of T relative to the standard basis of P 3 and P 2. So, how we can do that? Now, here D 3 D is from P 3 to P 2 and D of f is simply f dash or T of D of P is simply P dash ok, this for the standard basis. So, standard basis of P 3 is 1 x x square and x cube. The standard basis of B 2 is 1 x and x square. Now what will be D of 1, D of this element 1 because it is a differential operator it is 0 and 0 can be written as 0 times 1 plus 0 times x plus 0 times x square. The linear combination of these 3 elements Differential of x is 1 which can be written as 1 times 1 plus 0 times x plus 0 times x square plus 0 times x cube sorry 0 times x square because this element this element is in P 2.

Now, what is D of x square D of x square is 2 x which is 0 times 1 plus 0 plus 2 times x plus 0 times x square and what is D of x cube it is 3 x square which is equals 2 0 times 1 plus 0 times x plus 3 times x square. So, what is the correspondingly corresponding matrix? Corresponding matrix will be you see here the coordinates are 0 0 0. So, as a column 0 0 0, here coordinates are 1 0 0 1 0 0. Here coordinates are 2 0 2 0 which is 0 2 0. Here coordinates are 0 0 3, 0 0 3. So, this will be the matrix corresponding to it is of it of 3 row 4 columns yeah. So, this is a corresponding matrix respect to this linear transformation correspond to these 2 standard basis of P 3 and P 2.

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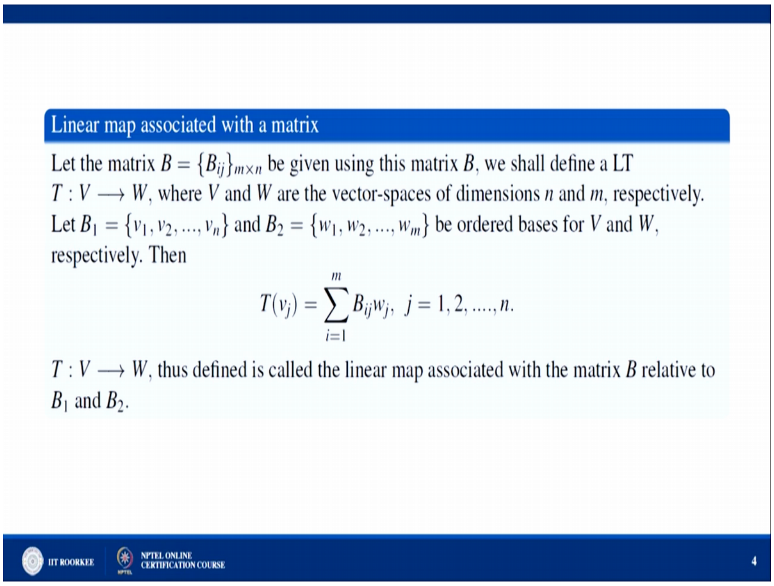

Now, if a linear map is known and a standard and the basis are known, then we then we can find out the corresponding matrix ok. Now, if if a matrix is known and basis are known, then can we find out the correspondingly linear map from V to W? So, the answer is yes. So, how we can find it, let us see here.

So, let us suppose B a matrix B which is given by B i j of order m cross n be given. Now using this matrix B, we can shall define a linear transformation T from V to W where V and W are the vector spaces of dimensions n and m respectively. Now to understand this, let us suppose B 1 which is a V which is given by v 1, v 2 up to v n and B 2 which is w 1, w 2 up to w m be the ordered basis of V and W respectively. Then how we can find out the correspondingly linear transformation from v to w?

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$T: V \to W \qquad B = (B_{ij})_{m \times n} \qquad \underline{B}_1 \ \& \ B_2$

$$= \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & & & \\ B_{m1} & & \cdots & B_{mn} \end{pmatrix}_{m \times n}$$

$$T(v_1) = B_{11} w_1 + B_{21} w_2 + \cdots + B_{m1} w_m$$

$$T(v_j) = \sum_{i=1}^{m} B_{ij} w_i \,, \quad j = 1, 2, \ldots n$$

You see, we are having T from V to W, a matrix B which is a B i j a of m cross n is known to you that matrix which is associated with the linear map cause for the basis B 1 and B 2. B 1 and B 2 are the basis of V and W ok.

Now again so, this matrix be something like you see, it is B 1 1 it is B 1 2 and so on up to B 1 n. It is B 2 1 B 2 2 and so on up to B 2 n and it is B m 1 and so on up to B m n. It is of order m cross n. Now if you if you want to write the first element T of v 1 which is the first element of the basis of v 1. So, this is the some element of W. So, again this can be written as linear combination of elements of basis of W which elements of B 2 and the coordinates we know that the coordinates of this vector is always in the column vector is always from the column.

So, this T of v 1 can be find as B 1 1 of w 1 plus B 2 1 of w 2 and so on up to B m 1 of w m. So, this is this is known the coefficients all B i j’s are known and w i w i’s are known. So, similarly if you want T of v j’s or v j’s yeah, So, this will be say B of B of i j and it is w i and summation i is varying from 1 to m. It is T of v j’s and i is yeah j is running from 1 to n. So, if if we know that T of all v j’s, then we can easily find out the correspondingly linear map. How we can find this? Let us discuss this by an example.

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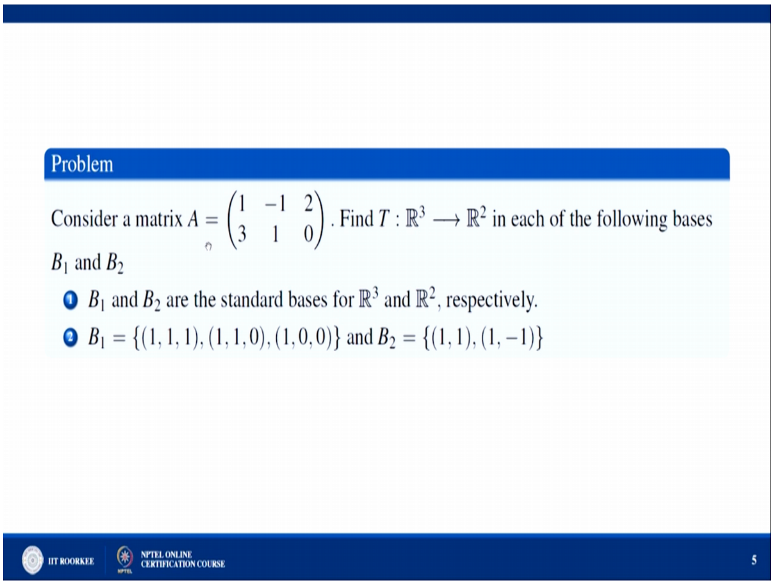

You consider a matrix A of 1 minus 1 2 3 1 0. We have to find out the linear transformation T from R 2 to R R 3 from R 2 in each of the following basis B 1 and B 2. The first of all, the first the first 1 is standard basis of R 3 and R 2. So, how we can see this?

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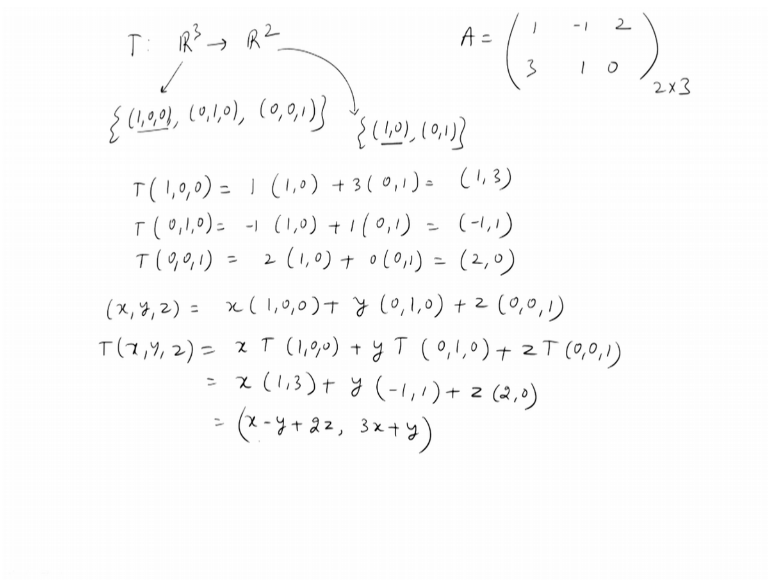

T is from R 3 to R 2. Matrix A for both the problems are is same ok. Now standard basis of R 3 is 1, 0, 0 0, 1, 0 and then 0, 0, 1. The standard basis of R 2 is 1, 0 and 0, 1 ok; that that you already know.

Now, what is T of 1, 0, 0, T of this element? Now T of this element will be given by the matrix associated with this linear map correspond to these standard basis, this matrix which is given to us and the first column correspond to the coordinates of the first vector that is T of v 1. So, this is nothing, but 1 times 1, 0 the first the first element of bases, quadrate bases plus 3 times 0, 1. So, that is simply 1 coma 3. Again T of 0, 1, 0 will be written as the second column that is minus 1 times 1 0 and 1 times 0, 1 that will be minus 1 coma 1.

Now T of 0, 0, 1 will be equal to 2 times 1, 0 plus 0 times 0, 1 which is 2 coma 0. Now, if you write any x, y, z in R 3. So, it can be written as x times 1, 0, 0 plus y times 0, 1, 0 plus z times 0, 0, 1. So, T of x, y, z will be equal to x of T of 1, 0, 0 because T is a linear map ;y of T of this element and z of T of this element and this is x into T of 1, 0, 0 is this is 1, 3 given to us, y of this is minus 1, 1 and z of this is 2 coma 0. So, this will be x minus y plus 2 z coma 3 x plus y. So, this is a required linear map. So, matrix is known and a correspondingly basis are known, then also we can find out the linear map ok.

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$T: \mathbb{R}^3 \to \mathbb{R}^2 \qquad A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}$

$B_1 = \{(1,1,1), (1,1,0), (1,0,0)\}$

$B_2 = \{(1,1), (1,-1)\}$

$T(1,1,1) = 1(1,1) + 3(1,-1) = (4,-2)$

$T(1,1,0) = -1(1,1) + 1(1,-1) = (0,-2)$

$T(1,0,0) = 2(1,1) + 0(1,-1) = (2,2)$

$(x,y,z) = \alpha(1,1,1) + \beta(1,1,0) + \gamma(1,0,0)$

$\alpha + \beta + \gamma = x \longrightarrow \cancel{\gamma = x - \beta - \gamma}$

$\alpha + \beta = y \Rightarrow \beta = y - z \qquad \gamma = x - \alpha - \beta$

$\alpha = z \qquad\qquad = x - z - (y - z)$

$\qquad\qquad = x - y$

$(x,y,z) = (z)(1,1,1) + (y-z)(1,1,0) + (x-y)(1,0,0)$

$T(x,y,z) = z(4,-2) + (y-z)(0,-2) + (x-y)(2,2)$

$= (\quad , \quad)$

In the second example, if you see a second example where basis are this and this. So, how we can find out the corresponding linear map? You again you can see here. Again T is from R 3 to R R 2 and A is given to us which is 1 minus which is same B 1 is 1, 1, 1 1, 1, 0 and 1, 0, 0 B 2 is 1, 1 and 1 minus 1.

Now, first you write T of 1, 1, 1; the first element the coordinates are 1, 3. So, 1 times 1, 1 plus 3 time 1 minus 1 which is 4 coma minus 2. Now T of 1, 1, 0 will be equal to the coordinates are minus 1 times 1, 1 plus 1 times 1 minus 1 which is the 0 and minus 2 T of 1, 0, 0 will be equal to 2 times 1, 1 plus 0 times 1 minus 1 which is 2 comma 2. Now you express any x y z in R 3 as a linear combination of these 3 elements ok. You try to find out alpha, beta and gamma in terms of x, y, z. How we can do that? You can simply have 3 equation you see alpha plus beta plus gamma from here is equal to x alpha plus beta is equal to y and gamma is equal to z.

So, from here, we will obtain as beta equal to y minus z and from here we obtain as alpha equal to x minus beta minus gamma. You see alpha plus beta plus gamma equal to x alpha plus beta equal to y and gamma equal to an alpha equal to an alpha equal to sorry because it is alpha. Alpha equal to z means alpha is z. So, beta will be is equal to y minus z it is and how we will find gamma. So, gamma will be equal to from this equation gamma will be equal to x minus alpha minus beta. It is x minus alpha is z and beta is y minus z. So, this is x minus y z z cancels out. So, this is x minus y. So, what is x what is x, y, z? It is it is z times 1, 1, 1 plus y minus z times 1, 1, 0 plus x minus y times 1, 0, 0 and T of x, y, z will be equal to z of T of 1, 1, 1 which is 4 minus 2 plus y minus z times T of 1, 1, 0 which is 0 coma minus 2 plus x minus y; T of this which is 2 coma 2.

Now, you can simplify this and you can find out the correspondingly linear map from R 3 to R 2 respect to these 2 bases B 1 and B 2. So, if we know the linear transformation T and the corresponding basis of V and W, then we can find out A matrix associated of matrix associated with T correspond to those basis B 1 and B 2. And conversely, if you know the matrix and the basis of V basis of V and W, then we can find out the linear map ok. So, in the next lecture we will see some more properties of vector space and linear transformations.

Thank you.