Matrix Analysis with Applications
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Lecture – 11
Inverse of a Linear Transformation

Hello friends welcome to lecture series on Matrix Analysis with Applications. In the last lecture we have seen that what linear transformation is and what are the fundamental properties of linear transformation. Also, we have seen that what the rank of linear transformation and how we can define null space or nullity of a linear transformation T. Now, in this lecture we will see that how we can find inverse of a linear transformation ok.

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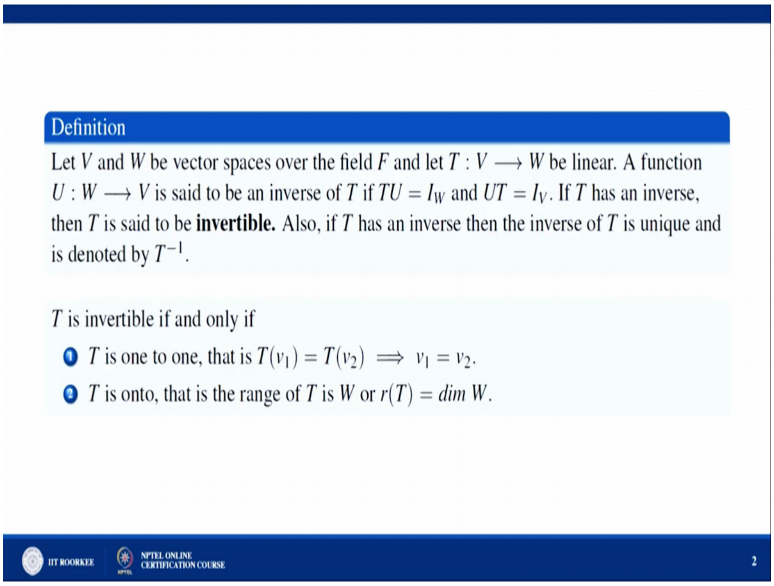

Now, first here definition that let V and W be vector spaces over the field F and let T is a linear transformation from V to W. A function U from W to V is said to be an inverse of T if T T of U is identity of W and U of T is equal to identity of V. If T has an inverse, then T is said to be an invertible ok.

Invertible means they are exist linear transformation U from W to V such that T U equal to identity of W and U T equal to identify of V. Also, if T has an inverse then the inverse of T is unique and is denoted by T inverse ok.

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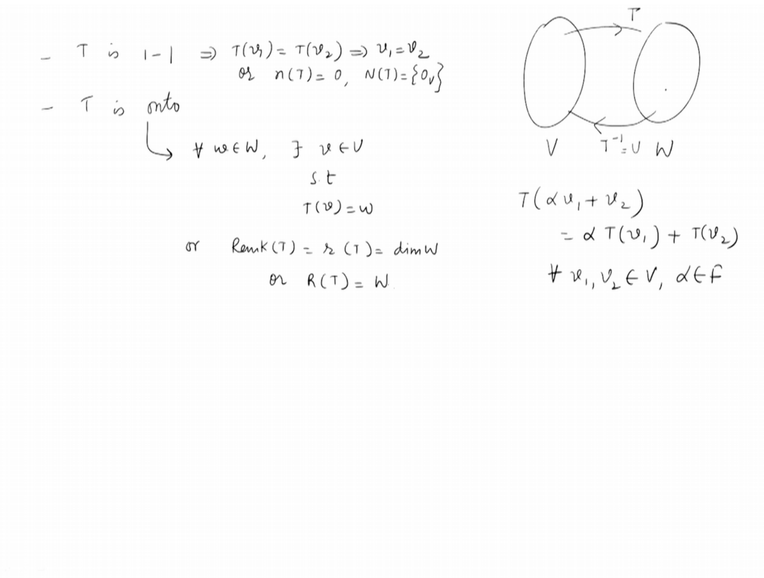

So, we have already seen that if we have a linear transformation T from say V to W T then we say that this T is a linear transformation, if T of alpha v 1 plus v 2 is equal to alpha T v 1 plus T v 2 for all v 1 v 2 in V and alpha belongs to feet ok. Now, T inverse is a linear transformation from W to V ok, this is T inverse T is the linear transformation from V to W and T inverse is the linear transformation from W to V. Now, when this T inverse exist? These T inverse exist if we have already seen that this is equal to U basically what we have say U, then this will exist if T of U equal to identity of V and U of T equal to identity of W.

Now, T inverse will exist if and only if the number 1 point is first point is T must be 1 to 1 and the second is T is onto that means, ah; that means, bijective something like bijective mapping ok.

Now, T is 1 to 1 means T of v 1 equal to T of v 2 implies v 1 equal to v 2 or nullity of T is 0 or null space of T is simply singleton 0 of V. Now, T is onto this means T is onto means for every w in W there exist v in V such that T of v equal to W. If you take any W in here they will always exist a p image v in V such that T of v equal W or we can defined that rank of T; rank of T which is r T should be equal to dimension of W or range of T is equals to W then the then the mapping will be onto. So, if these two conditions hold then we say that then T is said to be an invertible linear transformation.

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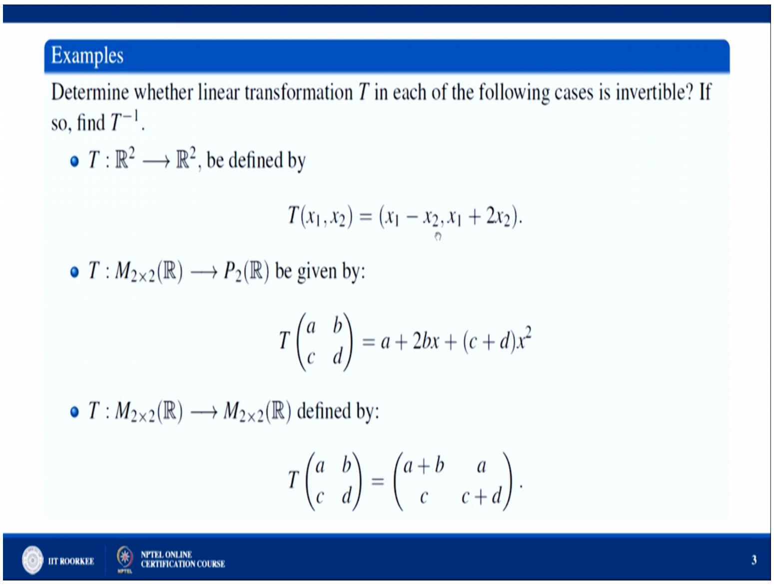

Now, let us let us solve this problem. Determine whether linear transformation T in each of the following cases invertible? And if yes find T inverse. So, the first example is T is from R 2 to R 2 we defined by this ok. So, what is T here let us see.

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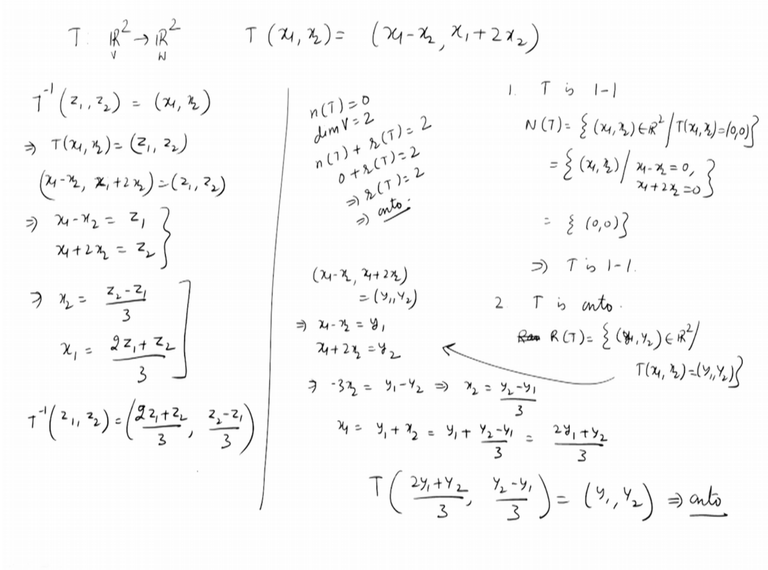

In the first example T is from R 2 to R 2 and this given by T of x 1 x 2 is equal to x 1 minus x 2 and x 1 plus 2 x 2.

Now, to verify whether T inverse exist or not or T is invertible or not we have to see two conditions. Number 1 T must be 1 to 1 and number 2 T must be onto. So, first is T is 1 to

1 or not, first we will verify whether T is 1 to 1 or not. Now, for T 1 to 1 we have to find the null space of a T if null space of T is a singleton 0 of a V, V here is this R 2 then we say that T is 1 to 1. So let us find nullity of T, nullity of T is are all those x 1 x 2 in R 2 such that T of x 1 x 2 is 0 0. So, this is equal to all x 1 x 2 in R 2 such that T of x 1 x 2 means this mapping x 1 minus x 2 is equal to 0 and x 1 plus 2 x 2 is equal to 0.

So, when we solve these two equation we will simply get x 1 equal to 0 and x 2 equal to 0; that means, 0 0 is the is the only point which is in the null space of T that is the singleton 0. So, this implies T is 1 to 1. Now, second is T must be onto T is onto. So, for T onto means we have to see that rank of T must be equal to W ok. Now, if you see if you find rank of T I mean R T, R T is simply range of T sorry it is all s all y 1 y 2 in all y 1 y 2 in W such that T of x 1 x 2 is equals to y 1 y 2.

So, this implies what is T of x 1 x 2? T of x 1 x 2 is simply x 1 minus x 2 and x 1 plus 2 x 2 should be equals to y 1 y 2. So, this implies x 1 minus x 2 should be equal to y 1 and x 1 plus 2 x 2 should be equal to y 2. Now, when you solve these two equation is simply subtract is these two. So, we will obtain minus 3 x 2 will be equals to y 1 minus y 2 or this implies x 2 equal to y 2 minus y 1 upon 3 ok. And when you when you find x 1 from the first equation it is y 1 plus x 2 which is equals to y 1 plus y 2 minus y upon 3 which is equals to 2 y 1 plus y 2 upon 3.

So, what we have shown we have shown that for every for every y 1 y 2 in W there exist x 1 x 2 in V such that x 1 is given by this expression, x 2 is given by this expression; that means, that simply means T of T of 2 y 1 plus y 2 upon 3 which is x 1 and x 2 is y 2 minus y 1 upon 3 is equal to y 1 y 2 ok.

So, for every y 1 y 2 there exist x 1 x 2 x 1 x 2 in V; that means, this mapping is onto. So, this implies onto or we can apply or we can simply apply rank nullity theorem; you see here nullity is here nullity of T is 0 ok. Dimension of V is 2 from here and by rank nullity rank nullity of T plus rank of T must be dimension of V, it is 0 plus rank of T equal to 2. So, this implies rank of T equal to 2 and this implies mapping is onto ok. So, this is a simple illustration to showed at T is onto. Now, T is onto and 1 1 this implies T is invertible ok.

Now, if T is invertible how we can find T inverse. So, to find T inverse we we let we take say T inverse of say z 1 z 2 is equal to say x 1 x 2 ok. So, this implies T of x 1 x 2 is equal to z 1 z 2 x 1 x 2 is T of x 1 x 2 is this thing and this is further equal to x 1 minus x 2 equal to z 1 and x 1 plus 2 x 2 is equal to z 2.

Now, when you solve these two you will simply obtain x 2 as we did here as x 2 as z 2 minus z 1 upon 3 and x 1 as 2 z 1 plus z 2 by 3 ok. So that means, T inverse of z 1 z 2 is equal to z 2 z 1 plus z 2 by 3 and z 2 minus z 1 upon 3. So, this is T inverse of this map. So, first we have to validate whether this T is invertible or not. In order to validate this you have to showed at T is 1 to 1 and onto. If it is not 1 to 1 or onto this means T is not invertible and if it is invertible then we can find T inverse using this processor ok. Now, the see the next example we have consider ah linear map T from considering all matrixes of order 2 cross 2 to P 2 over field R and it is given by T of a b c d as this expression.

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$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2$$

$$N(T) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2} \;/\; T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 + 0\cdot x + 0\cdot x^2 \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;/\; a = 0,\ 2b = 0,\ c+d = 0 \right\}$$

$$= \left\{ \begin{pmatrix} 0 & 0 \\ c & -c \end{pmatrix} \;/\; c \in \mathbb{R} \right\} = c \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} : c \in \mathbb{R}$$

$$\Rightarrow \text{Not } 1\text{-}1$$

So, first we will see whether this is invertible or not. So, here T of a b c d matrix of order 2 cross 2 maps to a plus 2 bx plus c plus d x square. So, first we will find out null space of this linear transformation. So, we will take all those a b c d in M of 2 cross 2 such that T of a b c d is 0. Here 0 means 0 polynomial 0 of P 2, 0 of P 2 means 0 plus 0 x plus 0 x square. So, these are all a b c d such that a this expression T of a b c d is simply this expression and this equal to 0 means the constant term equal to 0, the coefficient of x is equal to 0 and c plus d equal to 0. And this means a 0 b 0 if it is c then it is minus c, where c belongs to R.

So, clearly the null space null space is what null space is all the linear combinations of this type of matrix where c belongs to R. So, that null spaced of this is not singleton 0, this implies this is not 1 to 1 say this map is not 1 to 1. So, this mapping is this linear transformation is not invertible ok. Now, the third example we consider all matrixes for a 2 cross 2 2 matrixes of order 2 cross 2 and defined by this expression. So, let us see whether this is 1 1 and onto or not.

(Refer Slide Time: 13:21)

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$$

1. T is 1-1, $N(T) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big/ \ T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big/ \ a+b=0,\ a=0,\ c=0,\ c+d=0 \right\}$

$= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \Rightarrow n(T) = 0 \Rightarrow T$ is 1-1.

$n(T) + r(T) = \dim V$ $\qquad V = M_{2\times 2}$ $\quad \dim V = 4$

$0 + r(T) = 4 \Rightarrow r(T) = 4 = \dim W$ $\qquad W = M_{2\times 2}$

$\Rightarrow T$ is onto.

So, here T of a b c d is equal to a plus b then a c then c plus d ok. Now, we have to see whether first we have to verify whether T is 1 to 1 or not. For 1 to 1 we have to find out the null space of T, null space of T are all those a b c d such that T of a b c d is equal to 0 matrix 0 0 0 0 ok. Because here mapping going to all matrix of order 2 cross 2.

So, these are all those a b c d such that now T of this is this matrix and this is equal to 0. So, this means a plus b equal to 0 a equal to 0 c equal to 0 and c plus d equal to 0. If a equal to 0 then from here b equal to 0 and if c equal to 0 from here d equal to 0; that means, only singleton 0 matrix and this implies null space of T is only 0 and this implies T is 1 to 1. Now, we have to see whether T is onto or not. So, that we can see very easily by using rank nullity theorem, you see nullity of T plus rank of T must be equals to dimension of V. Here V is matrixes of order 2 cross 2.

So, its dimension is 4 ok, nullity is 0 rank of T and it is 4. So, this implies rank of T is 4 which is equals to dimension of W, W here is also M of order 2 cross 2 whose dimension is 4. So, this implies T is onto. Now, T is 1 to 1 and T is onto this implies T is invertible, now if T is invertible then what will be T inverse. So, we can find T inverse you see here.

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$$T^{-1}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \Rightarrow \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$\Rightarrow a+b=\alpha, \quad a=\beta$$
$$c=\gamma, \quad c+d=\delta$$

$$\Rightarrow a=\beta, \quad b=\alpha-\beta, \quad c=\gamma, \quad d=\delta-\gamma$$

$$T^{-1}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \beta & \alpha-\beta \\ \gamma & \delta-\gamma \end{pmatrix}$$

Let T inverse of some alpha beta gamma delta is equal to suppose a b c d ok. So, this implies T of a b c d will be equals to alpha beta gamma delta and this implies T of a b c d is given by a plus b a c and c plus d. And it is equal to alpha beta gamma delta this implies a plus b is alpha a is beta c is gamma and c plus d is delta. And this implies a is beta and b is alpha minus beta c is gamma and d is delta minus gamma.

So, what will be T inverse of alpha beta gamma delta; this is a given by a is beta b is alpha minus beta c is gamma and this is delta minus gamma. So, this would be the T inverse of this linear transformation ok.

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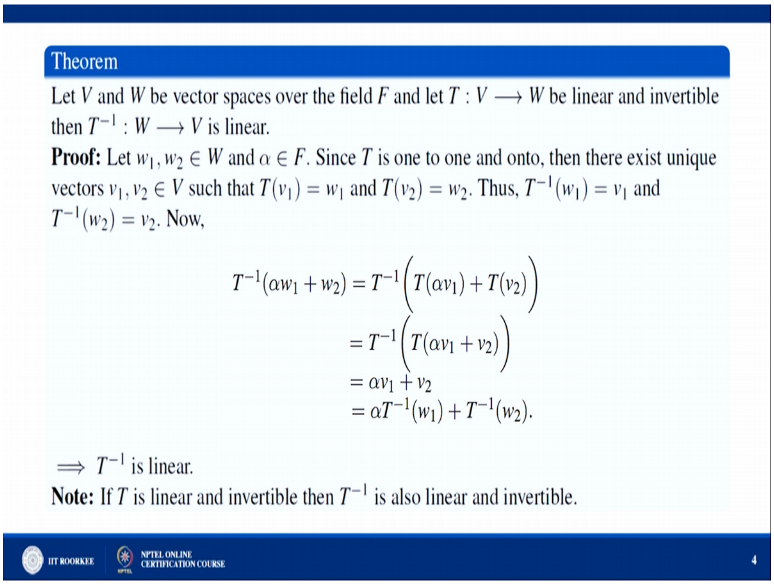

Now, we will see some properties of inverse of a linear transformation. The first property is it is it is a linear map; the T inverse of linear transformation is again linear transformation. So, what is the statement to the theorem? Let V and W be vector spaces over the field F and let T from V to W be linear and invertible ok, then T inverse which is which is from W to V is also linear ok. So, in order showed at T inverse is linear we have to take two elements w 1 and w 2 in W and alpha in field and we have to showed that T inverse of alpha w 1 plus w 2 is equals to alpha of T inverse of w 1 plus T inverse of w 2.

So, in the proof we have taken two vectors w 1 and w 2 and W and alpha in field. And since T is one and onto, because T is invertible now then there exist a unique vectors v 1 and v 2 in V such that T of v 1 equal to w 1 and T of v 2 equal to w 2. Therefore, T inverse of v 1 w 1 equal to v 1 and T inverse of w 2 equal to v 2. Now, we have to for in how to showed that it is linear we have to show the T inverse of alpha w 1 plus w 2 is this expression.

So, T inverse of this will be given by now, w 1 is T inverse of w 1 is T of v 1 w 2 is T of v 2 and since T is linear so, it is T of alpha v 1 plus v 2. Now, since it is inverse of this than T inverse of T is an identity. So, it gives a same expression alpha v 1 plus v 2 and v 1 is T inverse of w 1 and v 2 is T inverse of w 2. So, we have shown that this T inverses also linear ok.

So, if T is linear and invertible then T inverses also linear and invertible invertible also ok.

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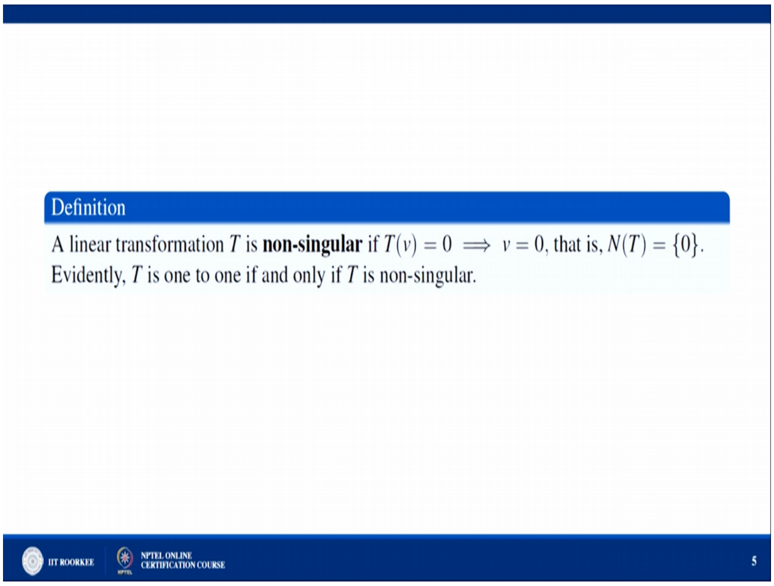

Now, a linear transformation T is set to be non-singular if T of v equal to 0 implies v equal to 0; that means, the null space of T contain singleton 0 or nullity of T is 0 ok. Evidently T is one to one if and only if T is non-singular ok, because you already know that if nullity of or null space of T is a singleton 0 this means T is one to one. So, we can say that T is one to one in a if and only if T is non-singular.

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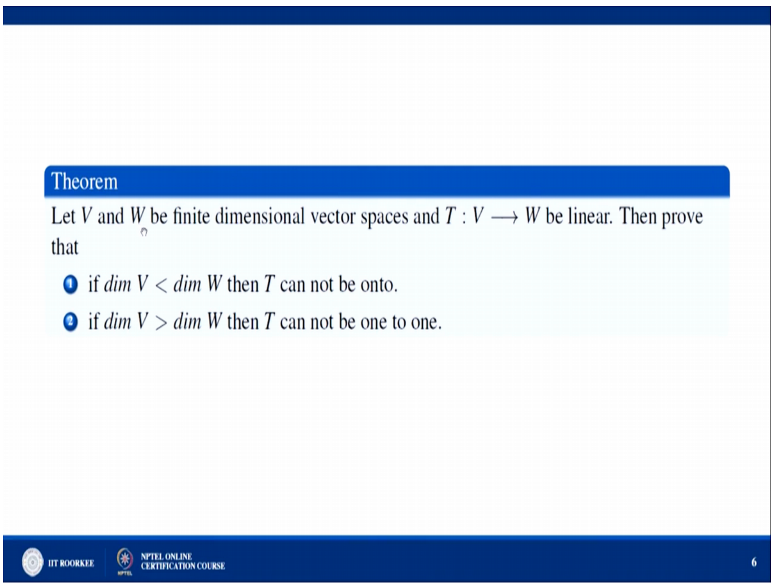

Now, the next result is if V and W be finite dimensional vector spaces and T is a linear map from V to W. Then we have to prove that we have to prove the first and two results.

So, what is the first result? The first result is if dimension of V is less than dimension of W then T cannot be onto. So, let us try to prove this result.

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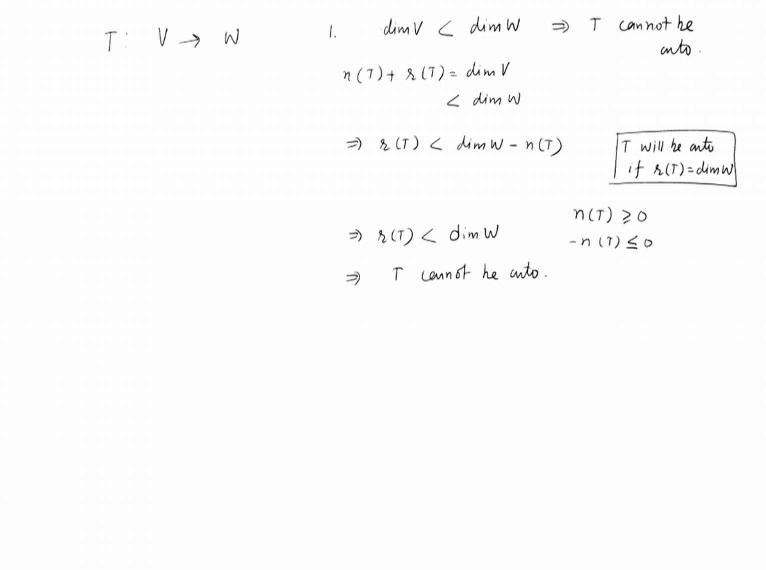

Now, T is a linear map from V to W. So, in the first part dimension of V is less than dimension of W and we have to show that T cannot be onto. Now, when T will be onto? T will be onto when r of T, T will be onto when T will be onto if r of T equal to dimension of W ok.

So, by the rank nullity theorem you know that nullity of T plus rank of T equal to dimension of V and dimension of V is less than dimension of W it is given to us. So, this implies rank of T will be less than dimension of W minus nullity of T. Now, nullity of T is always greater than equal to 0 we know this. So, negative of nullity of T will always be less than equal to 0. So, this implies rank of T because this is always less than equal to 0 with negative sign. So, this will again be less than 0; I mean less than dimension of W.

So, this means rank of T is always less than dimension of W it will never be equal to dimension of W. So, this means T cannot be onto ok. So, the first part of the theorem is over. So, the next part is if dimension of V is more than dimension of W then T cannot be one to one ok. Now, again T is a linear map from V to W and dimension of V is more than dimension of W and we have to show that T cannot be onto; I mean T cannot be one to one that means, nullity of T cannot be 0 or nullity of T is always strictly greater than 0, this you have to show ok.

(Refer Slide Time: 21:53)

$T: V \to W$, $\dim V > \dim W$, T cannot be 1-1.

$n(T) \neq 0$ or $n(T) > 0$

$n(T) + r(T) = \dim V$
$> \dim W$

$\Rightarrow$ if $n(T) = 0 \Rightarrow r(T) > \dim W$ which is not possible.

$\Rightarrow n(T) \neq 0$ or $n(T) > 0$

$\Rightarrow T$ cannot be 1-1.

Now, again apply rank nullity theorem it is nullity of T plus rank of T is equals to dimension of V ok, which is which is more than dimension of W. So, this implies now if nullity of T equal to 0 then this implies r T is more than dimension of W from this expression which is not possible. This is not possible because rank of T can never exceed dimension of V because range of T is a subspace of W ok.

So, this implies nullity of T can never be 0 or nullity of T will be a strictly greater than 0 and implies T cannot be 1 to 1 ok. So, this proof of this is over.

(Refer Slide Time: 23:30)

Theorem

Let T be a LT from V into W. Then T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.

Proof: First, let T be non-singular. Suppose $S = \{v_1, v_2, ..., v_k\}$ be any LI subset of V. Then, we have to show that $\{T(v_1), T(v_2), ..., T(v_k)\}$ is also LI subset of W.
Let $\alpha_1 T(v_1) + \alpha_2 T(v_2) + ... + \alpha_k T(v_k) = 0$
$\implies T(\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k) = 0$
and since T is non-singular, $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_k v_k = 0 \implies \alpha_i = 0$ for all $i = 1, 2, ..., k$ since S is an LI set.
Now, suppose that T carries independent subset of V onto independent subsets of W. Let α be a non-zero vector in V. Then, the set S containing of the α vector is independent. The image of this S (containing $T(\alpha)$) is also independent. Therefore, $T(\alpha) \neq 0$. (otherwise the set becomes dependent). This implies null space of T is a zero subspace $\implies T$ is non-singular.

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Now, the next result is let T let T be a linear transformation from V into W. Then T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W. That means, if you if T is a linear map and is non-singular also then you take any linearly independent subset of V; it always map to linearly independent subset of W or if this happens for any subset from T to W, I mean from V to W then T is non-singular.

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$T: V \to W$. T is non-singular $\Leftrightarrow$ T carries each LI subset of V onto LI subset of W.

let T be non-singular. $S = \{v_1, v_2, .. v_k\}$ be a LI subset of V.

Then, we have to show that $\{T(v_1), T(v_2), ..., T(v_k)\}$ is also LI of W.

$\alpha_1 T(v_1) + \cdots + \alpha_k T(v_k) = 0$

$\Rightarrow T(\alpha_1 v_1 + \cdots + \alpha_k v_k) = 0$

$\Rightarrow \alpha_1 v_1 + \cdots + \alpha_k v_k \in N(T) = \{0\}$

$\Rightarrow \alpha_1 v_1 + \cdots + \alpha_k v_k = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$ $(\because S$ is LI$)$

Let v be a non-zero vector in V. Suppose S is a set containg v is LI. Then ~~a~~ set containg $T(v)$ ~~of~~ of W is also LI

$T(v) \neq 0$.

$0 \neq v \Rightarrow T(v) \neq 0$

So, how we can show this? The proof is simple you can see here you see here T is a linear map from V to W ok. We have to show that T is non-singular if and only if T carries each linearly independent subset of V onto linearly independent subset of W ok. So, first of all let us let us take T is non-singular, let T be non-singular. So, we have to show that if you take any subset say we take S as v 1 v 2 up to say v k be a linearly independent subset of V, then we have to show that then we have to show that T of v 1 and T of v 2 and so on T of v k is also LI of W.

So, to show that it is LI we have to take a linear combination of these elements put it equal to 0 and try to showed at all scalars are equal to 0. So, let alpha 1 T V 1 and so on up to alpha k T V k equal to 0. Now, since T is linear so, this implies alpha 1 V 1 and so on up to alpha k V k equal to 0 and this implies alpha 1 V 1 and plus and so on plus alpha k V k belongs to null space of T. Because T of this element is equal to 0, this mean this belongs to null space of T.

But T is non-singular given to us and T is non-singular this means null space will contain only singleton 0 and this implies alpha 1 V 1 plus and so on up to alpha k V k equal to 0. But this set is linearly independent, it is given to us S is linearly independent. So, since S is linearly independent this implies alpha 1 equal to alpha 2 equal to and so on up to alpha k equal to 0. Since S is LI and this shows that this set is also linearly independent. So, the first part of the proof is over. Now, let us suppose let us suppose T carries each LI subset of V onto LI subset of W. We have to show the converse part now and we have to show that T is non-singular, T is non-singular means we have to show that the null space of T will contain only singleton 0 ok.

Let v be a non-zero vector in V ok. Now, let us consider a set containing this v ok. If we consider a set containing this v and suppose it is LI ok, suppose S is a set containing v is LI. So, by this assumption that T carries each LI subset of V onto LI subset of W then a set containing T of v in of W is also LI, is also LI means T of v will not equal to 0 then only it will be LI.

So, we have shown that v not equal to 0 implies T v is not equal to 0; we have shown that T of v not equal to 0 implies T of v is not equal to 0; that means, null space of v contains only a singleton 0, no other element other than other than 0 and that may these non-singular ok. The next definition is isomorphism.

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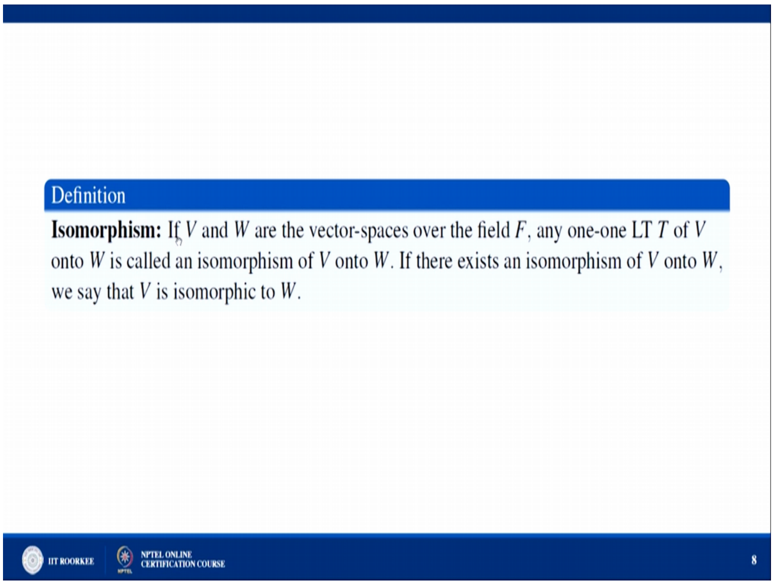

What do you mean by isomorphism? You see if V and W are the vector spaces over the field F, then any one-one linear transformation T of V onto W is called an isomorphism of V on to W. I mean if T is if a linear transformation T is one-one and onto from V to W than that linear map is called an isomorphism of V onto W. If there exists in a isomorphism of V onto W then we say that V is isomorphic to W it is ok.

So, if we have a linear transformation T from V to W such that T is one-one and onto then we say that then we say that V is isomorphic to W ok. So, in this lecture we have seen, we have discuss some of the important properties about inverse of a linear transformation. In the next lecture we will see a matrix associated with a linear transformation.

Thank you.