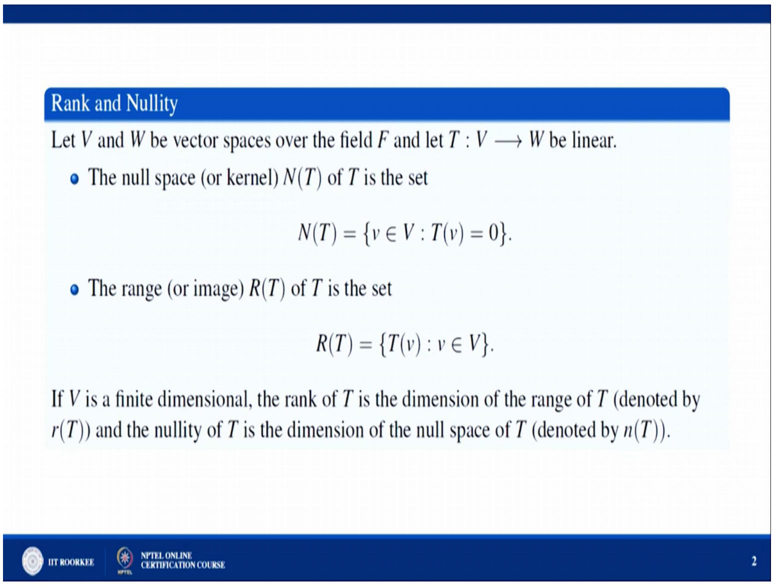
Matrix Analysis with Applications
Dr. S. K. Gupta
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 10
Rank and Nullity

Hello friends welcome to lecture series on Matrix Analysis with Applications. In the last text lecture we have seen that what linear transformations are and what are the fundamental properties of linear transformation. In this lecture we will see that what is mean by Rank and Nullity of a linear transformation ok.

(Refer Slide Time: 00:45)

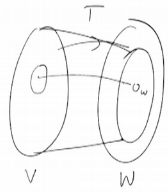


So, let us start let V and W be vector spaces over the field F and let T from V to W be linear ok.

(Refer Slide Time: 00:56)

$$N(T) = \{ v \in V \mid T(v) = 0_W \}$$

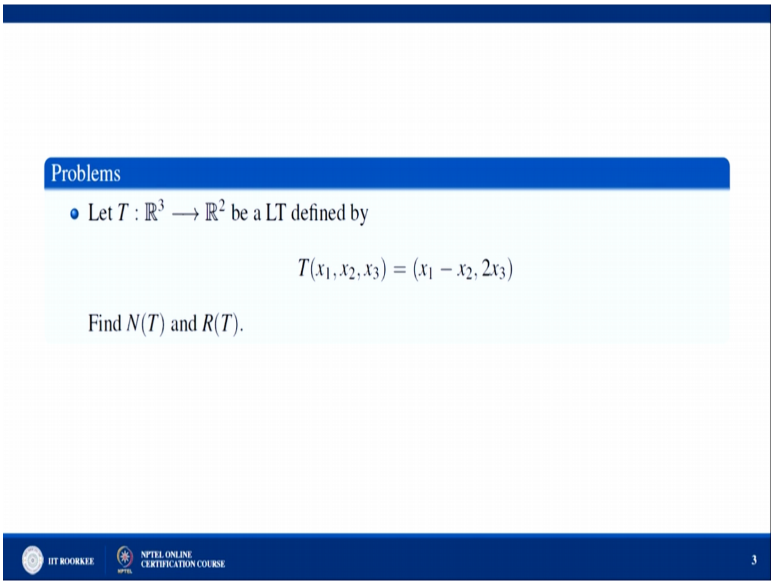
$$R(T) = \{ w \in W \mid T(v) = w,\ v \in V \}$$



Then, basically what we are having we are having a linear transformation T from V to W V is a vector space W is the vector space over the field F ok. Now the null space of T of this linear transformation T are all those elements in V, such that T of v equal to 0 of W. All those elements here ok, which map to 0 of W, it may be 1 or it may be a set ok. This set is called null space of T and the dimension and the range of T is defined as all those all those w in W such that T v is equals to w for v equals to V. I mean the image of this in W, the image of this in W will be the range of T ok.

Now if V is a finite dimensional vector space then rank of T is a dimension of range of T denoted by the small r T and the nullity of T is a dimension of null space of T denoted by n T. We later on show that this n T and r T are the subspaces of the vector spaces V and W respectively; So, they have the dimension, so dimension of the null space of T will be it is called nullity and the dimension of range of T is called rank of T ok.

(Refer Slide Time: 02:38)



Say we have this example and we have to find for this example null space say N T and R T and R of T, so what is the linear transformation here.

(Refer Slide Time: 02:51)

$$T : \mathbb{R}^3 \to \mathbb{R}^2, \quad T(x_1, x_2, x_3) = (x_1 - x_2, 2x_3)$$

$$\begin{aligned}
N(T) &= \{ v \in V \mid T(v) = 0 \} = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid T(x_1, x_2, x_3) = (0,0) \} \\
&= \{ (x_1, x_2, x_3) \mid (x_1 - x_2, 2x_3) = (0,0) \} \\
&= \{ (x_1, x_2, x_3) \mid x_1 = x_2,\ 2x_3 = 0 \} \\
&= \{ (x_1, x_1, 0) \mid x_1 \in \mathbb{R} \} \\
&= \{ x_1 (1,1,0) \mid x_1 \in \mathbb{R} \} = [(1,1,0)]
\end{aligned}$$

$$\begin{aligned}
R(T) &= \{ (y_1, y_2) \in \mathbb{R}^2 \mid T(x_1, x_2, x_3) = (y_1, y_2) \} \\
&= \mathbb{R}^2
\end{aligned}$$

$$T(y_1, 0, \tfrac{y_2}{2}) = (y_1, y_2)$$

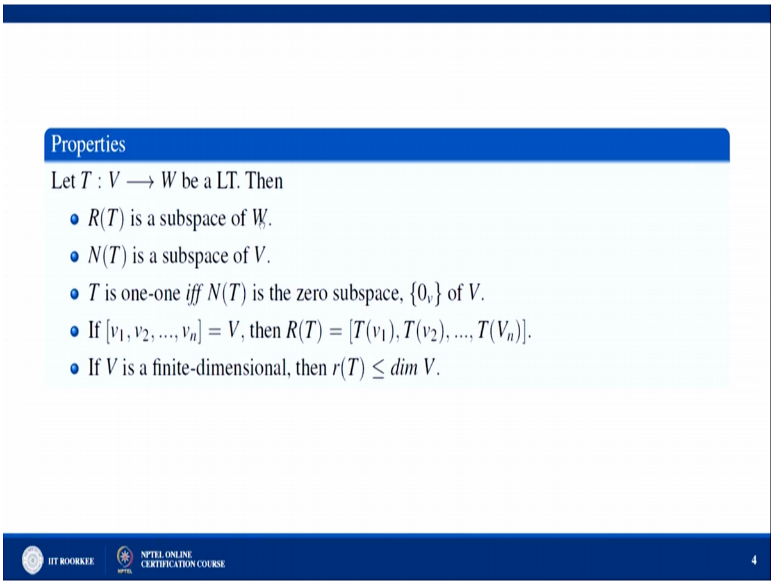
So, the linear transformer here is R 3 to R 2 and it is given by T of x 1 x 2 x 3 as x 1 minus x 2 and 2 x 3. Now, first we have to find null space of T by the definition it is all these are all those v in V such that T of v equal to 0. Now here it is all those x 1 x 2 x 3 in R 3 x 2 x 3 in R 3, such that T of x 1 x 2 x 3 is 0 0, because this is images an R 2.

So, these are all those x 1 x 2 in R 3 such that now T of x 1 x 2 x 3 is nothing, but x 1 minus x 2 comma 2 x 3 and it is 0 comma 0. So, this is all those x 1 x 2 x 3 in R 3 such that x 1 equal to x 2 and 2 x 3 equal to 0; that means, x 3 equal to 0. So, this is simply all those x 1 x 1 because x 1 and x 2 are equal and x 3 is 0 such that x 1 belongs to R. So, this is all x 1 times 1 1 0 such that x 1 belongs to R. So, this is basically the null space of T or we can say that it is equals to the span of 1 1 0 ok.

Now, to find range of T range of T are all those y 1 y 2 in R 2 such that T of x 1 x 2 x 3 ok, is y 1 y 2, for all x 1 x 2 x 3 in you see for all x 1 x 2 in it is x 2 ok. So, we have to we have to see find out R of T of all y 1 y 2 in this R 2 such that T of x 1 x 2 x 3 is equals toy y 1 y 2. Now, you can simply observe here that that T of if you take T of y 1 comma 0 comma y 2 by 2. So, it is equal to y 1 comma y 2 you guys simply observe here it is y 1 minus 0 is y 1 and y 2 y 2 into 2 is simply y 2 so; that means, that means if you take; that means, for any y 1 y 2 here in W which is R 2 there exist a there exist a pre image here in R 3.

So, that; that means, this function map is onto this is an onto map and this since this is an onto map onto map. So, the so the range of T is nothing, but interior 2 ok, because any vector in R 2 because if you have any vector y 1 y 2 in R 2 there exist y 1 comma 0 comma y 2 by 2 in R 3 such that T of this equal to this. So, if you take any y 1 y 2 here. So, there exist a pre image here in R 3 so; that means, range of T is the entire R 2 ok.

(Refer Slide Time: 06:33)



Now, there are some basic properties the first property is range of T is a subspace of W ok.

(Refer Slide Time: 06:38)

$T: V \to W.$ $\qquad R(T) = \{ w \in W \,/\, T(v) = w, \ v \in V \}$

let $w_1, w_2 \in R(T)$, $\alpha \in F.$

$\Rightarrow \exists \ v_1, v_2 \in V$ s.t

$T(v_1) = w_1, \quad T(v_2) = w_2$

$$T(\underbrace{\alpha v_1 + v_2}_{\in V}) = \alpha T(v_1) + T(v_2)$$

$$= \alpha w_1 + w_2$$

$\Rightarrow \alpha w_1 + w_2 \in R(T) \Rightarrow R(T)$ is a subspace of W.

Now what we are considering, here basically T is a map linear map from V to W over the field F, the first is range of T which is defined as all those w in W such that T of v equal to W for v belongs to V. For all I mean you vary v you vary v in V and the collection of all those w in W will be a range of T and we have to show that this is nothing, but a subspace of W. So, how can we show that? You take let w 1 and w 2 in R 2 and for sub space we have to show that alpha times w 1 plus w 2 is also in R 2 and you take alpha in field ok. So, if these are in R 2 this means there exist some v 1 v 2 in V, such that T of v 1 is w 1 and T of v 2 is w 2 ok.

Now, if you take if you take T of alpha v 1 plus v 2, so since a linear map. So, it is T of alpha v 1 plus alpha v 2 T of v 2 which is alpha w 1 plus w 2 by this expression. So, what we have we have shown that there exist a element alpha v 1 plus v 2 whose image is alpha w 1 plus w 2 that this implies alpha w 1 plus w 2 belongs to W, because this element is in V and we have shown that there exist a element in V whose image is whose image is alpha w 1 plus w 2; that means, this is in R T by the definition of R T

So, this means this is in R T yes, so this implies R T is a sub space of W then next is N T is a sub space of W the nullity how we can show this.

(Refer Slide Time: 08:57)

$$N(T) = \{ v \in V \;/\; T(v) = 0 \}$$

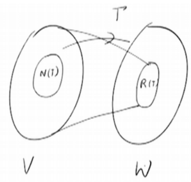
$$v_1, v_2 \in N(T), \quad \alpha \in F$$

$$\Rightarrow T(v_1) = T(v_2) = 0$$

$$T(\underbrace{\alpha v_1 + v_2}) = \alpha T(v_1) + T(v_2)$$

$$= \alpha 0 + 0 = 0$$

$$\Rightarrow \alpha v_1 + v_2 \in \underline{N(T)}$$



Now, nullity of T or all those V in V such that T V equal to 0, so again you take some v 1 v 2 in l T and alpha belongs to field and we have to show that alpha v 1 plus v 2 also belongs to N T for N T to be a sub space. Now, this implies T of v 1 is equal to T of v 2 equal to 0.

So, now take T of alpha v 1 plus v 2 since is a linear map, so it is alpha times T v 1 plus T v 2. So, it is alpha dot 0 plus 0 which is equal to 0, so this implies alpha v 1 plus v 2 is belongs to nullity of T, because this element is in V and we have shown that T of this element equal to 0 and T of all those V in V whose image is 0 are in null space of T or nullity of i mean null space of T. So, we have shown that null space is a subspace of V.

So, basically what we have shown that if you are having 2 subspaces V and W and T is a linear map from any linear map from V to W. The nullity of if this is a nullity of T then this is not only a subset of this V this is a subspace of this V, and the image of this V if it is say here which is which we are denoting by R T which has a sub. So, it is not only a subset of W it is a subspace of W, the next step is if T is 1 to 1 T is 1 to 1 if and only if null space is a 0 sub space or the single tone says 0 of V, so the proof is again simple let us discuss.

(Refer Slide Time: 10:54)

let T be 1-1

$v \in N(T) \Rightarrow T(v) = 0$

$\Rightarrow T(v) = T(0)$

$\Rightarrow v = 0$

$\Rightarrow N(T) = \{0_V\}$.

T is 1-1

$T(v_1) = T(v_2) \Rightarrow v_1 = v_2$

let $N(T) = \{0_V\}$. $T(v_1) = T(v_2)$

$\Rightarrow T(v_1) - T(v_2) = 0$

$\Rightarrow T(v_1 - v_2) = 0 \Rightarrow v_1 - v_2 \in N(T) = \{0_V\}$

$\cancel{\Rightarrow T(v_1 - v_2) = T(0)}$ $\Rightarrow \underline{\underline{v_1 = v_2}}$

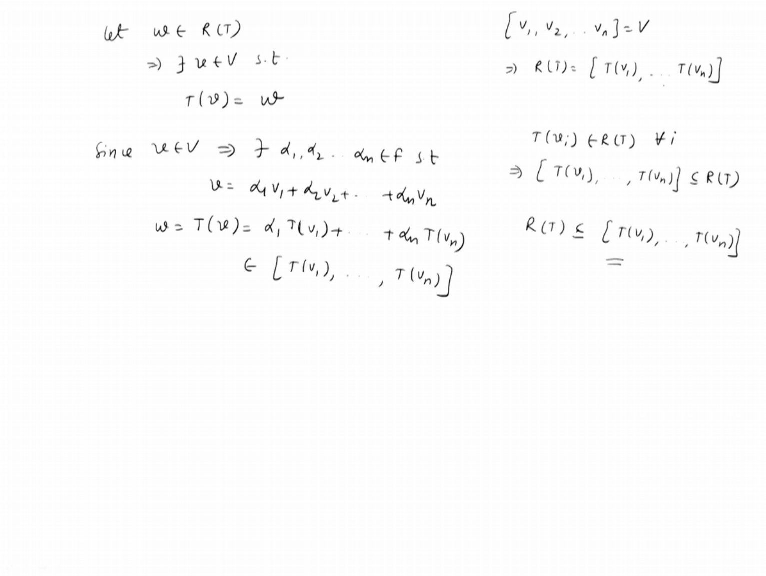
Now, where now when we can say that T is 1 to 1 T is 1 to 1 means T of v 1 equal to T of v 2 should implies v 1 equal to v 2 for all v 1 v 2 in V ok. Now, let us let us first take let T be 1 to 1 and we have to show that null space T is a singleton 0 of V ok. Now T is 1 to 1 let, let V belongs to null T of T, so if we have if we have shown that V is nothing, but 0 of V then we have done.

Now if this is in this belongs to null space; that means, T of V equal to 0, now this implies T of V equal to T of 0 because we know that T of 0 0 maps with 0 only. Now, since T is 1 to 1 and by this definition we can simply say that V equal to 0. So, that this implies nullity of T is nothing, but singleton 0 of V. Now the converse let null space of T is the single ton 0 and we have to show that T is 1 to 1.

So, let T of v 1 equal to T of v 2 and we have to show that v 1 equal to v 2 for T to be 1 to 1. So, this implies T of v 1 minus T of v 2 will be 0 you can add additive inverse of T of v 2 both the sides. Now, this is equal to V know that this is equal to T of v 1 minus v 2 equal to 0 this implies T of v 1 minus v 2 equal to T of 0 ok, any way this is not required now T of v 1 minus v 2 equal to 0. So, this implies v 1 minus v 2 is in the null space of T and null space of T is the singleton 0 of V, so this implies v 1 equal to v 2.

So, we have shown that T is 1 to 1 if and only if null space of T is a singleton 0 of V, the next result is if the span of v 1 v 2 up to V n equal to V, then the range of T is nothing, but span of T of v 1 T of v 2 up to T of V n again it is easy to show.

(Refer Slide Time: 13:23)



You see here this span of v 1 v 2 up to V n is equal to V, and we have to show that R of T is nothing, but T of v 1 and T of V n is span of this ok. Now, T of V i is belongs to R 2 R T for all i because, T of V i is simply the image of V i. So, by the definition of R T the T of V i will belongs to R T and since R T is the sub space of W. So, we can easily say that span of T of V i is span of T of v 1 and T of V n will be definitely contained in R T because R T has sub space of W.

So, to show that it is equal to this we have just to show that R T is a subset of span of T of v 1 up to T of V n; Now, to show that this is a subset of this take an arbitrary element in R T and try to show that it is also in the span of this set. So, let some W belongs to R T if W belongs to R T this implies there exist some V belongs to V such that T of V is W.

Now, since this V is in V and V is a capital V is equals to span of v 1 v 2 up to V n; that means, this V can be expressed as linear combination of element of this set. So, since V belongs to V this implies there exist alpha 1 alpha 2 up to alpha n belongs to field such that we will be equals to alpha 1 v 1 plus alpha 2 v 2 and. So, on up to alpha n V n, now T of V will be equal to since T is linear, so it is alpha 1 T of v 1 and so on alpha n T of V n; that means, belongs to span of T of v 1 and so on up to T of V n and T of V is what T of V is W ok. So, we have shown that is W belongs to span of this ; that means, ; that means, R T is the subset of span of this.

So, from these two results we can say that R T is equal to the span of T of v 1 T of v 2 up to T of V. Now the next result is if V is a finite dimensional vector space then R of T that is a rank of T is always less than equal to dimension of V. So, it is easy to show you see in the from this if these vectors are of linearly independent. So, if these vector are linearly independent, so and span of equal to v; that means, V is n dimensional and from this result we know that R T is equal to span of this, but span of, but T of V i T of V i is may not be linearly independent ok. So, basically the dimension of this R T will be less than equal to dimension of this that is dimension of V, the largest the largest allay set is itself if this is the

The whole set is ally then this will be the dime then this will be the basis of V ok, but the image of T i is image of T i is T of V i is may not be linearly independent, there may they are may be possible there some of the T T of V i s are linearly dependent. So, the dimension of this will be less than equal to dimension of V if you have a counter example say you are taking a span of 1 0 and 0 1 as R 2.

(Refer Slide Time: 17:33)

$$[\underset{v_1}{(1,0)}, \underset{v_2}{(0,1)}] = \mathbb{R}^2$$

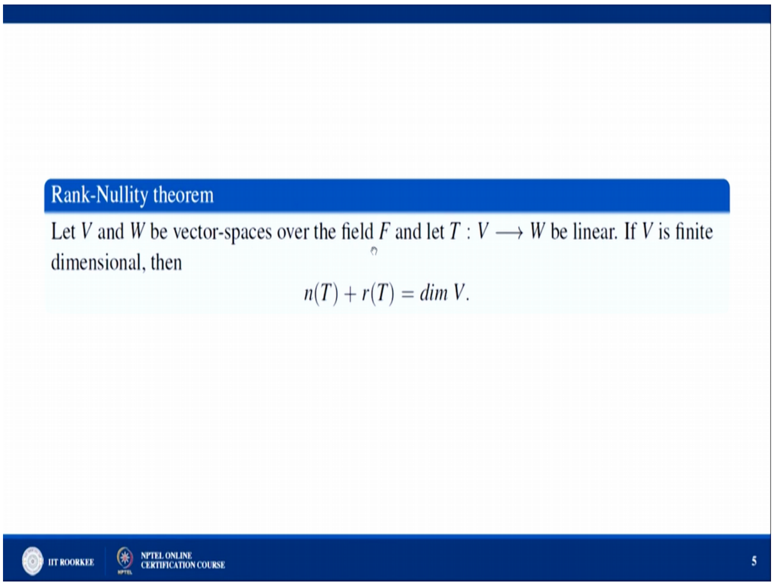
$$T(x_1, x_2) = (x_1+x_2, x_1+x_2)$$
$$T(1,0) = (1,1)$$
$$T(0,1) = (1,1)$$

$$R(T) = [T(1,0), T(0,1)]$$
$$= [(1,1)]$$

It is v 1 it is v 2 now you define you can define the linear map as this T of x 1 x 2 as say x 1 plus x 2 and say x 1 plus x 2 again. Now, T of 1 0 is simply 1 1 and again T of 0 1 is again 1 1. So, the span of T of 1 0 and T of 0 1 is of course, equal to R T by the definition by the result, but that is simply san of 1 comma 1 because both the elements are same.

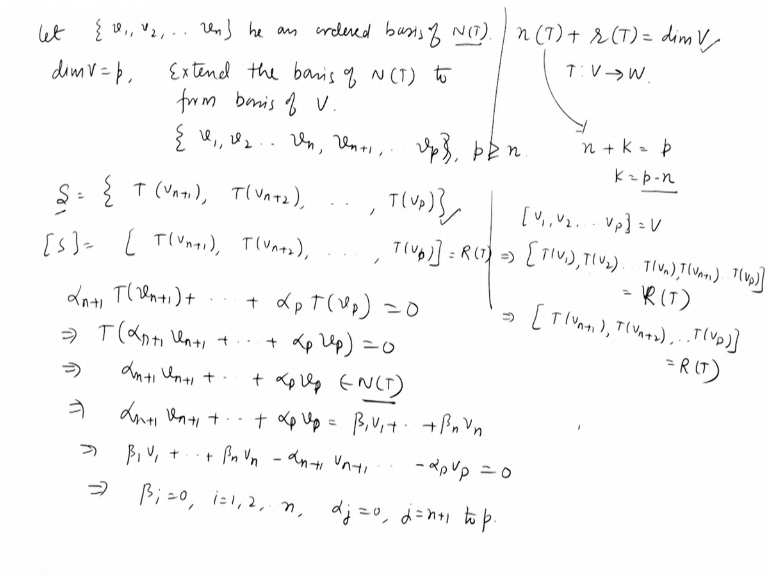
So, dimension of this is less than equal to dimension of this it cannot be more than this now the next result is the rank nullity theorem.

(Refer Slide Time: 18:26)



It states that if V and W are 2 vector spaces of field f and let T from V to W be linear if V is finite dimensional then nullity of T plus rank of T is always equal to dimension of V. So, the result is very easy to show let us see the proof we have to show that nullity of T plus rank of T is equal to dimension of V.

(Refer Slide Time: 18:46)



Here T is from V to W this we have to show. Now, let us consider let us consider that v 1 v 2 up to v n be an ordered basis of, ordered basis of N T null space of T let us suppose. Now since it is a subspace of sub space of V subspace of V. So, we can always extend this basis to form the basis of V it is a sub space of V n T is the sub space of V. So, we can always extend this sub space to form the basis of V. So, let us suppose the dimension of V is say p ok, dimension of V is say p. So, extend the basis of N T to form basis of V.

So, let us suppose this is a basis of V, we are where of course, p is greater than equal to n. Now, we have to show that now let us consider a set say we consider set s which is nothing, but T of V n plus 1 T of n plus 2 and so, on up to T of V p. Now, if we any show that this is nothing, but range of T I mean this is nothing, but a basis of range of T then the dimension of this then we have shown because here nullity of T is n ok.

And say it is k and it is p then k will be p minus n, and if any how we have shown that this is this set S is a basis of R T whose dimension is p minus n; that means, we have shown. So, we have just to show now that this S is nothing, but a basis of R T. So, the basis of R T we have to show the two properties number one it is linearly independent and number two is span of this generates entire R T.

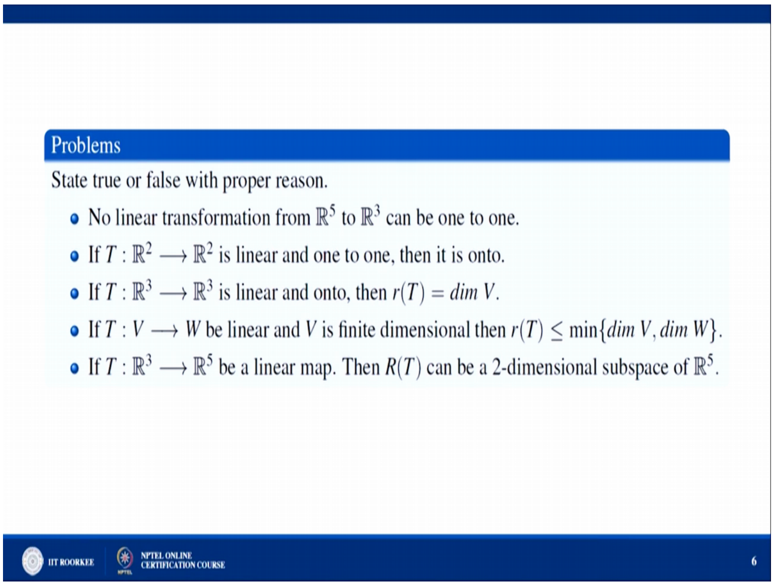
So, first let us see the span of this the span of this is T of V n plus 1 T of V n plus 2 and T of V n V p. We know that if V equal to the span of v 1 v 2 up to V n then R T is equals to the span of T v 1 up to T V n ok. So, now here a span of v 1 v 2 up to V p V p is equals to V T because this is a basis of because this is a basis of V. So, the span of T v 1 T v 2 up to TV n T V n plus 1 and so on T V p is equals to is equals to V now since v 1 v 2 V n R n R n n T. So, T of V i up to n is 0, so these are these all are 0.

So; that means, its equal to T of V n plus 1 T of V n plus 2 and so on T of V p and that is equals to sorry it is R T and that is equal to R T. So, the first property we have shown that the san of this is nothing, but span f R T, I mean is equal to R T that is a span of this is entire R T. Now the second thing we have to show that the this set is linearly independent. So, take some linear combination of this elements put it equal to 0, and we have to show that all alpha is from n plus 1 to p R 0.

So, this implies T of alpha n plus 1 V n plus 1 and so on up to alpha pvp equal to 0 because T is linear. So, this implies alpha n plus 1 V n plus 1 and so on up to alpha pvp belongs to null space of T and since it belongs to null space of T and that basis of null space of T is v 1 v 2 up to V n. So, this element can be written as linear combination of element of basis of n T. So, this implies alpha n plus 1 V n plus 1 up to alpha n alpha pvp will be beta 1 v 1 and so on up to beta 1 V n.

So, this implies beta 1 v 1 and so on up to beta n v n minus alpha n plus v 1 V n plus 1 and so on minus alpha p v p will be equal to 0, but this set v 1 v 2 up to V p is a basis of v. So, it will be linearly independent so this implies beta i s are 0 from i from 1 to n and alpha j s are 0 for j from n plus 1 to p. So, we have shown that alpha j s from j varying from 1 plus 1 to p R 0 this means this set is linearly independent. So, we have shown that this is linearly independent and the span of this is equal to what you did; that means, that means this is a basis of are R T, and hence we have shown the result now let us come to few problems based on this.

(Refer Slide Time: 24:54)



State true or false the proper reasoning proper reason the first one is no linear termination from R 5 to R 2 can R 3 can be 1 to 1. Now, let us let us see whether it is true or false. So, for all these things we will apply a rank nullity theorem by the rank nullity theorem you see here T is from R 3 to R 2 ok.

(Refer Slide Time: 25:17)

$T: \mathbb{R}^5 \to \mathbb{R}^3$

$T: \underset{V}{\mathbb{R}^5} \to \underset{\underline{W}}{\mathbb{R}^3}$, $\dim V = 5$

$R(T) \subseteq W$

$n(T) + r(T) = \dim V = 5$

if T is 1-1, $n(T) = 0$

$\Rightarrow r(T) = 5$ which is not possible.

It is from R 3 to R 2 sorry yes R 5 to R 3 it is from R 5 to R 3 ok, let us write again T is from R 5 to R 3. So, it is V and it is W, so what a dimension of V dimension of V is clearly 5 by the rank nullity nullity of T plus rank of T will be equals to dimension of V which is equals to 5. If this T is 1 to 1, we know that T is 1 to 1 if and only if nullity is 0 nullity is 0 or null space contain the singleton set 0 single ton 0. So, if it is 1 to 1 if T is 1 to 1 then n T will be 0.

So, this implies from here r T will be 5 which is not possible since the maximum dimension of R T maximum value of R T is 3, W is a see R T see R T is a sub subset subspace of W. So, its dimension cannot exceed the dimension of W it is a sub space of W. So, R T is 5 here which is not possible because the maximum value of R T is 3; So, no we cannot have any linear transformation from R 5 to R 3 which is 1 to 1.

So, the statement is false yeah statement is true no linear transformation yeah. So, the first statement is true now second 1 if T is from R 2 to R 2 is linear and 1 to 1 then it is onto you see here T is R 2 to R 2.

(Refer Slide Time: 27:18)

$$T : \underset{V}{\mathbb{R}^2} \rightarrow \underset{W}{\mathbb{R}^2}, \quad \dim V = 2, \quad \dim W = 2, \quad n(T) = 0$$

$$R(T) = \underline{W} \quad \text{or} \quad r(T) = 2$$

$$n(T) + r(T) = 2$$

$$0 + r(T) = 2 \Rightarrow r(T) = 2 \Rightarrow \underline{\text{onto}}$$

$$T : \underset{V}{\mathbb{R}^3} \rightarrow \underset{W}{\mathbb{R}^3}, \quad \dim V = 3, \quad r(T) = \dim W = 3$$

$$n(T) + r(T) = \dim V = 3$$

$$0 + \underline{r(T) = \dim V}$$

It is V it is W, so what is dimension of V is 2 what is dimension of W is 2 if R T is if this T is onto means range of T will be entire W, then it is onto or we can say that rank of T will be 2 rank of T will be 2 because rank of W dimension of W is 2 ok.

Now, it is given to 1 that it is 1 to 1 if it is 1 to 1; that means, nullity of T is 0 by the rank nullity nullity plus rank of T will be dimension of V which is 2 and 0 plus rank of T is equal to this implies R T equal to 2; that means, it is onto. So, yes it is true in the next 1 if T is from R 3 to R 3 and it is linear and onto then rank of T is equal to dimension of V here dimension of V is 3 ok. So, now, it is linear and onto now T is from R 3 to R 3 it is V it is W dimension of V here is 3 T is linear and onto means R of T is equal to dimension of W which is 3. So, nullity of T a plus rank of T will be equal to will be equal to dimension of V which is 3.

So, what is v 1 2 plus we have it is given to it is onto now here he it is 1 to 1. So, it is 0 plus rank of T equal to dimension of V, so it is clear. So, in fact if u and V are of same dimensional vector space, here also in the second example also V and W are same dimensional vector space and T is linear if it is linear and 1 1, it implies on to if T is linear and onto it implies onto if it is same dimensional vector space. The next is if T is from V to W is linear and a finite dimensional then rank of T is always less than equals to minimum of dimension of V and dimension of W you see, here T is a linear transformation from V to W ok.

(Refer Slide Time: 29:53)

$T : V \to W$

$\underline{r(T)} \leq \dim W$

$n(T) + r(T) = \dim V$

$r(T) = \dim V - n(T)$

$n(T) \geq 0$

$-n(T) \leq 0$

$\dim V - n(T) \leq \dim V$

$\Rightarrow \boxed{r(T) \leq \dim V}$

$\Rightarrow r(T) \leq \min\{\dim V, \dim W\}$

Now, rank of now r T is definitely less than equals to dimension of W because r T is a r T is a sub space of W, if r T is s sub space of W. So, its dimension its value will always be less than equal to dimension of W, if it is onto then it nullity holds and by the rank nullity nullity of T plus rank of T equal to dimension of V. So, rank of T will be equal to dimension of V minus nullity and nullity of T is greater than equal to 0 it is 0 if T is 1 to 1. So, minus nullity of T will be less than equal to 0; that means, dimension of V minus nullity of T will be less than equal to dimension of V.

So, this implies r T is also less than equals to dimension of V, so since it is less than equals to dimension of V and also less than equal to dimension of W. So, this implies r T will be less than equal to minimum of dimension of V and dimension of W. The last one is if T from R 3 to R 5 be a linear map then R T can be a 2 dimensional sub space of R 5, so let us see now here T is.

(Refer Slide Time: 31:29)

$$T: \underset{V}{\mathbb{R}^3} \to \underset{W}{\mathbb{R}^5} \qquad \dim V = 3$$

$$\underbrace{n(T)}_{1} + \underbrace{r(T)}_{2} = 3$$

$$r(T) = 3 - n(T)$$

$$R(T) \qquad n(T) \geq 0$$

$$\boxed{r(T) \leq 3}$$

Here T is from R 3 to R 5, so it is V it is W, so dimension of V is 3 now nullity of T plus r T is equals to dimension of V which is 3. So, we have to see that there exists R T of 2 dimensional.

So, yes there exist if T is 2 and nullity is 1 for some T then 1 plus 2 is 3 yes there exist. But there, there cannot exist R T I mean, I mean R T of more than 3 dimensional vector more than 3 dimensional subspace of R 5, because sum must be 3 the minimum value of n T is the minimum value of n T is 0. So, definitely r T will be r T is equals to 3 minus n T and n T will be greater than equal to 0; that means, r T will be less than equal to 3; it cannot be more than 3. So, for this problem it is true.

So, we have seen that what rank a nullity of a linear transformation is and what are the few properties based on this.

Thank you.