Ordinary and Partial Differential Equations and Applications
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Lecture - 09
Solution of Homogeneous Linear System with Constant Coefficients-I

Hello friends welcome to the lecture on solution of homogenous linear system with constant coefficients. Let us consider the vector differential equation x dot = Ax in what follows we will reduce the problem of finding all solutions of this vector differential equation to solving the system of equations Ax = b.

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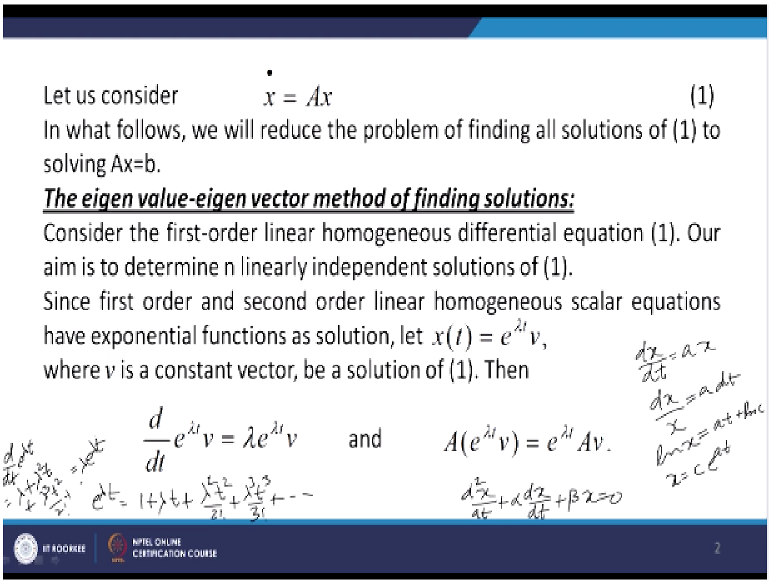

The eigen value-eigen vector method we are going to discuss to find all solutions of the vector differential equation x dot = Ax. So let us consider the first-order linear homogeneous differential equation 1 which is x dot = x. Our aim is to determine n linearly independent solutions of this vector differential equation. Now we know that first order and second order linear homogenous scalar equations have exponential functions as solution.

For example, you consider the first order liner homogenous scalar equation then it is dx/dt = some constant times x, okay, which we can write as dx/x = adt and which gives us lnx = at + some constant which we can write as lnc so then x = c times e to the power at. Same is the situation when you consider second order linear homogenous scalar equation, d square x/dt square + some constant say alpha times dx/dx + beta times x = 0.

So we have exponential functions solutions of these first order and second order homogenous scalar equations therefore, what we do is we consider xt = e to the power lambda t * v where v is the constant vector and take it to be a solution of equation 1. Now let us see that when you put e to the power lambda t * v in the equation 1, we have on the left side dx/dt, so d/dt of e to the power lambda t * v gives you lambda times e to the power lambda t * v.

This is because e to the power lambda t we know = 1 + lambda t + lambda square t square/2 factorial, lambda cube t cube/3 factorial and so on. So when you differentiate this with respect to t, what you will get is d/dt of e to the power lambda t will give you lambda + 2 lambda square t/2 factorial, okay, so we will get lambda square t and then we will get 3 lambda cube t square/3 factorial so we will get lambda cube t square/2 factorial and so on.

Which is = lambda times 1 + lambda t + lambda square t square/2 factorial and so on and so it is lambda times e to the power lambda t. So when you differential e to the power lambda t * v with respect to t where v is the constant vector what we get is lambda e to the power lambda t * v and by the matrix properties A is a matrix * e to the power lambda t * v, e to the lambda t is the scalar quantity * v will give you e to the power lambda t * av.

So what we will get if you put it in this equation 1, xt = e to the lambda t * v what we get is? We get the following.

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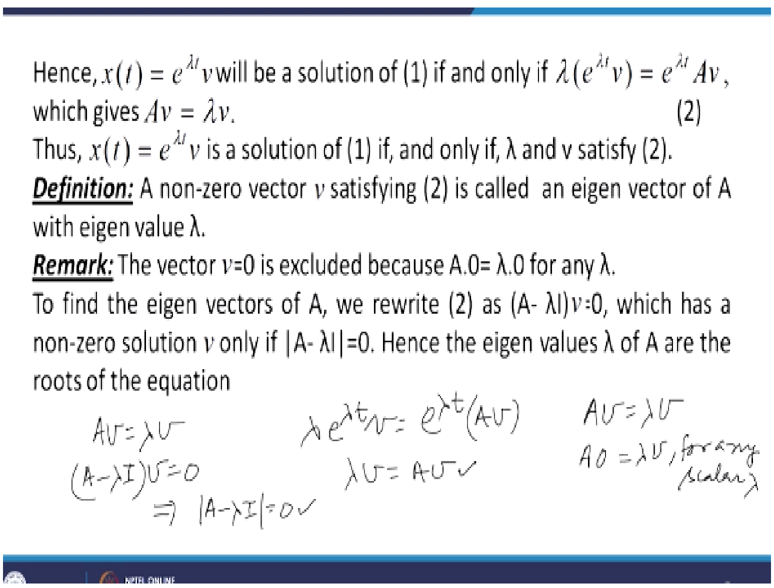

So we get lambda e raise to the power lambda t * v = lambda e to the power lambda t * v = e to the power lambda t * ab, this is what we get. Now e to the power lambda t is never 0 for any value of t, okay, so we can divide this equation by e to the power lambda t and we get lambda v = av. So xt = e to the power lambda t * v will be a solution of equation 1 if and only if we have this equation lambda * e to the power lambda t * v = e to the power lambda t * Av which gives us Av = lambda v.

Thus xt = e to the power lambda t * v is the solution of the equation 1 if and only if lambda and v satisfy Av = lambda v. Now a non-zero vector v satisfying this equation Av = lambda v is called an eigen vector of the matrix with eigen value lambda. Now here the case v = 0 is excluded because it is an uninteresting case. You can see that we have Av = lambda v, if you allow v to be 0 vector then A * 0 vector = lambda * 0 is true for any scalar lambda.

So we discard this case and so the vector v = 0 is not allowed to be an eigen vector. So we will say that a nonzero vector v which satisfies Av = lambda v will be an eigen vector of a associated with the eigen value lambda. To find the eigen vectors of A, we rewrite the equation Av = lambda v as now when you have Av = lambda v we have A - lambda I v = 0. Now remember we always write A – lambda I because we cannot subtract a scalar lambda from A.

So we multiply lambda by identity matrix of the same order as A, okay. So A – lambda I * v = 0, this is what we get, now this equation has a nonzero solution only if determinant of A – lambda = 0, so determinant of A – lambda I must be 0. Because if determinant of A – lambda is nonzero then A - lambda inverse will adjust and you can pre-multiply this equation A – lambda I v = 0/A – lambda I inverse and what you will get, v = 0, but v is != 0.

So A – lambda * v = 0 will give us a nonzero solution only when determinant of A – lambda I = 0. Hence the eigen values lambda of A are the roots of the characteristics equation determinant of A – lambda I = 0.

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$$0 = |A - \lambda I| = \begin{bmatrix} a_{11}-\lambda & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22}-\lambda & . & . & . & a_{2n} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn}-\lambda \end{bmatrix}$$

and the eigen vectors of A are then the non-zero solutions of the equation $(A-\lambda I)v=0$, for these values of λ.

The determinant of the matrix (A- λI) is clearly a polynomial in λ of degree n, with leading term $(-1)^n \lambda^n$.

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If you expand this equation determinant A – lambda = 0 what you get is this determinant, a11 – lambda a12 that is from the matrix A the diagonal elements of the matrix A are subtracted by lambda. We are subtracting from A lambda times identity matrix of order n. So this is what you get, so determinant of this = 0 gives us a polynomial equation in lambda of degree n. The eigen vectors of A are then the nonzero solutions of.

So once you find the values of lambda which satisfy this characteristic equation the corresponding eigen vectors are given by the nonzero solutions of the equation A – lambda I v = 0, now the determinant of the matrix A – lambda I is clearly a polynomial in lambda of degree n with the leading coefficient will be -1 to the power n lambda to the power n.

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This polynomial is called the characteristic polynomial of A and is denoted by p(λ). For each root λ_j of p(λ) there exists at least one non-zero vector v^j such that $Av^j = \lambda_j v^j$. Since p(λ) has at most n distinct roots, every $n \times n$ matrix has at most n eigen values. Finally, since the space of all vectors $v = [v_1, v_2, \ldots, v_n]^t$ has dimension n, every $n \times n$ matrix has at most n linearly independent eigen vectors.

Remark: Let v be an eigen vector of A with eigen value λ. Then cv, c≠0 is also an eigen vector of A, with the same eigen value. For each eigen vector v^j of A with eigen value λ_j, $x^j(t) = e^{\lambda_j t} v^j$ is a solution of (1). If A has n linearly independent eigen vectors $v^1, v^2, \ldots, v^n$ with eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively ($\lambda_1, \lambda_2, \ldots, \lambda_n$ need not be distinct), then $x^j(t) = e^{\lambda_j t} v^j$, $j = 1, 2, \ldots, n$ are n linearly independent solutions of (1).

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Now this polynomial is called the characteristic polynomial determinant of A – lambda I is called the characteristics polynomial of A and be denoted by p lambda. So p lambda = determinant of A – lambda I. For each root lambda j of p lambda there are just at least one nonzero vector vj such that Avj = lambda j vj. Suppose these is a value of lambda say lambda j for which p lambda is 0 that is p lambda j = 0.

Then the corresponding nonzero eigen vector will be given by Avj = lambda jbj or you can say A-lambda j1 bj =0. So from here we will get at least one nonzero eigen vector corresponding to lambda j since p lambda is utmost n distinct roots, p lambda being a polynomial n lambda of degree n, so it has utmost and distinct roots, every n*n matrix has utmost an eigen values. Finally, since these space of eigen vectors v = v1, v2, vn it has got n components.

So the space of vectors has n dimension and dimension n and therefore every n*n matrix has n linearly independent eigen vectors at the most. Now if v is an eigen vector of A with eigen value lambda then c * v where c is != 0 is also an eigen vector of A with the same eigen values. This is very simple to see. Suppose lambda is an eigen value of A and v is the corresponding eigen vector.

Then we have Av = lambda v. Now let us see A cv, c is a scalar which is nonzero we can write it as c time Av by the matrix properties so this is c times lambda v. So we have this is = lambda times cv. So we have Acv = lambda times cv. So if v is an eigen vector of A corresponding to the eigen value lambda then cv where c is nonzero, c is taken to be nonzero because if you allow c to be 0, cv will be a zero vector.

So it cannot be an eigen vector so c is != 0 then if you take then cv is also an eigen vector of A with the same eigen value lambda. Now for each eigen vector vj of A with eigen value lambda j, xjt = e to the power lambda t lambda jt * vj is the solution of 1. We have seen that xt = e to the power lambda t * v when we assume that as a solution of the equation x dot = Ax then we arrive at the equation where why we need to find the eigen vectors and eigen values of the matrix A.

So when you find corresponding to the eigen value lambda j, vj as an eigen vector then corresponding to that xjt will be a solution of equation 1. If A has n linearly independently eigen vectors v1, v2 vn with eigen value lambda 1, lambda 2, lambda n respectively. Note here that lambda 1, lambda 2, lambda n need not be distinct, when lambda 1, lambda 2, lambda n are all distinct eigen values we shall see that their corresponding eigen vectors are linearly independent.

But even in case where lambda 1, lambda 2 and lambda n are not all distinct we might have n linearly independent eigen vectors, so that is why suppose v1, v2, vn are n linearly independent eigen vectors of the matrix A corresponding to the eigen value lambda 1, lambda 2, lambda n respectively which need not be distinct then xjt = ejt * vj, j = 1, 2 and so on up to n are n linearly independent solutions of the vector differential equation 1 that is x dot = Ax.

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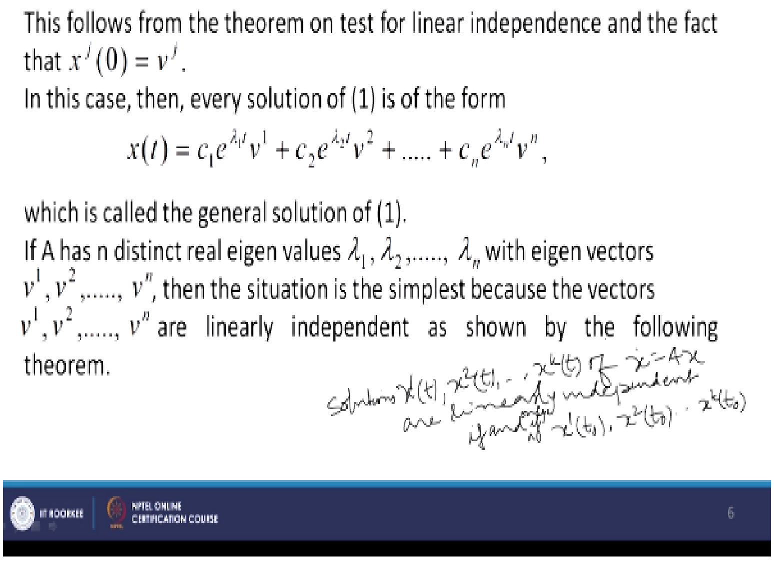

Now this follows from the theorem on test for linear independence, if you remember we did in the last lecture, we said that the solutions, x1t x2t xkt, the k solutions of x dot = Ax are linearly independent if and only if x1 t0, x2 t0 ends on xk t0 are linearly independent vectors so table value of t = t0. So here if you choose t = t0 to be 0 then we see that xjt = vj and v1, v2, vn are linearly independent. So xjt for j = xj0 for j = 1, 2 ends on up to n are linearly independent.

And therefore the solutions xjt for j = 1, 2 and so on up to n are linearly independent and therefore general solution of the vector differential equation x dot t = Ax t can be written as xt = c2 e to the power lambda 1t * v1, c2 e to the power lambda t * v2 and so on cn e to the power lambda nt * vn which is called the general solution. Now if A has n distinct real eigen values lambda 1, lambda 2, lambda n will be the eigen vectos v1, v2, vn.

Then the situation is the simplest because the eigen vectors v1, v2, vn are linearly independent as shown by the following theorem.

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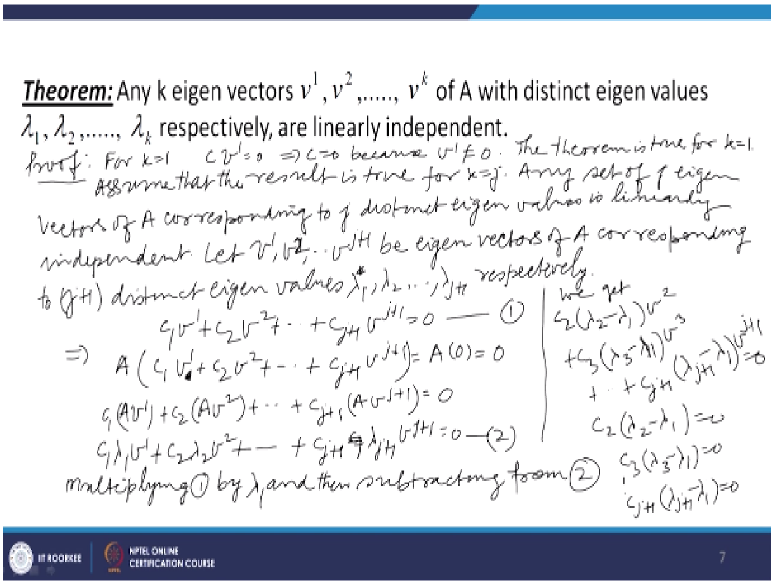

Any k eigen vectors v1, v2, vk of A with distinct eigen values lambda 1, lambda 2 ends on lambda k respectively are linearly independent. Let us prove this theorem. We shall prove this theorem by induction on k. For k =1 the eigen vector v1 is clearly linearly independent because if you take constant times v1 =0 then this implies c = 0 because v1 is an eigen vector so v1 is != 0.

So for k = 1 the theorem is true. The theorem is true for k = 1. For k =1 we have seen whenever we consider cv1 = 0 we get c = 0 and so v1 is linearly independent. Now let us assume that the result is true for k = 0 okay, so assume that which means that any set of j eigen vectors of A corresponding to distinct eigen values corresponding to j distinct eigen values is linearly independent.

So and then we shall show it for j + 1, okay, we shall show that if you consider j + 1 eigen vectors corresponding to j + 1 distinct eigen values then they are linearly independent. So let us assume that let v1, v2, vj+1 be j+1 eigen vectors of A corresponding to j +1 distinct eigen values lambda 1 we write here, lambda 1, lambda 2, lambda j+1 respectively, okay.

Now in order to prove that v1, v2, vj+1 are linearly independent let us consider the equation c1 v1 + c2 v2 and so on cj+1 vj+1 = 0 vector, okay, where c1, c2, cj+1 are scalar. Now if we want to show that v1, v2, vj+1 are linearly independent then we shall have to show that c1, c2, cj+1 are all 0s. Now what we do is let us operate on this equation by the matrix A, so when we operate by A we get this.

Now A matrix and this is 0 vector so we get 0 vector here okay, and lefthand side using the properties of matrixes we have c1 times Av1 + c2 Av2 and so on cj+1 Avj+1 = 0 and since v1, v2, vj+1 are eigen vectors of A corresponding to eigen values lambda 1, lambda 2, lambda j+1 Av1 will be lambda 1 v1, this will be lambda 2 v2 and so on, this will be lambda j+1 vj+1 = 0.

Now what we do is let us take it as equation 1, we multiply it by lambda 1, okay, we multiply equation 1 by lambda 1 and then subtract from 2, so multiplying 1 by lambda 1 and then subtracting from 2, what we notice is, let me write here, we will get, okay, so c1 lambda 1 v1 will cancel with c1, lambda 1 v1 and we will get c2, lambda 2 - lambda 1 * v2 + c3 lambda 3 - lambda 1 * v3 and so on.

Cj +1 lambda j + 1 – lambda 1 * vj +1 = 0 vector. Now v2, v3, vj+1 are j eigen vectors of A corresponding to the distinct eigen values lambda 2, lambda 3 lambda j+1 respectively, okay. So by our hypothesis since we have assumed that the result is true for k = j okay, it follows that v2, v3, vj+1 are linearly independent so we have c2 lambda 2 – lambda 1 = 0, c3, lambda 3 - lambda 1 = 0 and so on cj+1 lambda j+1 – lambda 1=0.

Now we have assumed lambda 1 lambda 2 lambda j+1 to be distinct eigen values, so lambda 1 is != lambda 2, lambda 1 is != lambda 3, lambda 1 is != lambda j+1 so what we get here.

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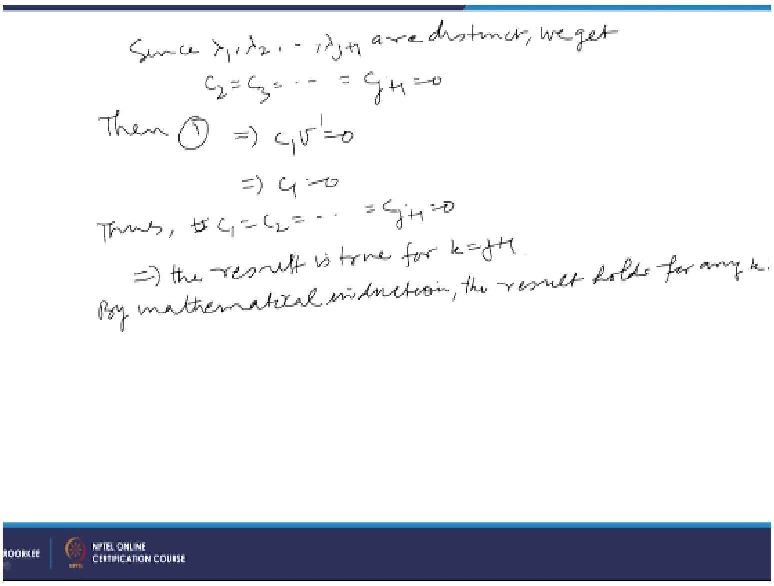

We get here c2 since lambda 1, lambda 2, lambda j+1 are distinct we get c2, c3, and so on, cj+1 = 0, now still we have to show that c1 is 0, so what we do, we go back to the equation 1 okay, we go to the equation 1 put here c2 = 0, c3 = 0, cj+1 = 0. So then equation 1 gives us c1 v1 = 0 and we know that since v1 is nonzero eigen vector c1 v1 = 0 implies c1 = 0, so thus c1 = c2 and so on.

Cj+1 = 0, which imply that that result is true for k = j+1, so by mathematical induction on k, the result holds for any k. So this is the proof of this result. Now let us take the example.

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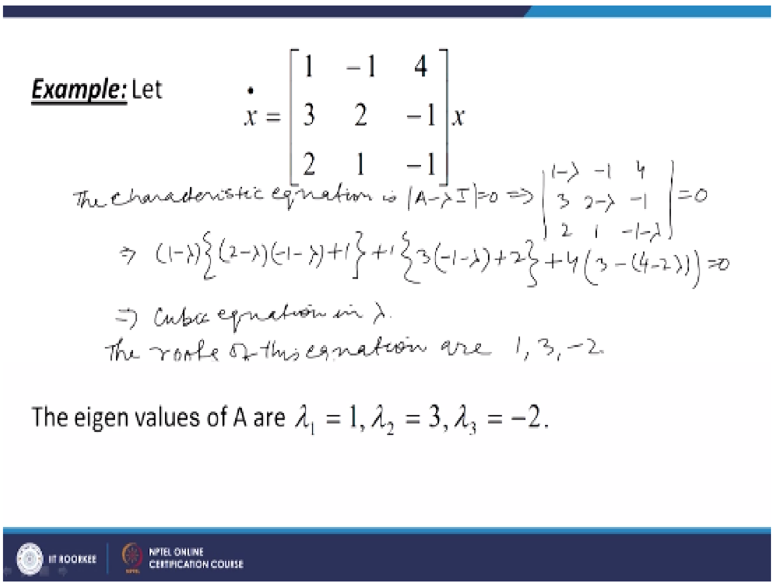

Suppose we have the vector differential equation x dot = Ax where A is the 3/3 matrix, 1 -1 4, 3 2 -1, 2 1 -1 then we want to solve this vector differential equation using the method of eigen values eigen vector. So what we will do? We will find the characteristic equation, we write the characteristic equation as determinant of A - lambda = 0, which gives us from the diagonal elements of A we subtract lambda.

So 1-lambda -1 4, 3 2-lambda, -1, 2 1 -1-lambda this determinant = 0, then we expand this determinant okay, what we will get, 1 - lambda times 2 - lambda * -1 – lambda and then we get + 1, okay, then we get minus of this so + 1 and then we have 3 * -1-lambda and then +2 and then we get 4 times 3 - 2 * 2 - lambda that is 4 - 2 lambda, okay.

We can then simplify this equation. We will get a cubic equation in lambda, so this will give us a cubic equation in lambda. We can easily solve the cubic equation in lambda, it will turn out that the roots are 1, 3, -2, okay. So the eigen values of A are lambda 1 = 1, lambda 2 = 3 and lambda 3 = -2.

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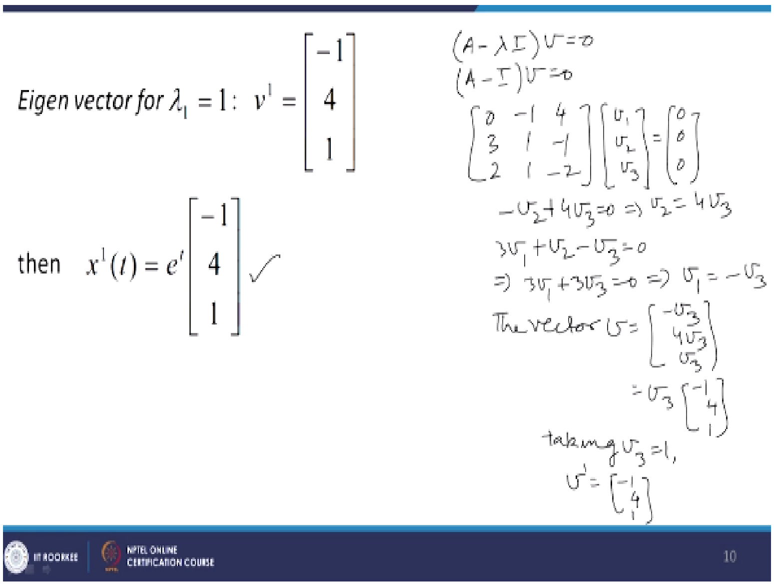

Now what we will do corresponding to the eigen value lambda 1 = 1 let us find the eigen vector for of the matrix A, okay. So we have to solve the equation A - lambda I v = 0, okay, lambda I = 1 here, so we get A - I v = 0, now A - I means from the matrix A, we subtract the identity matrix so when you subtract the identity matrix from here the diagonal element will be subtracted by 1 each.

So we will get 0 -1 4 that is the first row, then we get 3 1 -1, and then we get 2 1 -2. Let us say the vector v is v1 v2 v3. Righthand side is 0 vector. So now we have to solve this equation, so the first we have a system of 3 equations here, homogenous system. So first equation gives you v1 * 0 – v2 + 4v3 = 0 that is to say v2 = 4 v3. Second equation is 3v1 + v2 - v3 = 0.

And so we can put the value of v2 here so this gives us 3 v1 v2 is 4v3 so 4v3 - v3 is 3v3 = 0 which gives us v1 = -v3. Okay, so now we need not solve the third equation 2 v1 + v2 - 2v3 = 0 because the 3 equations are not linearly independent, one equation is always 0 equation, it is the linear combination of the other 2. This is because the determinant of A - lambda = 0. So what we do is, let us write the vector v then.

The vector v is v1 v2 v3, v1 is –v3, v2 is 4v3 and v3 is arbitrary. So I can write it as v3 times -1 4 1 okay. Now choosing v3 = any nonzero scalar value we can choose for v3. So let us choose the convenient value v3 = 1 taking v3 = 1 we get v1. We get the vector v, v = -1 4 1, okay, so this is vector v1 corresponding to eigen value lambda 1 and therefore one solution of the vector differential equation we can find for the eigen value lambda 1 and eigen vector v1.

It is e to the power lambda 1t, lambda 1t means e to the power t and then v1 – 141, so this is one solution of the given differential equation.

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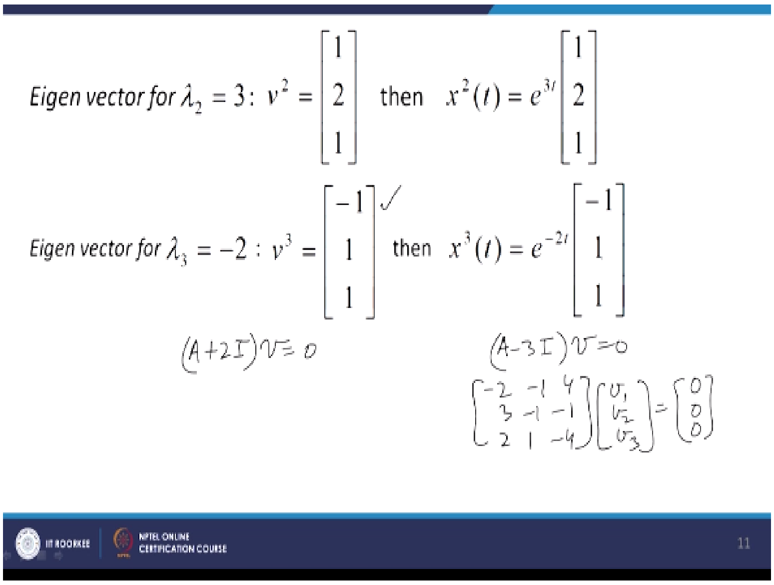

Now similarly we can find for the eigen value lambda 2 = 3 let us find the eigen vector. So for lambda 2 = 3 we have to solve A - 3I v = 0, A - 3I if you find, A - 3I will be you subtract 3 from the diagonal elements of the matrix A so when you subtract 3 from here you get -2 -1 4, so you get -2 -1 4 and then you have here 3 -1 -1 and then here you have 2 1 -4. Now we can see here first equation is -2v1 – v2 + 4v3 = 0 and third equation is just minus times the first equation.

Because it is 2v1 + v2 - 4v3 = 0 so we can consider only first 2 equations that is first 2 rows -2 -1 4, 3 -1 -1 and the 2 equations in 3 unknowns v1, v2, v3 can be solved okay, and we will get the eigen vector for lambda 2 = 3 is v2 that is 1 2 1 and the corresponding solution will then be = x2t = e to the power lambda 2t that is e to the power 3t * 1 2 1 and for lambda 3 = -2 similarly we can find for lambda 3 = -2 we need to solve the equation A - lambda I.

That is A + 2I v = 0, so to the matrix A, we add 2 times the identity matrix that means we add in the diagonal elements of A2 and then the matrix that we get that will be A+2I, so A+2I 2v = 0 can be solved and we get v3 = -1 1 1. So the third solution is then given by x3t = e to the power -2t and then -1 1 1, okay.

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Then solution of the given system is of the form

$$x(t) = c_1 e^{t}\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{-2t}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -c_1 e^{t} + c_2 e^{3t} - c_3 e^{-2t} \\ 4c_1 e^{t} + 2c_2 e^{3t} + c_3 e^{-2t} \\ c_1 e^{t} + c_2 e^{3t} + c_3 e^{-2t} \end{bmatrix}$$

The solutions of the given system is then of the form xt = c1 e to the power t okay, so c1 times x1t, c2 times x2t, c3 times x3t, so we get, if you multiply these c1 e to the power t, c2 e to the power 3t, c3 e to the power -2t to these 3 column vectors and then aid what you get is this –c1 e to the power t + c2 e to the power 3t – c3 e to the power -2t and then 4 c1 e to the power t + 2 c2 e to the power 3t + c3 e to the power -2t.

And then you get here the third component is c1 e to the power t + c2 e to the power 3t + c3 e to the power -2t so this is the general solution of the given vector differential equation. Now let us take another problem this is the initial value problem, here we are given the vector differential equation x dot = Ax.

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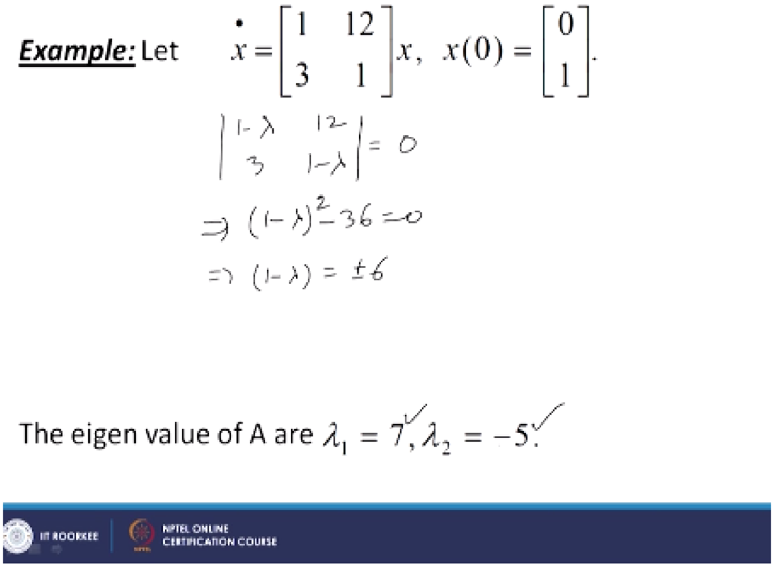

And we are also given an initial condition that is at x t = 0 x = 01 okay. So again we shall write the characteristic equation here 1 - lambda 12, 3 1 - lambda = 0 and this will give you quadratic equation in lambda 1 - lambda whole square -36 = 0. So you can say that 1 - lambda = +/-6 okay, so taking 6, when you take 6 here then we get lambda = 1-6, so -5 okay.

And when you take -6 here then 1 - lambda = -6 gives you lambda = 7, so we get 2 eigen values of lambda 7 and -5 and they are both real and distinct so corresponding eigen vectors will be linearly independent.

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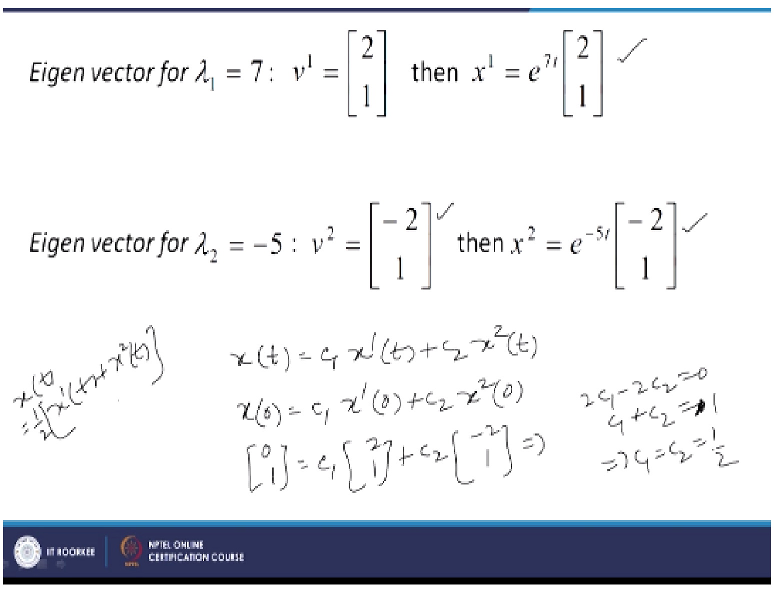

So for lambda 1 = 7 it turns out that v1 is 2 1 so x1t is e to the power 7t * 2 1 and lambda 2 = -5 gives you v2 = -2 1, so x2 = e to the power -5t -2 1 okay. So the general solution is what, x t = c1 x1 t + c2 x2t and we then use the initial condition that is the value of x at t = 0 which is given as 0 1. So if you do that okay, what you get is this.

This will be x0 put t = 0 here so x0 = c1 x10 + c2 x20, okay, x0 is given to be 01, so 01 = c1 times x1 0 we can find from here when you put t = 0 you get 2 1 here and then you get c2 times x20 when you put t = 0 you get -2 1 okay, so this gives you 2c1 - 2c2 = 0 and then c1 + c2 = 1. So it is clear, c1 = c2 and so c1 + c2 = 1, so this gives you c1 c2 = 1/2 each, okay. So put the value 1/2 here so then xt = 1/2 x1t + x2t, so this is the particular solution in given initial value problem.

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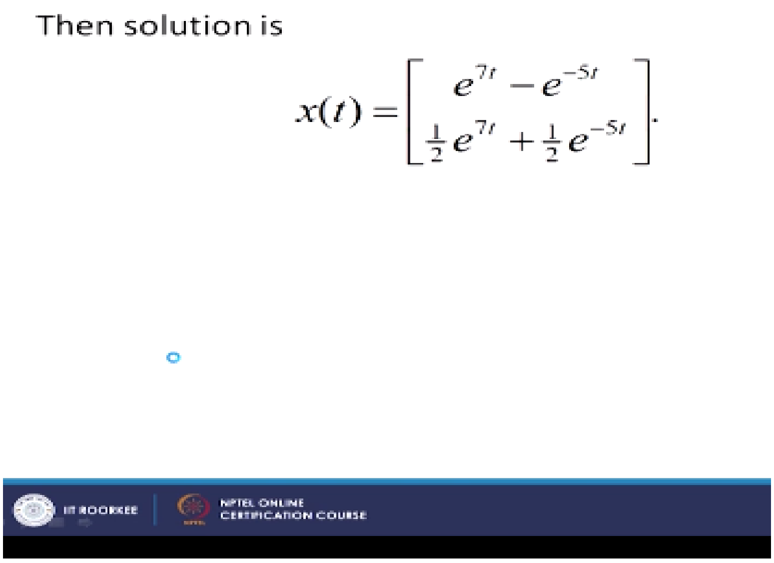

So xt turns out to be e to the power 7t - e to the power -5t 1/2 e to the power 7t + 1/2 e to the power -5t. That is all for this lecture. Thank you very much for your attention.