Ordinary and Partial Differential Equations and Applications

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Lecture – 07
Linear System

Hello friends, I welcome you to my course on ordinary and partial differential equations and applications, in this course, in this lecture we shall discuss linear systems of ordinary differential equations.

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One of the most important concepts of analysis is that of a system of n simultaneous first order differential equations in several variables i.e. equations of the form

$$\frac{dx_1}{dt} = f_1(t, x_1, x_2, \ldots, x_n)$$
$$\frac{dx_2}{dt} = f_2(t, x_1, x_2, \ldots, x_n) \quad \ldots(1)$$
$$\vdots$$
$$\frac{dx_n}{dt} = f_n(t, x_1, x_2, \ldots, x_n)$$

One of the most important concepts of analysis is that of a system of n simultaneous first order differential equations in several variables, such a system can be expressed as dx1/dt = f1 t, x1, x2, xn, dx2/dt = f2, t, x1, x2, xn and so on, the nth equation is dxn/dt = fn t, x1, x2, xn. Here, t is the independent variable x1, x2, xn are n dependent variables of the independent variable t.

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where $x_1, x_2, \ldots, x_n$ are unknown functions of a single independent variable t.

Systems of differential equations arise quit naturally in many scientific problems.

A solution of system (1) is n functions $x_1(t), x_2(t), \ldots, x_n(t)$ such that

$$\frac{dx_j(t)}{dt} = f_j(t, x_1(t), x_2(t), \ldots, x_n(t)), \quad j = 1, 2 \ldots, n$$

For example, let $x_1(t) = t$ and $x_2(t) = t^2$, then we provide a solution of the simultaneous first order differential equations

$$\frac{dx_1}{dt} = 1 \text{ and } \frac{dx_2}{dt} = 2x_1$$

Since,

$$\frac{dx_1}{dt} = 1 \text{ and } \frac{dx_2}{dt} = 2t = 2x_1$$

We will be; our aim will be to solved the system of differential equations, so, x1, x2, xn are unknown functions of the single variable t, systems of differential equations arise quite naturally in many scientific problems, a solution of system 1 is a set of n functions; x1t, x2t, and so on xnt such that dxj/dt = fj t x1t, x2t, xnt that means the n functions x1t, x2t and xnt satisfy the system of differential equations given by 1.

Now, for example, let us consider x1t = t and x2t = t square then these 2 functions provide us a solution of the simultaneous first order differential equations; dx1/dt = 1 and dx2/dt = 2x1, we can see, we have a system of 2 differential equations of first order, so here we can; if you take x1t = t and x2t = t square, we can show that these 2 functions satisfying this system of differential equations. How it follows?

You can see dx; x1t = t implies dx1/dt = 1, okay and x2t = t square gives you dx2/dt = 2t, now t = x1, so dx2/dt = 2x1. Therefore, x1t = t and x2t = t square satisfy the system of differential equations of first order; dx1/dt = 1 and dx2/dt = 2x1, so they form a solution of this system of differential equations.

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If we impose the initial condition on the functions $x_1(t), x_2(t), \ldots, x_n(t)$ as $x_1(t_0)=x_1^0, x_2(t_0)=x_2^0, \ldots, x_n(t_0)=x_n^0$ (1′)

then the system (1) together with the initial conditions (1′) is called as **initial-value problem**.

A solution of this initial value problem is 'n' functions $x_1(t), x_2(t), \ldots x_n(t)$, which satisfy (1) and the initial condition (1′).

For example, $x_1(t)=e^t$ and $x_2(t)=1+\frac{e^{2t}}{2}$ is a solution of the IVP

$$\frac{dx_1}{dt}=x_1, \quad x_1(0)=1$$
$$\frac{dx_2}{dt}=x_1^2, \quad x_2(0)=\frac{3}{2}$$

Now, if we impose the initial condition on the functions; x1t, x2t and so on xnt as x1t0 = x10, x2t0 = x20 and so on xnt0 = xn0, then these system 1 together with the initial conditions 1 dash, okay is called an initial value problem. So, here we, you can see at t = t0, the value of x1 is x10 at t = t0, the value of x2 is x20 and so on, the value of xn at t = t0 is xn0, so if we are given these conditions together with the system 1, then the system 1 is called; is an initial value problem.

A solution of this initial value problem is n functions, x1t, x2t, xnt, which satisfy the system 1 and the initial conditions given by 1 dash. For example, let us consider x1t = e to the power t and x2t = 1+ e to the power 2t/2, then it is a solution of the IVP; IVP means initial value problem and the initial value problem here is dx1/dt = x1, dx2/dt = x1 square, the initial conditions are x1 at t = 0 is 1, x2 at t = 0 is 3/2.

Now, we have to show that x1t = e to the power t and x2t = 1 + e to the power 2t/2 is a solution of this initial value problem, so if you; first we have to show that x1t = e to the power t and x2t 1 + e to the power 2t/ 2 satisfy the system of differential equations and then we have also to show that the satisfy the initial conditions, x10 = 1, x20 = 3/2. Now, x1t = e to the power t implies dx1/dt = e to the power t, e to the power t is x1, so dx1/ dt = x1 is satisfy.

X2t is 1+ e to the power 2t/2, when you differentiate this with respect t, what you get is dx2/dt = 2 times e to the power 2t/2 that is e to the power 2t, so dx2/dt is = e to the power 2t which is = e to the power t whole square and so, we can write that as x1 square, so dx2/dt =

x1 square, so x1t = e to the power t and x2t = 1 + e to the power 2t/2 satisfy the differential equations of the system and also you can see that x1 at t = 0 = e to the power 0, which is = 1.

So, x10 = 1 is satisfied and x2 at t = 0 is 1+ e to the power 0/2, which is = 1+ 1/2 so, we have 3/2, so x2 at t = 0 is 3/2 and therefore, x1t = e to the power t and x2t = 1 + e to the power 2t/2 satisfies this system of differential equations together with the given initial conditions and therefore, they give us a solution of the initial value problem.

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Since $x_1(t)=e^t \Rightarrow \frac{dx_1}{dt}=e^t=x_1$

$$\frac{dx_2}{dt}=e^{2t}=x_1^2(t),\ x_1(0)=1,\ x_2(0)=\frac{3}{2}$$

First order systems of differential equations also arise from higher order differential equations for a single variable y(t). Every n[th] order differential equation for the single variable y, $y^{(n)}=f(t,y,y',\dots,y^{(n-1)})$ can be converted into a system of n first-order differential equations for the variables

$$x_1(t)=y,\ x_2(t)=\frac{dy}{dt},\ \dots,\ x_n(t)=\frac{d^{n-1}y}{dt}.$$

Then $\frac{dx_1}{dt}=x_2, \frac{dx_2}{dt}=x_3, \dots, \frac{dx_{n-1}}{dt}=x_n$

and $\frac{dx_n}{dt}=f(t,x_1,x_2,\dots,x_2)$, which is clearly a special case of (1).

Now, first order systems of differential equations also arise from higher order differential equations for a single variable yt, every nth order differential equation for the single variable yn, and this is nth derivative of y with respect to t, yn we are representing; yn represents dny over dtn = ft, y, y dash, y and -1 that is n – 1 of derivative of y, so it can be converted into a system of n first order differential equations for the variables.

Now, what we do is; we will convert this nth order differential equation to a system of differential equations of first order, so let us write, x1t = y, x2t = dy/dt and so on xnt = dn – 1/ over dtn - 1 that is n – 1th derivative of y with respect to t, then from here, we can see that dx1/dt will be = dy/dt and dy/dt is x2, so dx1/dt = x2, dx2/dt will be d square y over dx square but d square y over dx square is x3, so dx2/dt = x3 and so on, dxn – 1; xn – 1t is the n -2th derivative of y with respect to t.

So, dxn -1t over dt will be = n – 1th of the derivative of y and therefore it is = xn and dxn/dt; dxn/dt is; if you differentiate this with respect to t, then dxn/ dt gives you nth derivative of y with respect t, so dxn/dt can be written for this nth derivative of y with respect to t and we have ft, y is x1, y dash is x2 and so on, n – 1th derivative of y is = xn, so this is ft, x1, x2 and so on xn, which is clearly especially, so now, you can see there are n equations.

It is a system of n equations; first order n equations; dx1/dt = x2, dx2/ dt = x3, dxn – 1/dt = xn and that the nth equation is dxn/dt = ft x1, x2, xn, so this is a special case of the system given by equations 1.

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Example: Convert the differential equation

$$a_n(t)\frac{d^n y}{dt^n}+a_{n-1}(t)\frac{d^{n-1} y}{dt^{n-1}}+.....+a_0 y=0$$

into a system of n first order equations.

Solution: let $x_1(t)=y, x_2(t)=\frac{dy}{dt}, ..., x_n(t)=\frac{d^{n-1}y}{dt^{n-1}}$

Then $\frac{dx_1}{dt}=x_2, \frac{dx_2}{dt}=x_3, ... \frac{dx_{n-1}}{dt}=x_n$

and $$\frac{dx_n}{dt}=-\frac{1}{a_n(t)}\left[a_{n-1}(t)x_n+a_{n-2}(t)x_{n-1}+.....+a_0x_1\right]$$

Now, let us for example, convert the differential equation an t dny over dtn + n -1t, dn -1over dtn - 1 and so on, a0y = 0 * system of n first order differential equations. Now, as we have said, we will put x1t = y, x2t = dy/dt and xnt = n -1th derivative of y with respect to t, then dx1/dt is dy/dt, so xdxn/ dt is x2, dx2/dt is x3 and dxn – 1 over dt, dxn – 1 over dt is = xn because xn – 1 is n – 2th derivative of y with respect to t, so when you differentiate xn -1 with respect to t, you get n - 1 of the derivative of y with respect to t, which is assumed as xn t.

So, dxn - 1 over dt is = xn and dxn/ dt, now you can see dny over dtn; dny over dtn is dxn / dt, okay, so this is the xn/dt. What we do is; we transform all these terms to the other side and divide by ant, assuming that ant is != 0, for nt, so that dxn/dt will be -1 upon ant and then an -1, this is xn, then an -2 * xn -1 and so on a0 and y is x1, so we get n equations of first order, which are dx1/dt = x2, dx2/dt = x3 and so on dxn – 1/ dt = xn.

And the nth equation is dxn/dt = -1 upon ant, inside the bracket, we have an – 1 t * xn + n – 2t * xn - 1 and so on a0x1, so the bracketed expression or you can say the right hand side is a function of t and the n variables; x1, x2, xn, so this system is of the form 1, so you can see that the nth order differential equation which is given to us has been converted into a system of n differential equations of first order.

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Example: Convert the initial value problem

$$\frac{d^3 y}{dt^3}+\left(\frac{dy}{dt}\right)^2+3y=e^t;\quad y(0)=1,\ y'(0)=0,\ y''(0)=0$$

Into an initial value problem for the variables y, $\frac{dy}{dt}$, and $\frac{d^2 y}{dt^2}$.

Solution: Let $x_1(t)=y,\ x_2(t)=\frac{dy}{dt},\ x_3(t)=\frac{d^2 y}{dt^2}$

Then

$$\frac{dx_1}{dt}=x_2,\quad \frac{dx_2}{dt}=x_3,\quad \frac{dx_3}{dt}=e^t-x_2^2-3x_1,$$

$$x_1(0)=1,\ x_2(0)=0,\ x_3(0)=0.$$

Now, let us take an initial value problem, we can see this is the third order differential equation, d cube y over dt cube + dy/dt whole square + 3y = e to the power t, the values of the dependent variable y at t = 0 is given as 1, the derivative of y with respect to t; dy/dt at t = 0 is 0 and d square y over dt square at t = 0 is = 0, so we have to convert this initial value problem into a system of differential equations.

Now, again, let us say x1t = y, x2t = dy/dt and x3t = d square y/ dt square, okay so then d cube y over dt cube will be dx3/ dt, so this is dx3/dt, okay and we can then, this is dx3/dt and these terms dy/dt whole square + 3y, if you take to the other side, what you get? E to the power t – dy/dt is = x2, okay, so x2 square, so e to the power t – x2 square – 3 * y, y = x1, so we get dx3/ dt = e to the power t – x2 square – 3x1.

Now, you can see, we have 3 equations; first equation is dx1/dt; dx1/dt is = x2 because dx1/dt is dy/dt and dy/dt is x2, so dx1/dt = x2 and dx2/d2 is d square y over dt square, which is x3, so dx2/dt is = x3 and dx3/ dt, we have just now seen, it is e to the power t – x2 square – 3x1. Now, y at 0 is given = 1, so you can see x1t = y, so at t= 0, y =1, therefore x10 = 1, so x10 = 1at t = 0, dy/dt is = 0, so x20 = 0 and at t = 0, d square y / dt square = 0, so x30 = 0.

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If each of the functions $f_1, f_2, \ldots, f_n$ in (1) is a linear function of the dependent variables $x_1, x_2, \ldots, x_n$, then system of equations is called linear.

The most general system of n first-order linear equations is of the form

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + \ldots + a_{1n}(t)x_n + g_1(t)$$
$$\frac{dx_2}{dt} = a_{21}(t)x_1 + \ldots + a_{2n}(t)x_n + g_2(t) \qquad (2)$$
$$\vdots$$
$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + \ldots + a_{nn}(t)x_n + g_n(t)$$

So, the initial value problem, okay, the given initial value problem is converted into a system of first order differential equations with the initial condition, x10 =1, x20 = 0, x30 = 0. Now, if each of the functions; f1, f2, fn in the system 1, okay if each of the functions f1, f2, fn in the system 1 is the linear function of the dependent variables x1, x2, xn, then the system of equations is called linear.

Now, the most general linear system of n first order differential equations is of the form; dx1/dt = a11t * x1 and so on a1nt * xn + g1t, dx2/dt = a21t * x1 and so on a2nt * xn + g2t and the nth equation is dxn/dt = an1 t * x1 and so on annt * xn + gnt, so this is the most general system of n first order linear equations, a linear system it is, you can see the functions f1, f2, fn, these are the functions of tx1, x2, xn, this f1 tx1, x2, xn, this f2 tx1, x2, xn and this fnt x1, x2, xn.

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If each of the n functions $g_1, g_2, \ldots, g_n$ is identically zero, then the system (2) is called homogeneous; otherwise it is non-homogeneous.

Let us consider the case where the coefficients a_{ij} do not depend on t.

A homogeneous linear system with constant cofficients

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + \ldots + a_{1n}(t)x_n$$

$$\frac{dx_2}{dt} = a_{21}(t)x_1 + \ldots + a_{2n}(t)x_n \qquad (3)$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + \ldots + a_{nn}(t)x_n$$

In all these functions; f1, f2, fn, x1, x2 and so on xn, they occur in the first degree and separately, okay, so it is a linear system. Now, if each of the n functions; g1, g2, gn is identically 0, if it is so happens that g1t, g2t and so on gnt are identically 0, then for every value of t; g1t, g2t and so on gnt = 0 for every value of t, then we say that they are identically 0, then the system 2 is called homogeneous linear system otherwise, we call it a non-homogenous linear system.

Now, let us consider the case, where the coefficients aij do not depend on t, so we are telling a particular case of the homogeneous system, where the coefficients of x1, x2, xn that is a11 t and so on a1n t, a21t and so on a2nt and so on an1t, an2t, annt, they are just constants, they do not depend on t, so then such a system is called homogeneous linear system with constant coefficients.

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(3) can be expressed in the concise form

$$\dot{x} = \frac{dx}{dt} = Ax \qquad (4)$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix}$$

Moreover if $x_1(t), x_2(t), \ldots, x_n(t)$ satisfy the initial conditions $x_1(t_0) = x_1^0, x_2(t_0) = x_2^0, \ldots x_n(t_0) = x_n^0$, then x(t) satisfies the initial value problem

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So, this is of the form dx1/dt = a11 * x1 and so on a1n * xn, dx2/dt = a21 * x1 and so on a21 * xn and dxn / dt = an1 * x1 and so on ann * xn. Now, this system, we are writing it by equation 3, this system given by 3 can be expressed in the concise form. Now, you can see if the number of equations are very large, if the n is very large, then it will be very cumbersome to write these equations, so we can write them in a concise form using the vector notation.

What we do is; let if we denote xt = x1t, x2t, xnt, this column vector, then this xt is called vector valued function and if you differentiate this xt with respect to t, then dx/dt is another vector valued function whose components are given by dx1/dt, dx2/dt and so on dxn/dt, so therefore you can see the left hand side which is; in the first equation dx1/dt, in the second equation dx2/dt, in the last equation dxn/dt, it can be put in the form of a vector notation which is x dot.

X dot means dx/dt, so dx/dt will contain; is a column vector having n components dx1/dt, dx2/dt, dxn/dt, the right hand side of these n equations is a11 x1and so on an1 xn, a21 x1 + and so on a2n xn, an1x1 and so on ann * xn, it can be put in the form of a matrix a multiplied by the vector x, where the matrix a is a11, a12, a1n, a21, a22, a2n, an1, an2, ann, multiplied by; so the vector x that is x1, x2, xn.

You can see if you multiply this matrix a by the vector x; x1, x2, xn, then the first row of ax will be a11 x1 + a12 x2 and so on a1n xn and then the second row of the matrix ax will be a21 x1, a22 x2 and so on a2n xn, the last row of matrix ax will be an1 x1, an2 x2 and so on ann xn and so, the n equations; the n equations are given by this 3 can be put in the form of a vector equation; dx/dt = Ax.

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$$\dot{x} = Ax, \quad x(t_0) = x^0, \text{ where } x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \cdot \\ \cdot \\ \cdot \\ x_n^0 \end{bmatrix}$$

Now, if x1t, x2t, xnt satisfy the initial conditions that is at t = t0 x1 is x10, x2 is x20 and xn is xn0, then xt satisfies the initial value problem, then we can write the system in the concise form x dot = dx/dt = Ax, x at t0= x0, okay, because xt is = x1t, x2t, xnt, so at t = t0, x t0 will be x1t0, x2t0, xnt0 and we have assumed that x1t0 is x10, x2t0 is x20, xnt0 is xn0, so xt0 = x1t0, x2t0, xnt0 that is x10, x20, xn0.

And that we can write as x0, so xt0 = x0, where x0 = x10, x20 and so on xn0, so you can see that the initial value problem can be expressed in the concise form which is x dot = Ax, where x t0 = x0 and x0 is this column vector.

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Example: consider the system of equations

$$\frac{dx_1}{dt} = 3x_1 - 7x_2 + 9x_3$$
$$\frac{dx_2}{dt} = 15x_1 + x_2 - x_3$$
$$\frac{dx_3}{dt} = 7x_1 + 6x_3$$

then $\dot{x} = \begin{bmatrix} 3 & -7 & 9 \\ 15 & 1 & -1 \\ 7 & 0 & 6 \end{bmatrix} x$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Now, let us consider for example, the system of equations, dx1/dt = 3x1 - 7x2 + 9x3, dx2/dt = 15 x1 + x2 – x3 and dx3/dt = 7x1 +6 x3, then again if you take xt to be the vector value function, x1t, x2t, x3t, then x dot will be dx1/dt, dx2/dt, dx3/dt, so this left side can be represented by the vector valued function x dot and the right hand side, we have; here the coefficient of x1, x2, x3 are 3, – 7, 9, here x1, x2, x3 have got coefficients, 15, 1 and -1.

Here, x1 has coefficient 7, x2 has coefficient 0, x3 has coefficient 6, so the matrix a, okay will be say, 3, -7, 9, first row of the matrix a will be 3, -7, 9, second row will be 15, 1, -1 and the third row will be 7, which is the coefficient of x1, then coefficient of x2 is 0 and then coefficient of x3, which is 6, multiplied by the vector x; x is the vector; x1, x2, x3, so if you multiply this matrix a by the vector x1, x2, x3, you get these right hand side.

And left hand side is this x dot, which is dx1/dt, dx2/dt, dx3/dt, so this system of 3 equations; this you can see, this is the homogenous system of linear differential equations of first order, this can be expressed in the concise form, x dot = Ax, where A is this 3 by 3 matrix.

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Example: The initial value problem

$$\frac{dx_1}{dt} = x_1 - x_2 + x_3, \quad x_1(0) = 1$$

$$\frac{dx_2}{dt} = 3x_2 - x_3, \quad x_2(0) = 0$$

$$\frac{dx_3}{dt} = x_1 + 7x_3, \quad x_3(0) = -1$$

can be written as

$$\dot{x} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 1 & 0 & 7 \end{bmatrix} x \quad \text{and} \quad x(0) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Let us now take up another problem, here we are taking initial value problem, so we have 3 equations; dx1/dt = x1 – x2 + x3, dx2/dt = 3x2 – x3, dx3/dt = x1 + 7x3, we are given that at t = 0, x1 is 1, at t = 0, x2 is 0, at t= 0, x3 is -1. Now, as in the previous example, here also if you take xt to be the vector valued function having components x1t, x2t, x3t, then the left hand side of this systems of equations is x dot t because x dot t is dx1/dt, dx2/dt, dx3/dt, so x dot t and the coefficient matrix here.

The matrix of the unknown functions x1, x2, x3 is 1-11, okay, 1-11, then here, 0, 3,-1 because the coefficient of x1 is 0, so 0 * 0 then 3, then -1 and here we get 1, then 0, then 7, the coefficient of x2 is 0 here, so 1, 0, 7 and then the vector x that is which has got components x1, x2, x3. Now, let us count for these initial conditions, so xt = x1, x2, x3, so x0 is x10 which is 1, x20, which is 0, and then x30 which is -1.

So, we have seen in this lecture how to write a linear system of first order differential equations, which is homogenous in the concise form also, we have seen how we can write initial value problem where we are given a system of first order differential equations homogeneous system with the initial conditions in the concise form, so by using the vector notation, we can write the equations in a concise form.

Now, in our next lectures, we shall see how we can solve system of homogeneous linear differential equations with constant coefficients. Thank you very much for your attention.