Ordinary and Partial Differential Equations and Applications
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Lecture – 60
Duhamel's Principle

Hello friends, welcome to this lecture. In this lecture we will discuss about Duhamel principle and its application. Now Duhamel principle is very important say concept and it will help us to find out the solution of non-homogenous equation with the help of homogenous equation. So basically for example you have this kind of problem say.

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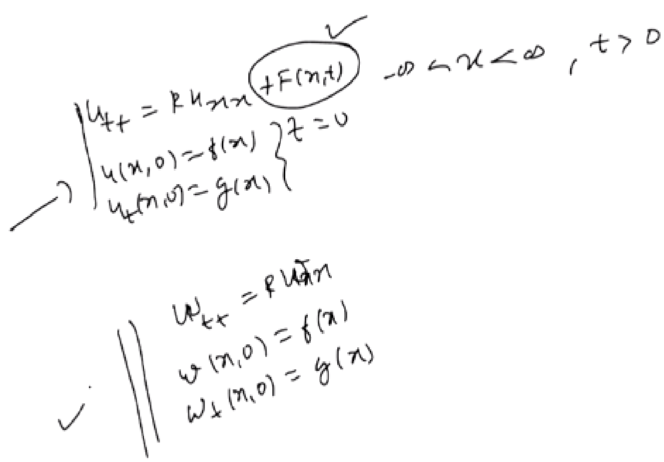

Utt = KUxx + F(x, t) and U(x, 0) = f(x), Ut(x, 0) = g(x). So you consider this as 1 dimensional wave equation and here x is lying between say - infinity to infinity and t > 0 and at t = 0 we have these initial condition. Now we know the solution when this forcing term F(x, t) is not there. So it means that we know the solution of Utt + Kuxx, U(x, 0) = F(x), Ut(x, 0) = g(x), but if you put any kind of force here that is f(x, t) then we do not know how to find out a solution.

Of course there are certain method available, but here we will discuss a method of solution solving this non homogenous problem with the help of say homogeneous problem that is KUxx. So here let us say W and here we have W(x0) = say f(x), Wt(x, 0) = g(x) and how we can relate this solution of this non homogenous problem as a solution of this homogenous problem that is the thing we wanted to discuss here. So let us proceed.

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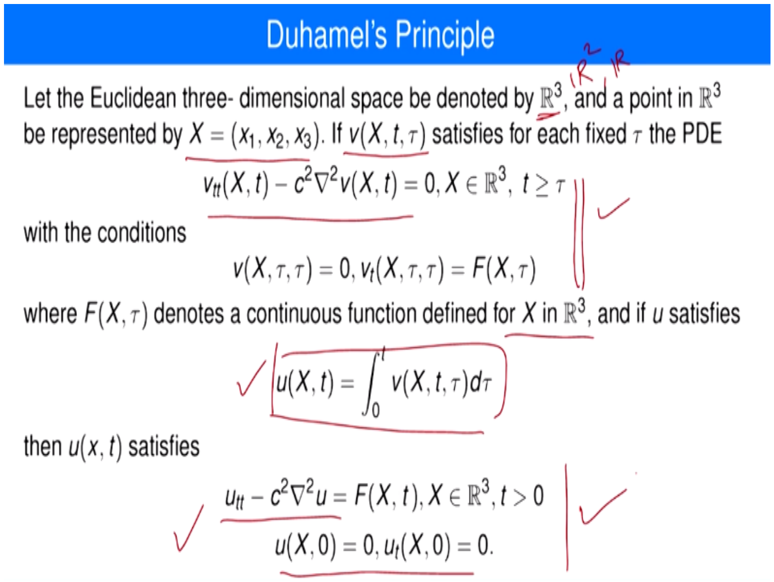

So let the Euclidean 3-dimensional space be denoted by R3. So here we just taken started with R3. You can work in R2 and R. The same concept will work and a point in R3 be represented by X = (x1, x2, x3) and if v(X, T, tau) satisfies for each fixed tau the PDE, vtt(X, t) tau - c square del square v(X, t) = 0, X belongs to R3 and t is >=tau with the condition that v(X, tau, tau) = 0 and vt(X, tau, tau) = F(X, tau).

So once we know the solution of this where F(X, tau) denote a continuous function defined for X in R3 and if u satisfies u(X, t) = 0 to t v(X, t, tau) d tau then this is the important thing. Then this u which is defined like this satisfy the non homogenous equation given as utt - c square * del square * u = F(X, t) where X belongs to R3 t > 0, u(X,0) = 0 and ut(X, 0) = 0. So it means that here if you look at this is what this is a non homogenous equation utt - c square * del square * u = F(X, t).

This is a non homogenous equation with homogenous initial condition. The solution of this we are finding by the homogenous equation with non homogeneous initial condition. So relation between these 2 problems is given by this relation that U(X, t) can be given as 0 to t v(X, tau) d tau. So here we are not giving the exact proof of this result, but we verify that if u(X, t) is given as 0 to t v(X, t, tau) d tau where v(X, t, tau) is a solution of this problem.

Then u(X, t) will be a solution of this problem. The actual proof involves lot of say concept involving this Laplace transform so which we are not going to give in this lecture rather than we are just showing that if we define u(X, t) like this then it will satisfy this non homogenous equation with homogenous initial condition.

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Proof. Consider the equation

$$u_{tt} - c^2\nabla^2 u = F(X,t) \tag{1}$$

with

$$u(X,0) = u_t(X,0) = 0$$

Assume that the solution of the problem (1) is of the form

$$u(X,t) = \int_0^t v(X, t-\tau, \tau)\,d\tau \tag{2}$$

where $v(X, t-\tau, \tau)$ is one- parameter family solution of

$$v_{tt} - c^2\nabla^2 v = 0 \text{ for all } t \geq 0. \tag{3}$$

Further, we assume that at $t = \tau$,

$$v(X, 0, \tau) = 0 \text{ for all values of } \tau \tag{4}$$

So let us consider the equation utt - c square del square u = F(X, t) with u(X, 0) = ut(X, 0) = 0 so here we have a non homogenous equation and with homogenous initial conditions are given and assume that the solution of the problem is of the form u(X, t) = 0 to t v(X, t - tau, tau) d tau, where v(X, t - tau, tau) is a 1 parameter family solution of this. vtt - c square * del square v = 0 for all t >= 0 and for t = tau, v(X, 0, tau) = 0 for all values of tau and vt(x, 0, tau) is given as F(x, tau).

Please look at here. If you look at the statement of the result here. Here we have assumed that v(X, t, tau) is a solution of this problem, but here your t >= tau. So, but whatever so it means that here your partial differential equation start from say arbitrary point tau, but if we make this kind of transition like t - tau, then the solution of this which is given as v(X, t, tau) will satisfy we can write v(X, t, tau) as v tilde (x, t - tau, tau).

Then this v tilde (x, t - tau, tau) will satisfy the following equation v tilde tt - c square del square v tilde = 0 and here with x belongs to R3 and t is >= 0 rather than starting from t >= tau now for this v tilde is t >= 0 and initial condition if you look at then v of x. Now at t = tau we have the initial condition 0 so I can write since are assuming v(X, t, tau) as v tilde(x, t - tau, tau). So at t = tau this value is what v tilde(x, 0, tau) is given as 0 and v tilde(x, 0, tau) is given as F(x, tau) here.

So it means that if v(X, t) is a solution of this problem then we can say that by suitable change we can say that v tilde(X, t, tau, tau) satisfy this problem is it okay and we already know how to solve this particular problem. So here we can say that by taking v tilde which is a solution of this problem we can write v(x, t) as v tilde(x, t - tau, tau). So here we simply say that in place of writing v(x, t, tau) now we are writing v tilde(x, t - tau, tau) d tau.

So here we that is what we are writing in equation number 2 that u(X, t) is given as 0 to t v(X, t - tau, tau) d tau where v(X, t - tau, tau) is a 1 parameter family solution of vtt - c square del square v = 0 for all t >= 0 and at initial condition is v(X, 0, tau) is 0 and vt(X, 0, tau) is F(x, tau). So it is one of the same result.

So here when we define u(X, t) as this then your v(X, t) will satisfy this equation but and it is valid for t >= tau and at t = tau initial conditions are given, but if we take u(X, t) as 0 to t, v(X, t - tau, tau) d tau, then in this case this v (X, t - tau, tau) is a solution of this equation and it is defined at t >= 0.

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By differentiating and using Liebnitz rule, from (2), we have

$$u_t = v(X, 0, t) + \int_0^t v_t(X, t-\tau, \tau)d\tau$$

Using equation (4), we get

$$u_t = \int_0^t v_t(X, t-\tau, \tau)d\tau$$

Differentiating this result once again with respect to t, we obtain

$$u_{tt} = v_t(X, 0, t) + \int_0^t v_{tt}(X, t-\tau, \tau)d\tau \quad (5)$$

Using equation (3), the above equation reduces to

$$u_{tt} = v_t(X, 0, t) + \int_0^t c^2 \nabla^2 v d\tau$$

$$u(x,t) = \int_0^t v(x, t-\tau, \tau)\, d\tau \qquad \nabla^2 u = \int_0^t \nabla^2 v(x, t+\tau, \tau)\, d\tau$$

So now let us verify this. So here we have u. now let us differentiate this with respect to t using Leibnitz rule. So ut is so here what is that, let me write it here. u(X, t) is equal to this. So we can write ut. So ut = so 0 to t and then it is vt(x, t - tau, tau) dtau + and then here you put value in place of tau you are putting value t, so we can write it x, 0, and t here and so here what is the value f v(x, 0, tau).

Here v(x, 0, t) = 0 for all values of tau. So this will be cancelled out. So we have vt = 0, t.Vt(x, t - tau, tau) d tau. Ss that is what it is here written that we can write ut as v(X, 0, t) + 0 to t vt(X, t - tau, tau) d tau and we can use initial condition then this part is simply vanish and we can write ut = 0 to t, vt(X, t - tau, tau) d tau. Now again you differentiate with respect to t, then if you differentiate with respect to then you have utt = vt(X, 0, t) + 0 to t vtt(X, t - tau, tau) d tau.

And using equation 3 the above equation is reduced to because here we already assumed that this v(X, t - tau) is a solution of vtt - c square del square v = 0 for all t >= 0 So we can replace the value of this vtt as c square del square u. So that is what we have written that is utt = vt(X, 0, t) = + 0 to t c square, del square v, d tau right.

And here if you remember what we have defined u(x, t) = 0 to t v(x, t - tau, tau) d tau. So if you find out del square u so del square u is defined as 0 to t del square v(x, t - tau, tau) d tau and we already know that in place of this we can write down in place of this we can write down c square del square u.

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Finally, using equation (1), the above equation reduces to

$$u_{tt} - c^2\nabla^2 u = v_t(X, 0, t) \qquad (6)$$

Comparing equations (1) and (6), we obtain

$$v_t(X, 0, t) = F(X, t) \qquad (7)$$

Therefore, if v satisfies the equation

$$v_{tt} - c^2\nabla^2 v = 0$$

with the conditions

$$v(X, 0, \tau) = 0, \; v_t(X, 0, \tau) = F(X, \tau) \text{ at } t = \tau$$

then, u defined by equation (2) satisfies the given inhomogeneous equation (1) and the specified conditions and the function $v(x, t)$ is called the pulse function or the force function.

That is what we have written that utt - c square del square u = vt(X, 0, t) and we already know that vt(X, 0, t) if you look at here that our vt(X, 0, tau) is fx(tau). So vt(X, 0, t) is going to be the function f(x, t). So using this we can write that vt(X, 0, t) is your F(X, t). So we satisfy this relation that is vtt - c square del square u = 0 and v(X, 0, tau) is 0. vt(X, 0, tau) is F(X, tau) at t = tau then u defined by this that u(X, t) = 0 to t v(x, t - tau) d tau.

Then this u will satisfy this relation that utt - c square del square u = f(x, t). So we can say that if v satisfy this equation, with initial v(X, 0, tau) is 0 and vt(X, 0, tau) = F(X, tau) at t = tau. Then u will satisfy this equation. Now we have to look at the initial condition satisfied by u. So let us look at your initial condition so here your u(x, 0) will be what? u(x, 0) who can find out that it is at put t = 0 then it is nothing but 0. So u(x, 0) is going to be 0.

Now what about ut(x, 0) so looking at ut(x, 0) look at this coefficient here and if you put t = 0 again then ut(x, 0) is again coming out to be 0. So here u will satisfy that if u(x, t) is defined by this then u satisfy this non homogenous partial differential equation with homogenous initial condition that is u(x, t) = 0 and ux(x, 0) = 0 and this is valid for any dimensions whether it is R, R square, or R cube. So this is the concept given by Duhamel principle and we are going to utilize this concept to find out the solution of non homogenous partial differential equations.

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One Dimensional Wave Equation

Consider the non- homogeneous wave equation

$$u_{tt} - c^2 u_{xx} = F(x,t), -\infty < x < \infty, t > 0, \quad (8)$$

with the homogeneous initial conditions

$$u(x,0) = u_t(x,0) = 0, -\infty < x < \infty. \quad (9)$$

Consider the function $v(x, t-\tau; \tau)$ which satisfies the following equation with respect to x and t for all τ,

$$v_{tt} - c^2 v_{xx} = 0, -\infty < x < \infty, t > 0, \quad (10)$$

and the following condition at $t = \tau$

$$v(x,0,\tau) = 0, v_t(x,0,\tau) = F(x,\tau). \quad (11)$$

So here let us take 1 example of 1 dimensional wav equation. So consider the non-homogeneous wave equation utt - c square uxx = F(x, t) and x is lying between - infinity to infinity t > 0 and initial condition is given that u(x, 0) = 0 and ut(x, 0) = 0 here. So here now again question may come that when we discuss our wave equation we have assumed our initial conditions as non homogenous initial condition, but here whenever we consider this kind of problem.

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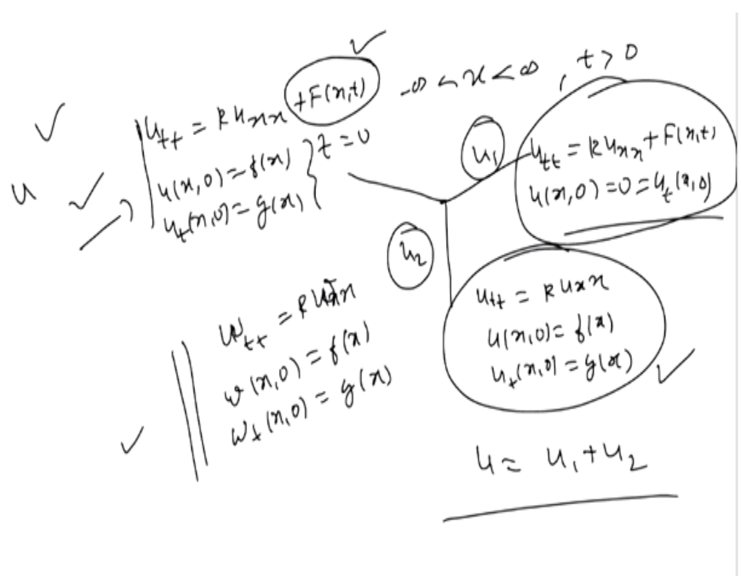

Where utt = k * Uxx + F(x, t) and u (x, 0) = f(x) and ut(x, 0) = g(x). Then we can always write this into 2 part, 1 part is that utt = k * uxx + F(x, t) and initial condition is simply 0 and let us say that u1 satisfy this and u2 satisfy this equation that utt = k uxx and u(x, 0) = f(x) and ut(x, 0) = g(x) then your original solution u I can write this u as u1 + u2. You can verify that if u1 satisfy this equation this system and u2 satisfy this system.

Then u1 + u2 will satisfy this partial differential equation non homogenous partial differential equation with non homogenous initial condition and we can always solve this equation given for u2 so you can always find out u2. So only say thing we want to solve is how to get u1. So how to get this solution of this non homogenous partial differential equation with homogenous initial condition? So our emphasis is to solve non homogeneous partial differential equation with homogenous initial condition.

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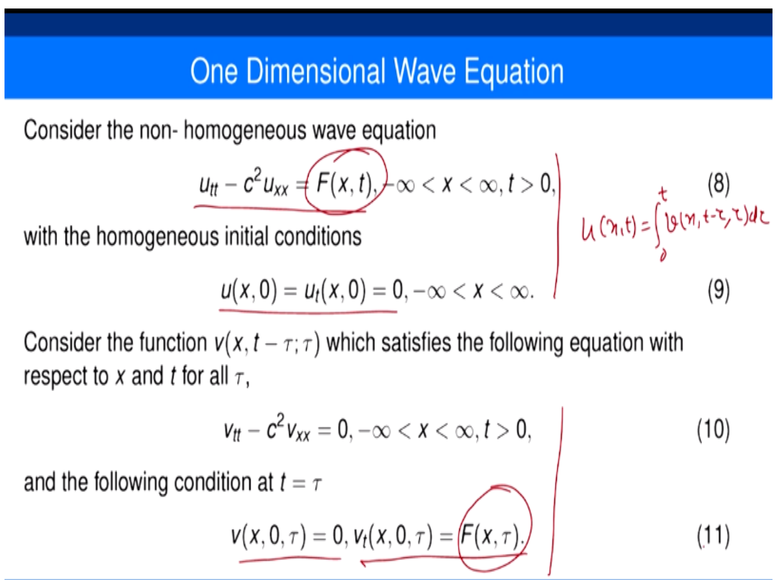

So that is why we are considering here non homogeneous wave equation with homogeneous initial condition. So now let us consider the function v(x, t - tau, tau) which satisfy the following equation that is vtt - c square vxx = 0 at t is > 0 and x is lying between - infinity to infinity and initial conditions are what. V(x. 0, tau) = 0 and vt(x, 0, tau) is F(x, tau).

So here this non homogenous (())(16:11) term is now we are taking here at initial condition of 1 less order. So here it is utt. So it is 1 less order means vt writing as F(x. tau). So if we can find out this solution of this problem, then solution of this problem I can write it u(x, t) = 0 to t v(x, t - tau, tau) d tau. So that is our claim and see how this claim is verified.

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The solution for this problem is

$$v(x,t;\tau) = \frac{1}{2c}\int_{x-ct}^{x+ct} F(s,\tau)ds. \quad (12)$$

Consider

$$u(x,t) = \int_0^t v(x,t-\tau;\tau)d\tau. \quad (13)$$

$\Rightarrow u(x,0) = 0$

We will now show that u is the solution we are looking for

$\Rightarrow u(x,t) = \int_0^t \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} F(s,\tau)ds\,d\tau$

$$u_t = v(x,0;t) + \int_0^t v_t(x,t-\tau;\tau)d\tau,$$

$$= \int_0^t v_t(x,t-\tau;\tau)d\tau (\text{ since } v(x,0;t) = 0),$$

$u_t(x,0) = 0$

So here the solution of this problem is v(x, t, tau) = 1/2c x - ct to x + ct * F(s, tau) ds. So here the solution of this problem equation number 10 along with the initial condition 11 is given by this that v(x, t, tau) = 1/2c x - ct to 2x + ct F(s, tau) ds. Now using this v(x, t, tau) let us write down the solution u(x, t) as 0 to t v(x, t - tau, tau) d tau. Now we already have the expression of v(x, t, tau) so we can always find out the expression for v(x, t - tau).

So let us verify that this will be a solution of our non homogenous equation with initial homogeneous initial conditions so find out ut. ut is v(x, 0 , tau) + 0 to t vt(x, t - tau, tau) d tau. Now since v(x, 0, tau), v(x, 0, t) is given as 0 that is what it is written here. v(x, 0, tau) = 0 at t = tau. So we can say that v(x, 0, t) is given as 0. So we can write 0, t, vt (x, t - tau, tau) d tau.

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$$u_{tt} = v_t(x, 0; t) + \int_0^t v_{tt}(x, t-\tau; \tau) d\tau,$$

$$= F(x, t) + \int_0^t v_{tt}(x, t-\tau; \tau) d\tau,$$

$$u_{xx} = \int_0^t v_{xx}(x, t-\tau; \tau) d\tau.$$

Therefore

$$u_{tt} - c^2 u_{xx} = F(x, t) + \int_0^t (v_{tt} - c^2 v_{xx}) d\tau,$$

$$= F(x, t).$$

So again differentiate 1 more time. Then utt = vt(x, 0, t) + 0 to t vtt(x, t - tau, tau) d tau. Now this value vt(x, 0, t) is again given here. vt(x, 0, tau) is F(x, tau) at t = tau. So vt(x, 0, t) is your F(x, t). So using this we can write F(x, t) + 0 to t vtt(x, t - tau, tau) d tau. Now here we already know that if u is given by this, then you can find out uxx by simple differentiating with respect to x variable.

And you can write uxx = 0 to t vxx(x, t - tau, tau) d tau. So we can write utt - c square u xx = utt is having this value that F(x, t) + 0 to t so F(x, t) + 0 to t vtt(d tau). Now ux - c square uxx means - c square Vxx. Since v satisfy this partial differential equation so we can say that this part is gone and we have utt - c square Uxx = F(X, t) here.

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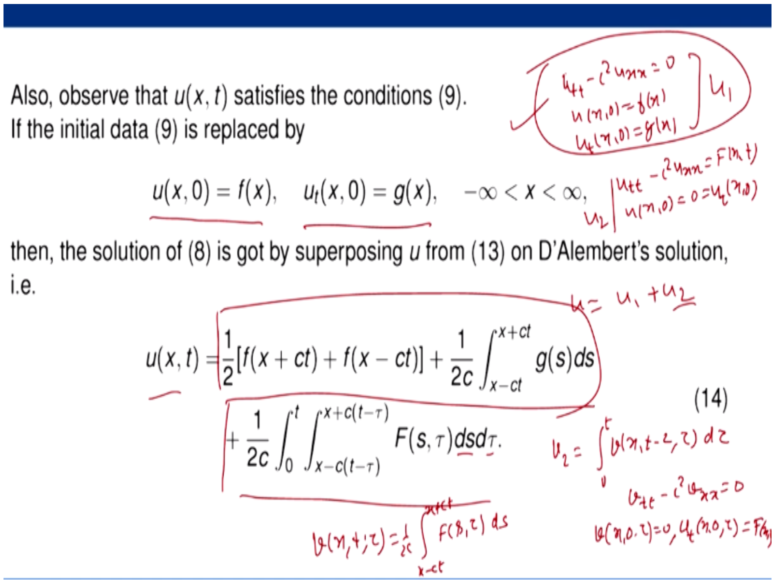

And also we observe that u(x, t) satisfy the conditions (9) that is initial condition look at here condition is that u(x, 0) = ut(x, 0) so that we can verify from this that here if we look at the equation number 13 if you put t = 0 then we will get the values 0. So this implies that u(x, 0) = 0 and if you want to find out the value of ut(x, 0) then we can look at this equation and put t = 0 then also we are coming out to the we are getting the value 0 here.

So it means that U satisfy the initial condition and u satisfy this non homogenous partial differential equation. So if the initial data 9 is replaced by now if we replace the initial condition by non homogeneous initial condition, then we can as we have pointed out then we can write down the solution u as solution given by u1 and u2. So what is u1? u1 is the solution of this utt - c square uxx = 0 and u(x, 0) = f(x), ut(x, 0) = g(x).

This is the u1 and u2 will be solution of this that utt - c square uxx = f(x, t) and u(x, 0) = 0 = ut(x, 0) that will be the part u2 and solution u is given as u1 + u2. So u1 we already know this solution given by D ' Alembert's solution that is 1/2 f(x + ct) + f(x - ct) + 1/2c x - ct to x + ct g(s) ds. Now to find out this u2. U2 is given as 0 to t v(x, t - tau, tau) d tau. So let us write it down here our u2 = 0 to t v(x, t - tau, tau) d tau. Now what is this v(x, t - tau) it is the solution of what.

It is the solution of this vtt - c square vxx = 0 and v(x, 0, tau) = 0 and vt(x, 0, tau) = your F(x, tau). So here solution of this your v(x, t, tau) is given as 1/2c * x - ct to x + ct F(s, tau) ds that is what we have written here as equation number 12. So once we have v(x, t, tau) then you can find out u(x, t, tau) as this implies that u(x, t) is given as 0 to t and now we are writing v(x, t, tau) so in place of t I am writing just t - tau.

So here you just replace t/t - tau. So it is 1/2c x - c(t - tau) and x + c(t - tau) F(s) tau ds and then we have d tau. So this we can write it here as this. Then 1/2c is root x - c(t - tau) x + c(t - tau) F (s, tau) ds and then d tau. So here this is the part u2 which we are looking at so u2 is written as this and u1 is already you obtain as a solution of D 'Alembert solution of 1 dimensional wave equation. So that is how we find out the solution u(x, t) of the problem that we have a non homogeneous problem with non homogenous initial condition.

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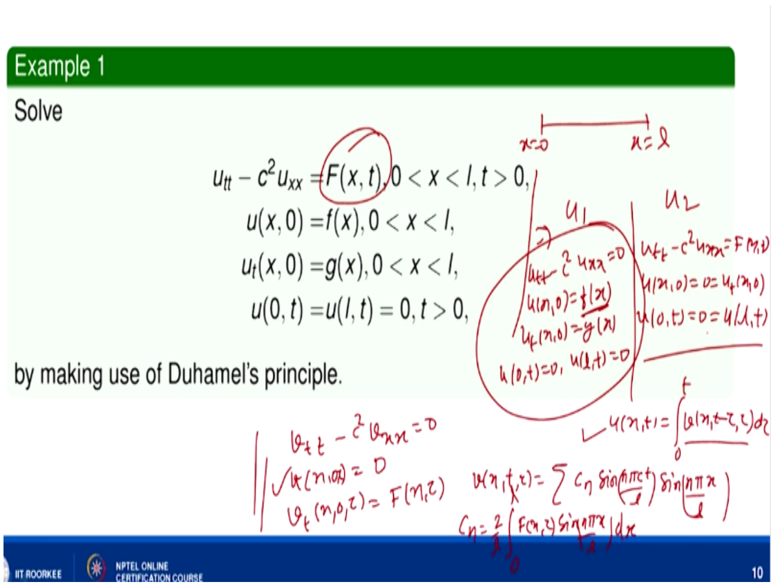

Now we will move to next problem that is solving the wave equation, but this time it is finite domain problem that is we are looking at the (()) (23:27) problem. Where 1 end is fixed at x = 0 and another end is fixed at x = l. So here we have the problem utt - c square uxx = F(x. t) where x is lying between 0 to l t > 0. U(x, 0) = f(x), x is lying between 0 to 1. Ut(x, 0) = g(x). u(0, t) = 0 and u(l, t) = 0. So again we truncate into 2 problem.

u1 and u2 we will satisfy what u1 will satisfy this, utt - c square uxx = 0 and u(x, 0) = f(x) ut(x, 0) = g(x), u(0, t) = 0 and u(l, t) = 0. So this we will contain everything other than the non homogenous (()) (24:23) setting here and u2 is satisfied this equation. utt - c square uxx = f(x, t) and u(x, 0) = 0 = ut(x, 0) and u(0, t) = 0 = u(l, t). Now this we already know. Now we want to find out u2.

For finding u2, we will use the Duhamel principle and we write down this as u(x, t) = 0 to t v(x, t - tau, tau) d tau. What is v(x, t - tau) is a solution of it is a solution of vtt - c square vxx = 0. u(x, 0) = 0 and vt(x, 0) let me write it tau here also because it is (()) (25:24) F(x, tau). So we all we know how to find out the solution using the form of u1. So then I can find out v and then put it here and it will give you u2 and we can write down solution u as summation of u1 and u2.

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Solution 2

From (14), we get

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos(\frac{n\pi ct}{l}) + b_n \sin(\frac{n\pi ct}{l}) + \int_0^t c_n(\tau) \sin(\frac{n\pi c(t-\tau)}{l}) d\tau \right) \sin(\frac{n\pi x}{l}),$$

where

$$a_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx, \quad b_n = \frac{2}{n\pi c}\int_0^l g(x)\sin(\frac{n\pi x}{l})dx,$$

$$c_n(\tau) = \frac{2}{n\pi c}\int_0^t F(x,\tau)\sin(\frac{n\pi x}{l})dx.$$

And if you look at u part is u1 is given by this that n = 1 to infinity an cos(n * pi * ct)/l + bn sin(n * pi* ct)l * sin(n * pi x/l). So here we have this thing where al is what? al is 2/i 0 to l f(x) sin n * pi * x/l * dx and bn is 2/n * pi * c 0 to l g(x) sin(n pi x/l) dx. Now if you look at u2. Now if you look at u2. u2 is what? u2 will satisfy this kind of equation. So let us find out the solution of this. So solution is given as v(x, t, tau) is a solution of summation here.

Let us say cn sin n pi ct/l * sin n pi x/l. Why because here v(x, 0, tau) is 0 so this corresponding to this cos n pi x/l will not be there. If you look at corresponding to cos n pi * ct, an will not be there because f(x) is 0 so an is going to 0. So we do not have any term corresponding to this. So here we get v(x, t, tau) as this. Now once we have v(x, t, tau) where cn is given as what. cn is given as 2/l f(x, tau) sin n * pi x/l and d of x here right.

So cl is given by this and 0 to l here then v(x, t, tau) is given by this then if we replace t by t - tau and integrate between 0 to t you will get u(x, t). So that is what we have written here it is 0 to t cn(tau) sin(n pic(t - tau)/l d tau * sin(n pix/l) and how to find out cn(tau). cn(tau) is 2/npi c look at your bn and if you look at it similarities it is given by 2/n * pi *c 0 to t F(X, tau) sin(n * pi* x/l) dx. So using the u1 you can find out the solution u2 in a similar way the only thing is here v(x, t - tau) so when we solve this v(x, t) we replace just t by t - tau and you can get the corresponding part of u2.

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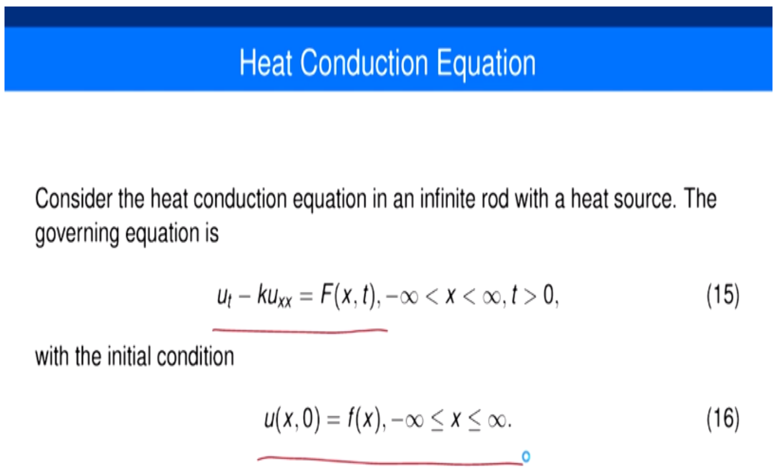

So we can write it like this. Now coming on to heat equation, so it means that if Duhamel's principle is known then we can find out the solution of non homogeneous problem using Duhamel's principle. So now let us do the same thing for heat condition equation. So here consider the heat equation in an infinite rod with a heat source ut = kUxx + F(x, t) x is lying between - infinity to infinity and t is > 0 and initial condition is given as u(x, 0) = f(x) and you want to find out the solution here.

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Let us consider the function $v(x, t; \tau)$ satisfying

$$v_t - kv_{xx} = 0, -\infty < x < \infty, t > \tau > 0, \quad (17)$$

and the following condition at $t = \tau$

$$v(x, \tau; \tau) = F(x, \tau). \quad (18)$$

This v is given by

$$v(x, t; \tau) = \frac{1}{2\sqrt{\pi k(t - \tau)}} \int_{-\infty}^{\infty} F(\xi, \tau) \exp(-\frac{(x - \xi)^2}{4k(t - \tau)}) d\xi. \quad (19)$$

So here we again use the part that u(X, t) can be written as integral of 0 to t v(x, t - tau). So we first find out the solution which V(x, t, tau) satisfy. So here let us say that we have a function v(x, t, tau) satisfying this equation vt - kvxx = 0. X is lying between - infinity to infinity and t is > tau > 0 and the following condition hold that v(x, t, tau) = F(x, tau). So here if we look at here we have 1 derivative so 1 less derivative is 0 derivatives.

So here condition v(x, tau, tau) is given as F(x, tau) and we already know that the solution of this 17 and 18 is given by this. Here you simply replace by so here t is given = tau so you just replace t - tau and we can find out this solution like vt - kvxx = 0 and x is lying between - infinity to infinity t > 0 and vx0 tau is f(x, tau).

We already know how to solve this and once we solve this you just replace by t - tau. You replace t by t - tau and you can see that v(x, t, tau) which is a solution of this is given by 1/2 root pi k(t - tau) - infinity to infinity F(xi, tau) exponential of - (x - xi) whole square/4k(t - tau) d xi. So here you see that here t is replaced by t - tau and we already know the solution is this.

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Consider

$$u(x,t) = \int_0^t v(x,t;\tau)d\tau. \tag{20}$$

Then

$$u_t(x,t) = v(x,t;t) + \int_0^t v_t(x,t;\tau)d\tau,$$
$$= F(x,t) + \int_0^t v_t(x,t;\tau)d\tau.$$

Therefore

$$u_t - ku_{xx} = F(x,t) + \int_0^t (v_t(x,t;\tau) - c^2 v_{xx}(x,t;\tau))d\tau,$$
$$= F(x,t).$$

So now we verify that this u(x, t) which is given by 0 to t v(x, t, tau) d tau is a solution of our problem. So again you differentiate ut(x, t) = v(x, t, t) + 0 to t vt(x, t, tau) d tau. Now v(x, t, t) is already given as F(x, t) = 0 to t vt(x, t, tau) d tau. Now we already know we can find out here uxx and uxx is what. uxx is 0 to t vxx(x, t, tau) d tau.

So using this we can write ut - kUxx = F(x, t) + 0 to t vt(x, t, tau) - k * vxx, x(t, tau) d tau. So this part is going to be 0 because v satisfy the heat equation that is what we have assumed in equation number 17. So from 17 we can say that this integral part is gone and we have ut - kUxx - F(x, t) and you can also verify here that at t = 0, u(x, 0) = 0 here. So here we have this.

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and

$$u(x,0) = 0.$$

Hence u given in (20) is the required solution of (15) and (16). Now suppose we consider the initial condition

$$u(x,0) = f(x),$$

instead of (16). Then the solution of (15) is

$$u(x,t) = \int_0^t \frac{1}{2\sqrt{\pi k(t-\tau)}} \int_{-\infty}^{\infty} F(\xi,\tau)\exp\left(-\frac{(x-\xi)^2}{4k(t-\tau)}\right)d\xi d\tau + \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} f(\xi)\exp\left(-\frac{(x-\xi)^2}{4kt}\right)d\xi. \tag{21}$$

So here we have u(x, 0) = 0 here, but we have assumed the initial condition that u(x, 0) = f(x). So here again we have to do this part. We have to use super position of principle. So we have to write a component corresponding to the non homogeneous initial condition that is 1/2 root pi kt - infinity to infinity f(xi) exponential of (x - xi whole square/4 kt) d xi + component which we have just obtained that is u2 here and what is u2 here?

u2 is 0 to t i/2 root * pi k(t - tau) - infinity to infinity F(xi, tau) exponential of - (x - xi) whole square/4k(t - tau) d xi * d tau. So here when we replace our homogeneous condition by non homogenous condition then we have to write our solution as u1 and u2 and u1 is here is a solution of homogenous solution condition that is ut - kuxx = F(x, t) and u(x, 0) = 0 and u2 is a solution of this ut - kuxx = 0 and u(x, 0) = f(x).

And this solution we have already obtained like this and this solution we just now obtained using Duhamel principle as 0 to t v(x, t - tau) d tau. So here we have written u1 like this. So by writing u1 + u2 you have a solution of the problem that ut - kUxx = F(x, t) and U(x, 0) = your F(x). So here we have a non homogenous equation with non homogenous initial condition that can be written as addition of u1 and u2.

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Example 3

Find the solution of the following heat equation $u_t = \kappa u_{xx} + F(x,t)$ with $u(x,0) = f(x)$, $u(0,t) = h_1(t)$ and $u(l,t) = h_2(t)$.

Proof. Let $v(x,t) = u(x,t) + A(t)x + B(t)$ be such that $v(0,t) = 0 = v(l,t)$. Then,

$$v(0,t) = 0 \Rightarrow B(t) = -h_1(t)$$

$$v(l,t) = 0 \Rightarrow A(t) = \frac{h_1(t) - h_2(t)}{l}$$

Thus,

$$v(x,t) = u(x,t) + \left(\frac{h_1(t) - h_2(t)}{l}\right)x - h_1(t)$$

Now let us look at a 1 more problem that is find the solution of the following heat equation that ut = kUxx + F(x, t) and with initial condition u(x, 0) = f(x) and this time we are taking boundary condition are also non homogeneous boundary condition that is u(0, t) = h1(t) and u(l, t) = h2 t. We have also considered this kind of problem already also, but again I am repeating it. So here let us assume that here first our thing is that we assume that boundary condition should be homogenous.

So first we convert whenever we have this kind of problem with non homogenous boundary condition the first work is to say make our boundary condition or homogenous boundary condition. So here let us assume that v(x, t) = u(x, t) + A(t)x + B(t) where A(t) and B(t) we can obtain in a way such that v(0, t) = 0 and v(l, t) = 0. So this we can find out by writing v(0, t) = 0. This implies that B(t) = - h1(t) and v(l, t) = 0.

So this means that A(t) is coming out to be h1(t) - h2(t)/l. So v(x, t) we can write it u(x, t) + this A(t) - h1(t). So once we have v(x, t) then we can so it means that our initial condition if we define v(x, t) as u(x, t) + A(t)x + B(t). The boundary condition must be homogenous. Now look at the initial condition and the equation itself.

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and hence $v_{xx} = u_{xx}$, $v_t = u_t + \left(\frac{h_1'(t)-h_2'(t)}{l}\right)x - h_1'(t)$.

$$v_t = u_t + \left(\frac{h_1'(t)-h_2'(t)}{l}\right)x - h_1'(t)$$

$$= \kappa u_{xx} + F(x,t) + \left(\frac{h_1'(t)-h_2'(t)}{l}\right)x - h_1'(t)$$

$$v_t = \kappa u_{xx} + \tilde{F}(x,t)$$

where $\tilde{F}(x,t) = F(x,t) + \left(\frac{h_1'(t)-h_2'(t)}{l}\right)x - h_1'(t)$.

So here we find out the vxx and Vxx is coming out to be Uxx that we can verify and vt we can find out vt as ut + h1 dash (t) - h2 dash(t)/l) x - h1 dash(t). So vt we can write it ut + this quantity this thing. Now here we already know that ut = kUxx + F(X, t). So using the expression for ut we can write vt as kUxx + F(x, t) + (h1 dash(t) - h2 dash(t)/l) x - h1 dash(t). Now this whole thing we write it a new function that is F tilde(x, t). So we can write Vt = kUxx + F tilde(X, t) where F tilde(x, t) is F(x, t) + this quantity.

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$$v(x,0) = u(x,0) + \left(\frac{h_1(0) - h_2(0)}{l}\right)x - h_1(0)$$
$$= f(x) + \left(\frac{h_1(0) - h_2(0)}{l}\right)x - h_1(0)$$
$$= \tilde{f}(x) \quad (say)$$

So, the given problem is reduced to

$$v_t = \kappa v_{xx} + \tilde{F}(x,t)$$
$$v(x,0) = \tilde{f}(x),$$
$$v(0,t) = 0, \quad v(l,t) = 0$$

So here our initial condition also we need to check v(x, 0) = u(x, 0) + (h1(0) - h2(0)/l)x - h1(0) and we can write it here u(x, 0) is f(x). So f(x) + this quantity we call this as quantity as f tilde (x). So it means that if we use v(x, t) as u(x, t) at A(t) x + B(t) to make our boundary conditions are homogenous boundary condition then v will satisfy this equation that is vt = KVxx + F tilde(x, t). V(x, 0) = f tilde(x). V(0, t) = 0 and v(l, t) = 0 here.

Now we want to solve this particular problem. Then as usual we can try to solve this problem by writing v as a addition of 2 function.

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Now, let $v = v_1 + v_2$, where

$$v_{1_t} = \kappa v_{1_{xx}}, \quad v_{2_t} = \kappa v_{2_{xx}} + \tilde{F}(x,t)$$
$$v_1(x,0) = \tilde{f}_1(x), \quad v_2(x,0) = 0$$
$$v_1(0,t) = v_1(l,t) = 0, \quad v_2(0,t) = v_2(l,t) = 0.$$

That is v1 and v2 where v1 satisfy this problem. v1t = kV1xx, v1(x,0) = f1 tilde(x). V1(0, t) = v1(l, t) = 0. So this is the usual problem which we have we know how to find out the solution, but this v2 will satisfy the non homogeneous equation with homogeneous initial condition and homogenous boundary condition that is v2t = kv2xx + F tilde(x, t) v2(x, 0) = 0. v2(0, t) = v2(l t) = 0.

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Here

$$v_1(x,t) = \sum_{n=1}^{\infty} \left[a_n \exp(-\frac{n^2\pi^2 kt}{l^2})\right] \sin(\frac{n\pi x}{l}),$$

where $a_n = \frac{2}{l}\int_0^l \tilde{f}_1(x)\sin(\frac{n\pi x}{l})dx$.

For v_2, we solve the following PDE

$$u_t - u_{xx} = \tilde{F}(x,t), 0 < x < L, t > 0,$$
$$u(0,t) = u(L,t) = 0,$$
$$u(x,0) = 0, 0 \leq x \leq L.$$

by making use of Duhamel's principle.

So v1t, v1 we can find out like this that n = 1 to infinity. [an exponential of - n square pi square kt/l square] sin(n pi x/l) and where an you can find out as 2/l 0 to l f1 tilde (x) sin(n * pi *x/l dx. Now to find out v2, we look at this we solve the following PDE that is ut - uxx = F tilde(x, t) and x is lying between 0 to L, t > 0. U(0, t) = u(L, t) = 0. u(x, 0) = 0 and for this we use the Duhamel principle.

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Solution. Using Duhamel's principle, the required solution is given by

$$u(x,t) = \int_0^t v(x, t-\tau, \tau) d\tau$$

where $v(x, t-\tau, \tau)$ is one- parameter family solution of

$$\begin{aligned} v_t - v_{xx} &= 0, 0 < x < L, t > 0, \\ v(0,t,\tau) &= u(L,t,\tau) = 0, t > 0, \\ u(x,0,\tau) &= \tilde{F}(x,\tau). \end{aligned}$$

For that using Duhamel's principle the required solution is given by u(x, t) = 0 to t v(x, t - tau, tau) d tau. Now what is x(x, t - tau, tau) is a solution of here we can say that v(x, t - tau) is a solution of vt - Vxx = 0. x is lying between 0 to L, t > 0. V(0, t, tau) = v(L, t, tau) = 0 and v(x, 0, tau) is F tilde (x, tau).

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The solution of the above homogeneous problem is given by

$$v(x, t, \tau) = \sum_{n=1}^{\infty} a_n e^{-(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

where Fourier coefficients a_n are given by

$$a_n = a_n(\tau) = \frac{2}{L}\int_0^L \tilde{F}(x,\tau) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Hence, the solution of the given IBVP is found to be

$$u(x,t) = \int_0^t \sum_{n=1}^{\infty} a_n(\tau) e^{-(n\pi/L)^2 (t-\tau)} \sin\left(\frac{n\pi x}{L}\right) d\tau.$$

S for this we can find out the solution as this that v(x, t, tau) = n = 1 to infinity an e to the power - n pi/l square t sin(n * pi* x/) and here the coefficient an you can find out as an(tau) as 2/L 0 to

L F tilde(x, tau) sin(n pi x/L) dx. So v(x, t, tau) is known to us then you can find out u(x, t) as 0 to t v(x, t - tau) d tau. Now the only thing is in place of t we write t - tau. So we can write n = 1 to infinity an(tau) e to the power - n * pi/L square (t - tau) sin (n * pi * x/L) d tau.

So in this way we can find out v2 like this and your v1 is already given by this. Then we can find out our v and once v is known you can find out u using the relation this that u(x, t) = v(x, t) - A(t)x - B(t). So you can solve a very general problem that is ut - kUxx + f(x,t) and u(x, 0) = f(x) u(0, t) = h1(t) and u(l, t) = h2(t) and the similar thing you can do for non homogenous wave equation with non homogenous boundary condition and non homogenous initial condition.

So here first you need to truncate your problem into 2 parts. First you make your boundary condition as homogenous boundary condition, then you truncate into 2 parts, 1 part we have already solved, another part we solve with the help of Duhamel principle. So here Duhamel principle plays very important role in the sense.

That how to find out a solution of a non homogenous partial differential equation means wave equation and heat equation in terms of solution of homogenous heat or wave equation. So with this I end our lecture and we will see some more examples in our exercise assignment and there we will get some information about it. So with this I end my lecture. Thank you very much for listening us. Thank you.