Ordinary and Partial Differential Equations and Applications
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Lecture – 59
Solution of Homogeneous Diffusion Equation - II

Hello Friend! Welcome to this lecture. In this lecture, we will continue our study of heat equation and if you recall in previous lecture, we have discussed heat equation in an infinite region and semi infinite region and there in case of semi infinite region we have discussed 2 case in which we have applied Fourier cosine transform and Fourier sine transform to solve the problem. Now in this lecture, we will discuss the finite heat equation given in a finite domain or we can say that here we have a finite rod is given.

And again we assume the same kind of condition that here the finite rod of finite length is situated along the X axis. So 1 end is given at X = 0 and other end is given at X = L and the material of the rod is homogeneous and rod is very thin so that temperature distribution is homogeneous along the rod and we assume that whatever condition we have assumed regarding the infinite rod in case of infinite case we assume here also. And the governing equation along with the boundary condition will take their following forms.

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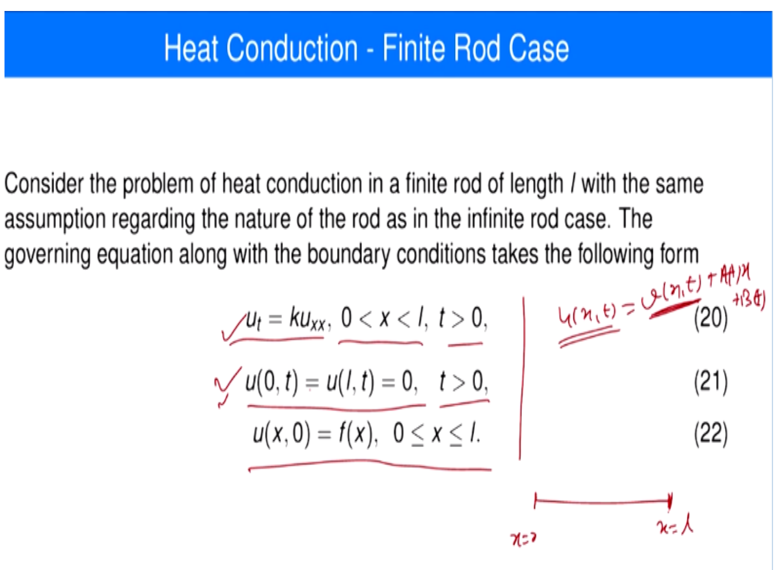

So here heat conduction in a finite rod case is given like this $U_t = kU_{xx}$ where x is lying between 0 to l, t is > 0 and our boundary condition is this that $U(0, t) = U(l, t) = 0$ at $t > 0$. And $U(x, 0)$ is given as $f(x)$ where x lying between 0 to l. So here we assume that both the end are kept at 0 temperature for all time t. So here we have this rod. This end is $x = 0$ and this end $x = l$ and both the end point are kept at 0 temperature and it may happen that both the ends are having some different, different temperature values.

And then we can always apply our linear transform and to convert our boundary condition, non homogeneous boundary condition as homogeneous boundary condition. I hope you recall this thing, I can write it $U(x, t) = V(x, t) + A(t)x + B(t)$. We can find $A(t)$ and $B(t)$ in the sense that this $V(x, t)$ will take homogeneous boundary condition. And of course we have to pay the price, and this homogeneous equation will convert into a non homogeneous equation and we will see how we can deal with these kinds of problems.

But, before that let us first solve the problem of this heat conduction in the finite rod case with the homogeneous boundary condition. So first let us solve this homogeneous problem with homogeneous boundary condition with non homogeneous initial condition. So here we can apply finite Fourier transform. But here we are not considering this integral transform method rather we are applying variable and separable method. Because your equation is now in separable form and your boundary condition is also in separable form.

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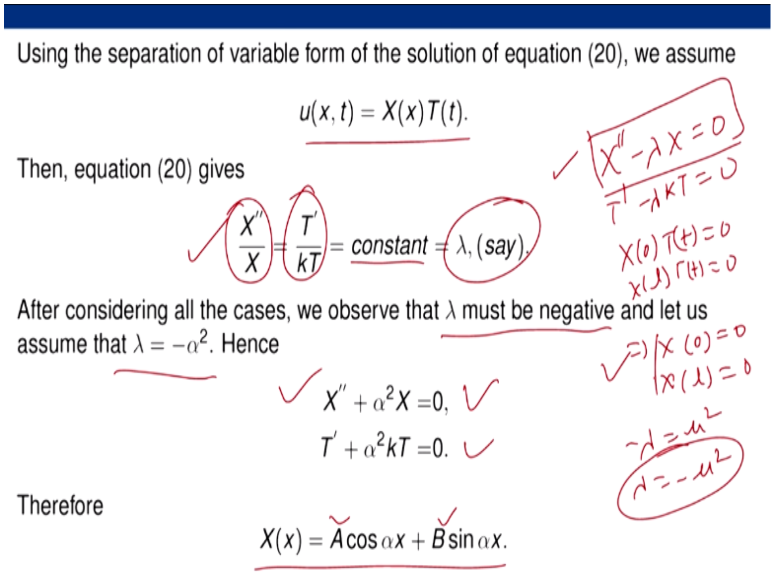

So here let us assume that using the separation of the variable form of the solution we assume this that U(x, t) is given as X(x) and T(t). So using this we can put it as, assume form in this equation number 20 and we can write this as X double das/X = T dash/kT and say this is function of X and this is function of T and this is equal only when, when we have a constant value which is separating both the equations.

So we have X double dash/X = lambda and = T dash/kT = lambda. Now again this constant lambda may take positive, negative or 0 values, but we will say that we have already considered this kind of problem X double dash – lambda X = 0 and T dash - lambda kT = 0. And we have already considered the different cases, and the boundary condition. Now what is the boundary condition given? Here you know that U(0, t)= U(l, t) = 0.

So it means that X(0)T(t) = 0 and X(l) T(t) = 0. So since T(t) = 0 (()) (05:03) so this implies that X(0) = 0 and X(l) = 0 and we have seen that this equation along with these boundary condition will have a non trivial solution only when, when this - lambda is a positive value. So it means that here we take that lambda must be negative, - lambda is taking a positive value say mu square it means that lambda = - mu square. So it means that lambda is a negative value.

So it means that based on this decision, we can say that, we observe that lambda must be negative, and let us assume that lambda = - alpha square. So when we take that lambda = - alpha square, then this can be written as 2 set of ordinary differential equation X double dash + alpha square X = 0 and T dash + alpha square kT = 0. And this we can solve by writing X(x) = A cos alpha x + B sin alpha x where A and B we can obtain using our initial condition that is X(0) = 0 and X(l) = 0.

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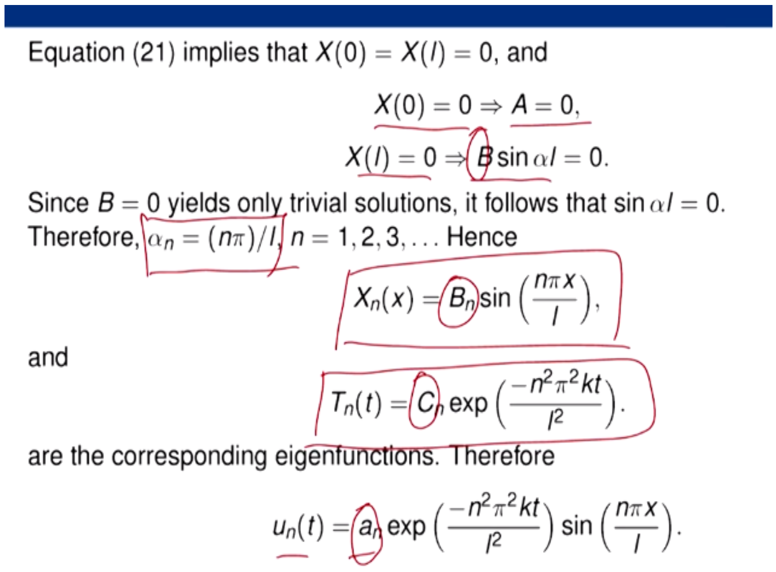

So we applying X(0) = 0 we have A = 0. When we take X(l) = 0 this implies that B sin alpha l = 0 and now if you want a non trivial solution then B has to be non 0 so this sin alpha l has to be 0 and this implies that your alpha must take this value n *pi/l here. So here this is valid for every n = 1,2, 3, and so on. So we can write Xn(x) as Bn sin(n * pi * x/l) for each n. So corresponding to each n we have a solution. So we write solution as Xn(x) = Bn sin(n *pi * x/l).

So taking the value alpha n and we can solve the other equation that is T dash + alpha square kT = 0. So we can write down the solution corresponding to that also. And we can write Tn(t) = Cn exp(- n square * pi square/n square kt). Now when you have Xn(x) and Tn(t) then we can write down our Un(t) as multiplication of this and we can write Bn * Cn as new constant that is An.

So we can write Un(t) as An exp(- n square * pi square kt/l square) sin(n * pi * x/1). Now still we have 1 constant here An for each n so to handle this we have to use initial condition because so far we have obtained this form only using the boundary condition. We have not utilized the initial condition. So to handle this An we will apply initial condition.

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By principle of superposition, we have

$$u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right)\sin\left(\frac{n\pi x}{l}\right). \qquad (23)$$

Now, $u(x,0) = f(x)$ implies that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right), \quad 0 \leq x \leq l, \qquad (24)$$

and

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx. \qquad (25)$$

The function $u(x,t)$ given in (23) represents the solution of equation (20) if $f(x)$ is twice continuously differentiable and f''' is piecewise continuous.

So here we have this thing. But before that since it is true for every n so we can apply our principles of superposition to write solution as this form that U(x, t) = Summation n = 1 to infinity An exp(- n square * pi square kt/l square) * sin(n * pi * x/l). Now use the condition that U(x, 0) = f(x) then we can write this as f(x) = n = l to infinity An Sin(n * pi * x/l) because this term is nothing but 1 at T = 0 this is 1.

So we can write f(x) = n = 1 to infinity An Sin(n * pi * x/l) and x is lying between 0 and l and this kind of thing we have already seen several times. So we can find out our an by using orthogonality of sin(n * pi * x/l) and we can write An as 2/l 0 to l f(x) sin(n* pi * x/l) dx. So it means that your An is given in terms of initial condition that is f(x). So our solution is given as U(x, t) = n = 1 to infinity An exp(- n square * pi square * kt/l square) sin(n * pi * x/l).

Where An is given by this formula that is 2/l 0 to l f(x) sin(n * pi * x/l) dx. So this solution is given in terms of 23. Now so far we have not discussed anything about convergence, uniform convergence anything we have not considered. So we will consider this solution given in 23 as a formal series solution. And we can prove, in fact I am not proving, but we can prove that the function U(x, t) given in 23 represent the solution of the equation if f(x) is twice continuously differentiable and f triple dash is piecewise continuous.

So this I am not giving. So I can simply say that 23 is given as a formal Fourier series solution. Now here we have found the solution in finite rod case using variable and separable. But as I pointed out that you can find out the same by applying the finite Fourier transform also. So here to make sure that whatever method you apply you have a unique solution we will consider the following result which says that.

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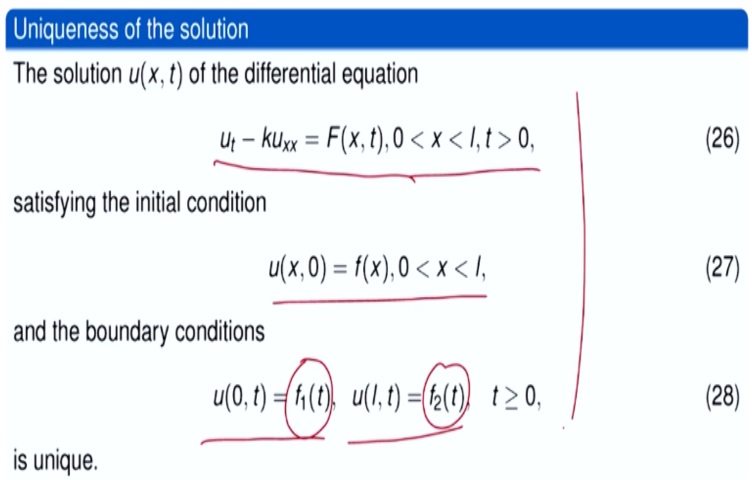

The solution U(x, t) of the differential equation Ut = Ut - kUxx = F(x, t) x is lying between 0 to l, t is > 0 satisfying the initial condition that U(x, 0) = F(x) and the boundary condition that U(0, t) = f1(t) and U(l, t) = f2(t) at t > 0. As I pointed out you need not to consider all the times boundary conditions are 0.

You need not to consider that both the end points are put at constant temperatures. You can take f1(t) and f2(t). And we will see that the solution of this problem if it exist must be unique. So whether I should apply integral transform method or apply the variable separable method we have a unique solution. So lets us prove this. And to prove this again we use generalized energy function.

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Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of the given problem. Then $v = u_1 - u_2$ satisfies

$$v_t = kv_{xx}, \quad 0 < x < l, t > 0, \tag{29}$$

$$v(0,t) = v(l,t) = 0, \quad t \geq 0, \tag{30}$$

$$v(x,0) = 0, \quad 0 \leq x \leq l. \tag{31}$$

Let

$$E(t) = \frac{1}{2k}\int_0^l v^2(x,t)dx.$$

Therefore $E \geq 0$. On differentiating this with respect to t, we get

$$\frac{dE}{dt} = \frac{1}{k}\int_0^l vv_t dx$$

So here we let us say that U1 and U2 be 2 solutions of the given problem and let us consider the difference of this, U1 - U2 so v satisfies the following equation. Vt = kVxx where x lying between 0 to l and t is > 0, v(0, t) = v(l, t) = 0 v(x, 0) = 0. So here we have homogenous equation, homogeneous boundary condition and homogeneous initial condition.

And we want to show that v is erringly = 0. And to show this let us assume as generalized energy function E(t) as 1/2k 0 to l v2(x, t)dt. Now this integrant is positive your this is positive and so we can say that this E(t) is > 0 for all t. So here let us differentiate this with respect to t. We have dE/dt = l/k vvt * dx. Now here vt I can replace by kVxx.

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$$= \int_0^l vv_{xx}dx$$

$$= vv_x\Big|_0^l - \int_0^l v_x^2 dx.$$

Since $v(0,t) = v(l,t) = 0$, we find that

$$\frac{dE}{dt} = -\int_0^l v_x^2 dx \leq 0.$$

Therefore, E is a decreasing function of t and the condition $v(x,0) = 0$ implies that $E(0) = 0$. Hence, $E(t) \leq 0$ for all $t > 0$. But, by definition, $E(t)$ is non- negative. Therefore

$$E(t) \equiv 0, \quad \forall t > 0 \Rightarrow v(x,t) \equiv 0 \quad \text{in} \quad , 0 \leq x \leq l, \ t \geq 0.$$

Therefore $u_1 \equiv u_2$. This shows the uniqueness of the solution.

So by replacing vt by kVxx we can write this as 0 to l VVxx dx. Here we can use intergration by part and we can write VVx 0 to l - 0 to l V x square * dx. Now here if you look at the boundary term, here we already know that v(0, t) and v(l, t) are given as 0. So it means that this boundary term will simply vanish. So it means dE/dt = - 0 to l Vx square dx . Now again we can say that this is something which is non negative.

But there is this minus sign here so dE/dt <= 0. So it means that E is a decreasing function of time t and the condition v(x, 0) = 0 implies that E(0) = 0. If you look at here, here we already know that this v(x, 0) = 0. So E(0) is you say 0. So E(0) is 0 and E is a decreasing function of t it means E(t) <= 0 for all time t > 0. But we already know that E(t) is non negative so E(t) is non negative means here by definition your E(t) which we have defined here E(t) is >= to 0.

And what we have achieved that E(t) <= 0. These 2 things happen together only when, when E(t) = 0. So it means that your E(t) is erringly = 0. Now but what it means that E(t) is erringly = 0 for all t > 0 means that the integrant that is V2(x, t) must be erringly = 0. Now what it means, means your v is erringly = 0 it means that u1 = u2 for all time t. So this shows the uniqueness of the solution.

So it means that solution of the heat equation in finite domain if it exists must be unique. Now as I pointed out that if we take a non homogeneous boundary condition and if we do a suitable change of say in equation to make our boundary condition as homogeneous boundary condition then we will get a homogeneous boundary condition but we may get a non homogeneous equation. I think let me look at here.

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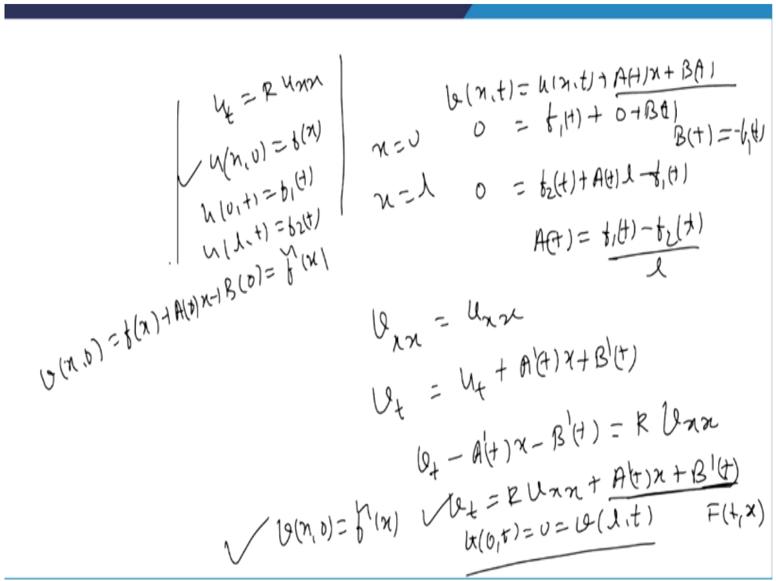

Here we have this Ut = kUxx right and U(x, 0) = f(x) is given no problem. U(0, t) is given as f1(t) and U(l, t) is given as f2(t) right. So here I really do not want, because we have solved this problem by taking f1 and f2 are 0, but we do not want to solve this, so let us assume that V(x, t) = U(x, t) + A(t)x + B(t) here. So now we want that x = 0 so at x = 0 this must be 0. But u(0, t) is f1(t). So f1t) = f1(t) this part is 0 + Bt.

So we can find out Bt as - f1(t) and at x = l we have this is 0 = f2(t) + A(t) * l - f1(t). So we can find out A(t) as f1(t) - f2(t)/l. So you can find out A(t), B(t), but this Vxx will be same as Uxx that you can see that because this is your linear in terms of x, but Vt will be what? Vt will be Ut + A dash(t)x + B dash(t). So we can write our equation as Vt - A dash(t)x - B dash(t) = k times in place of Uxx I can write Vxx so it is Vxx.

So we can write Vt = K * Vxx + A dash(t)x + B dash(t). The initial condition is what V(0, t) = 0 = V(l, t). Now we can find out U(x, 0). What is U(x, 0) v(x, 0) is basically v(x, 0) will be f(x) + A(0) x + B0 which we can write it f tilde x so here we have this condition V(x, 0) = f tilde x. So it means that our problem is now reduced to a non homogenous problem by writing this as f(t, x), but initial condition or boundary conditions are homogenous and initial condition is something. So it means that this problem when bounding condition are no homogenous.

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Example 1

The heat conduction in a thin round insulated rod with heat sources present is described by the PDE

$$u_t - \alpha u_{xx} = F(x, t), \quad 0 < x < l, t > 0 \tag{32}$$

subject to

$$BCs: u(0, t) = u(l, t) = 0$$
$$IC: u(x, 0) = f(x), 0 \leq x \leq l \tag{33}$$

where c are constant and F is a continuous function of x and t. Find $u(x, t)$.

Solution. Consider the associated homogeneous equation

$$u_t - \alpha u_{xx} = F(x, t), \quad 0 < x < l, t > 0 \tag{34}$$

It can be reduced to this following thing that ut- alpha Uxx = F(x. t) x is lying 0 to l and t > 0 and boundary condition is now homogenous and initial condition is given. So it means that if I can solve this then we can solve the homogenous equation with non-homogenous boundary condition. In fact, whenever we have a non homogenous boundary condition we have to handle in a previous case which we have written here and by doing this we can make our boundary condition homogeneous.

But there is some addition term is present in a equation which is given in this form. So now let us solve this problem. So consider the associated homogeneous equation Ut - alpha Uxx = 0 and x is lying between 0 to l and t > 0.

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Since the boundary conditions are homogeneous so by considering $u(x,t) = X(x)T(t)$, we get

$$\frac{T'}{\alpha T} = \frac{X''}{X} = -\lambda^2 \quad (say) \tag{35}$$

which gives $X'' + \lambda x = 0$. The corresponding boundary conditions are

$$X(0) = X(l) = 0.$$

On solving (35), we get eigenvalues and eigenfunctions which are given by

$$X_n(x) = \sin \lambda_n x, \quad \lambda_n^2 = \left(\frac{n\pi}{l}\right)^2, n \geq 1 \tag{36}$$

For the non- homogeneous problem (32), consider a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) \tag{37}$$

Since the boundary conditions are homogenous so we can consider u(x, t) as X(x) T(t) and we can get our T dash/alpha T = X double dash/X = - lambda square that we have just pointed out and looking at the boundary condition x = 0 = x(l) = 0. We say that Xn(x) = sin lambda n X and lambda n is given as n* pi/l whole square n is >= 0 and we can say that we can find out our solution x, Xn(x) like this and using getting idea from the homogeneous part.

Now we can write down that for the non homogenous equation consider the solution of the form u(x, t) = n = 1 to infinity Tn(t) Xn(x), where Xn(x) is given by this sine lambda nX where lambda n is given by n * pi/l here. So we need to find out Tn(t) so that this U(x, t) will satisfy the non homogeneous equation number 32 that is ut - alpha Uxx = F(x, t).

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From the orthogonality of eigenfunctions, we get

$$T_m(t) = \frac{2}{l}\int_0^l u(x,t)X_m(x)dx$$

However,

$$T_m(0) = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{m\pi}{l}x\right)dx \quad (38)$$

which is an initial condition for $T_m(t)$. By introducing equation (37) into equation (32), we get

$$\sum_{n=1}^{\infty} T_n' X_n - \alpha \sum_{n=1}^{\infty} T_n X_n'' = F(x,t) \quad (39)$$

Now on $0 < x < l, t > 0$, we shall expend $F(x,t)$ in a convergent series form

$$F(x,t) = \sum_{n=1}^{\infty} q_n(t) X_n(x) \quad (40)$$

From the orthogonality we can find out from the equation 40 Tm(t) that Tm(t) = 2/l 0 to l u(x, t) Xm(x)dx. So here if you write our U(x, t) as n = 1 to infinity Tn(t) Xn(x) and since we already know the Xn(x) is sine lambda nx then we can find out these expression for Tn(t) by using orthogonality of Xn(x) and we can write down Tm(t) as 2/l 0 to l u(x, t) Xm(x) dx. In particular for t = 0, I can write Tm(0) = 2/l 0 to l.

Now u(x, t) at t = 0 is given by f(x). So Tm(0) is given as 2/l 0 to l f(x) sin(m * pi * x/l) dx. So it means that we need to solve our expression for Tn(T) so that this U(x, t) will be a solution of non homogenous problem with the initial condition that is Tm(0) = 2/l 0 to l f(x) sin(m *pi * x/l) dx. So now by putting your expression this special number 37 into our equation and we can write our equation n = 1 to infinity Tn dash Xn - alpha from n = 1 to infinity Tn * Xn double dash = F(x, t).So this is what our equation is ut = alpha Uxx.

Now u(x, t) is what? u(x, t) is summation Tn(t) Xn(x). So when you differentiate this u(t) will be giving as Tn dash Xn and now here if you find out Uxx. So sine(m * pi * x/l) if you differentiate with cos(m * pi * x/l) * (m* pi/l) and again if you differentiate then we have a - sign present. So we have - alpha n = 1 to infinity Tn * Xn double dash = this. When you simplify this then this is term, which will create a problem. So we will write F(x, t) as a convergent series form of in terms of Xn(x). So we can write F(x, t) as n = 1 to infinity qn(t) Xn(x). Now what is qn(t).

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where

$$q_n(t) = \frac{2}{l}\int_0^l F(x,t)\sin\left(\frac{n\pi}{l}x\right)dx \qquad (41)$$

Now, by using equation (35) into equation (39), we get

$$\sum_{n=1}^{\infty} X_n(T_n' + \lambda_n^2\alpha T_n - q_n) = 0$$

Hence,

$$T_n'(t) + \lambda_n^2\alpha T_n(t) = q_n(t) \qquad (42)$$

and by using the initial condition (38), its solution is given by

$$T_n(t) = T_n(0)\exp(-\lambda_n^2\alpha t) + \int_0^t \exp[\lambda_n^2\alpha(\tau - t)q_n(\tau)d\tau] \qquad (43)$$

We can find out qn(t) as 2/l 0 to l F(x, t) sin(n * pi * x/l) dx. So using the equation number 39, 40 we can write down our equation like this n = 1 to infinity Xn(Tn dash + lambda n square * alpha Tn - qn) = 0. qn is given by this, Xn is sin(n * pi * x/l). Now this is 0 only when this term is going to be 0. So it means that we have Tn dash(t) + lambda n square * alpha * Tn(t) = qn(t). Now this is simple ordinary differential equation, homogeneous ordinary differential equation.

And we can find out the solution using whatever method in fact variation of parameter method you can apply and you can simply solve and this problem can be written as Tn(t) = Tn(0) exponential of (- lambda n square * alpha * t) + 0 to t exponential of [lambda n square alpha(tau - t) qn(tau) d tau. So once we have Tn(t) and xn we have already obtained as sin(n * pi * x/l).

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From equations (37) and (43), the complete solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[T_n(0) \exp(-\lambda_n^2 \alpha t) + \int_0^t \exp[\lambda_n^2 \alpha (\tau - t)] q_n(\tau) d\tau \right] X_n(x)$$

By using (38) and (41), we get

$$u(x,t) = \sum_{n=1}^{\infty} \left[\left\{ \frac{2}{l} \int_0^l f(\xi) X_n(\xi) d\xi \right\} \exp(-\lambda_n^2 \alpha t) + \frac{2}{l} \int_0^t \exp\left\{ \lambda_n^2 \alpha (\tau - t) \right\} \int_0^l F(\xi, \tau) X_n(\xi) d\xi d\tau \right] X_n(x) \quad (44)$$

$q_n(\tau)$

By changing the order of integration and summation, in equation (44), we get

Our solution u(x, t) is given as Tn(t) * Xn(x). So Tn (t) expression we have written here. Now we already know this expression for Tn(0) that is 2/l 0 to l f(xi) Xn(xi) d xi. So using this expression for Tn(0) we can write u(x, t) as n = 1 to infinity this is Tn(0) * exponential of - lambda n square * alpha * t + here 2/l 0 to l exponential of lambda n square * alpha(t - tau). We are just writing this expression for qn(tau) here that is 2/l 0 to l F(xi, tau) Xn(xi) d xi.

So here we write simply the expression for qn(tau). So writing the expression for qn(tau) and Tn(0) we can write down the solution u(x, t) as given in equation number 44 and here I am assuming that this series solution is uniformly convergent so that we can interchange the integral sign and summation sign and say by changing the order of integration and summation we can get the following form.

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$$u(x,t) = \int_0^l \left[\sum_{n=1}^{\infty} \frac{\exp(-\lambda_n^2 \alpha t) X_n(x) X_n(\xi)}{l/2} \right] f(\xi) d\xi$$
$$+ \int_0^l \int_0^l \left[\sum_{n=1}^{\infty} \frac{\exp(-\lambda_n^2 \alpha (t-\tau)) X_n(x) X_n(\xi)}{l/2} \right] F(\xi,\tau) X_n(\xi) d\xi d\tau$$

which can also be written in the form

$$u(x,t) = \int_0^l G(x,\xi;t) f(\xi) d\xi + \int_0^l \int_0^l G(x,\xi;t-\tau) F(\xi,\tau) X_n(\xi) d\xi d\tau \quad (45)$$

where

$$G(x,\xi;t) = \sum_{n=1}^{\infty} \frac{\exp(-\lambda_n^2 \alpha t) X_n(x) X_n(\xi)}{l/2}$$

is called Green's function.

That is u(x, t) 0 to l [n = 1 to xi exponential of (- lambda n square * alpha * t) Xn(x) * Xn(xi)/l/2] f(xi) d xi + 0 to l * 0 to l n = 1 to infinity this expression F(xi, tau) Xn(xi) d xi d tau and which can be simplified by introducing a new function G(x, xi, t) and we can write U(x, t) as 0 to l G(x, xi, t) f(xi) d xi + 0 to l 0 to l G(x, xi, t - tau) F(xi, tau) Xn(xi) d xi d tau.

Where this G(x, xi, tau) is given by this expression that is exponential of (- lambda n square * alpha t) Xn(x) Xn(xi)/l/2 summation n = 1 to infinity and we call this function as Green function. So it means that the solution of non homogenous finite domain problem can be written as this u(x, t) = this and with the help of Green function which is defined like this is it okay?

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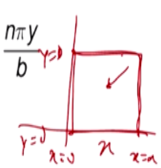
Heat Equation in 2-dimension

The boundaries of the rectangle $0 \le x \le a$, $0 \le y \le b$ are maintained at zero temperature. If at $t = 0$, the temperature T has the prescribed value $f(x,y)$, show that for $t > 0$, the temperature at a point within the rectangle is given by

$$T(x,y,t) = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n) \exp(-\alpha \lambda_{mn}^2 t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$



where

$$f(m,n) = \int_0^a \int_0^b f(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dxdy$$

and

$$\lambda_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$$

So now we are moving to 2 dimensions. So far we have discussed infinite 1 dimension, infinite domain 1-dimension finite domain, semi-infinite domain, and then we discussed the finite domain case and homogeneous and non homogeneous. Now let us consider 1 problem in 2 dimension. Now let us consider the heat equation in 2 dimension. So here we have for simplicity.

We have taken a very simple domain that is a rectangular domain so the boundaries of the rectangle x lying between 0 to a so it is x - 0, it is x = a and here it is y = 0 to y = b. So this is a rectangular plate we can consider that we have considered rectangular plate so the boundaries are kept at 0 temperature. So it means that the boundaries of the rectangle x lying between 0 to a, y lying between 0 to b are maintained at 0 temperature.

Now if at t = 0 the temperature T has the prescribed value some f(x, y) then we have to show that if you take any point here somewhere here then the temperature at this point at any given time T is given by 4/ab summation m = 1 to infinity n = 1 to infinity f(m, n) exponential of (- alpha lambda square mn t) sin m *pi * x/a sin n * pi * y/b.

Where this f(m, n) is given by this expression that 0 to a, 0 to b f(x, y) sin m * pi * x/a sin n * pi * y/b dx dy and this lambda is given by this that lambda square mn is given by pi square (m square/a square + n square/b square). So let us see that how we can obtain the solution at any given point at time t > 0 here.

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The problem is to solve the diffusion equation described by

PDE: $\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \quad 0 < x < a, \quad 0 < y < b, t > 0$

Boundary conditions: $T(0, y, t) = T(a, y, t) = 0, 0 < y < b, t > 0$

$T(x, 0, t) = T(x, b, t) = 0, 0 < x < a, t > 0$

Initial conditions: $T(x, y, 0) = f(x, y), 0 < x < a, 0 < y < b$

Let the separable solution be

$$T = X(x)Y(y)\beta(t)$$

Substituting into PDE, we get

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{\alpha}\frac{\beta'}{\beta} = -\lambda^2$$

So the problem is to solve the following diffusion equation dou T/dou t = alpha(dou 2 T/dou x square + dou 2 T/dou y square) x is lying between 0 to a, y is lying between 0 to b and t > 0 with the boundary condition that at x = 0 T(0, y, t) = 0 and x = alpha, T(alpha, y, t) is given as 0. Sorry this is a. So at x = 0 and x= a it is given as 0 and y is lying between 0 to b. Similarly at y = 0 and y = b your T(x, 0, t) and T(x, b, t) = 0 where x is lying between 0 to a.

Now initial condition is that at T = 0, T(x, y, 0) is f(x, y) and corresponding domain is given. So the solution is given by variable separable form and T = X(x) * Y(y) * and beta(t). Now when you take this and look at this equation that is Tt = alpha(Txx + Tyy). When you put it this variable separable form and divide by x, y, beta we have the following equation X double dash/X + Y double dash/Y = 1/alpha beta dash/beta and call this is as - lambda square. Now we have 2 set of equation, 1 set of equation is this and another set of equation is given by this.

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Then $\beta' + \alpha\lambda^2\beta = 0.$

$$\frac{X''}{X} = -\left(\lambda^2 + \frac{Y''}{Y}\right) = -p^2 \quad (say)$$

Hence,

$$X'' + p^2X = 0$$

$$\frac{Y''}{Y} = -\lambda^2 + p^2 = -q^2 \quad (say)$$

Therefore,

$$Y'' + q^2Y = 0$$

Thus, the general solution of the given PDE is

$$T(x, y, t) = (A\cos px + B\sin px)(c\cos qy + D\sin qy)e^{-\alpha\lambda^2 t}$$

where

$$\lambda^2 = p^2 + q^2$$

So we have beta dash + alpha lambda square beta = 0 and another we can write X double dash/X = - (lambda square + Y double dash/Y) = - p square. So we have equation X double dash + p square X = 0 and Y double dash/Y square is written as - lambda square + p square let us for simplicity we write it as - q square. So we can write Y double dash + q square Y = 0. So your PDE, this PDE.

When you take our solution in variable separable form is now reduced to 3 set of ordinary differential equation 1 is beta dash + alpha lambda square beta = 0 another one is X double dash + p square X = 0 and Y double dash + q square Y = 0 and how this lambda is related lambda we can write it as p square + q square. Here we have we can find out our lambda like this and solution of this equation is given as A cos px + B sin px.

Solution corresponding to this we can write c cos qy + D sin qy and solution of this equation is given as e to the - alpha lambda square t right. Now we need to find out constant A, B, c, D we need to find out here. So for that we have to look at the boundary condition because so far you have not applied any boundary condition.

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Using the boundary condition $T(0, y, t) = 0$, we get $A = 0$. Then the solution is of the form

$$T(x, y, t) = B \sin px (c \cos qy + D \sin qy) e^{-\alpha\lambda^2 t}$$

Applying the boundary condition $T(x, 0, t) = 0$, we get $c = 0$. Thus, the solution is given by

$$T(x, y, t) = BD \sin px \sin qy \; e^{-\alpha\lambda^2 t}$$

By using $T(a, y, t) = 0$, we get $\sin pa = 0$ implies that $pa = n\pi$ or $p = \frac{n\pi}{a}, n = 1, 2, \ldots$

By using principle of superposition, the solution can be written in the form

$$T(x, y, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} x\right) \sin qy e^{-\alpha\lambda^2 t}$$

By using $T(x, b, t) = 0$, we get $q = \frac{m\pi}{b}, m = 1, 2, \ldots$

So here let us apply boundary condition that is corresponding t o x = 0. So T(0, y, t) is given as 0. So we get A = 0 here. In fact we can do it together or we can do it say simplify it and we can simply say that your X(0) = 0 = X(a). Right so this implies that your A is 0 and this implies that your p is in terms of some n * pi/a. So we can simply write let us do one by one. So here when you take T(0, y, t) = 0 we get A = 0.

Then our solution is reduced to T(x, y, t) = B sin px(c cos qy + D sin qy) e to the power - alpha lambda square t. Similarly now look at the condition correspond to y = 0 that is T(x, 0, t) = 0 and we get c = 0. So this part is also gone. So now solution is reduced to T(x, y, t) = B * D sin px sin qy * e to the power - alpha lambda square t. Now let us find out the condition corresponding to x = a. Now when we put x = a condition what you will get?

You will get 0 = BD sin pa * sin qy * e to the power - alpha lambda square t. Now this if we take this as erringly 0 for all y we are having (()) (30:58) solution. Similarly, if we take this as say this cannot be non 0 so this part cannot be non 0. So only condition that sine of pa has to be 0 so if sine pa has to be 0 means your pa must be = some multiple of pi. So pa = n * pi where n is lying between 1 to n and so on.

So I can write p as n * pi/a where n is running between 1, 2, 3 and so on. So using principle of superposition we get our solution as T(x, y, t) = n from 1 to infinity An * sin(n * pi * x/a) sin qy

* e to the power - alpha lambda square t. Now again we have 1 more condition at y = b. So T(x, b, t) = 0 what we will get we have a 0 = summation n = 1 to infinity An sin n * pi * x/a.

And here we have sin q * b * e to the power - alpha lambda square t. Now this is non0, this is non0. So this is possible when sin qb = 0. So it means that sin qb = 0 means your qb is some multiple of pi. So we can write q as m * pi/b where m is running 1, 2, and so on. So using the expression for q.

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Hence, solution is

$$T(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) e^{-\alpha\lambda^2 t}$$

where

$$\lambda^2 = p^2 + q^2 = \pi^2\left(\frac{m^2}{b^2} + \frac{n^2}{a^2}\right)$$

By using initial conditions, we get

$$T(x,y,t) = f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right)$$

where

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} dxdy$$

We can write down our solution as a double series as T(x, y, t) = n = 1 to infinity m = 1 to infinity Amn sin(n * pi *x/a) sin(m * pi * y/b) * e to the power alpha lambda square t. Now we have p, we have q, then we already have a relation between lambda square as p square + q square. So using the value of p and q we can write that pi square (m square/b square + n square/a square). So, we have still 1 Amn is left.

So to identify to find out the value of Amn we use the initial condition that at t = 0 T(x, y, t) = f(x, y, 0). So at T = 0 this is f(x, y) and it is given as n = 1 to infinity, m = 1 to infinity Amn sin(n * pi x/a) sin(m * pi/b * y). So using orthogonality of sin m * pi y/b and sin m * pi * x/a you can find out Amn as follows 2/a * 2/b 0 to a, 0 to b f(x, y) sin m * pi * x/a sin n * pi * y/b dx dy.

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Hence, general solution is

$$T(x, y, t) = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m, n) \exp(-\alpha \lambda_{mn}^2 t) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

where

$$f(m, n) = \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dxdy$$

and

$$\lambda_{mn}^2 = \pi^2 \left(\frac{m^2}{b^2} + \frac{n^2}{a^2} \right).$$

So our solution is given as T(x, y, t) = we are writing the expression for this Amn like this. So we can write T = 4/ab m = 1 to infinity, n = 1 to infinity f(m, n) exponential of - (alpha lambda square mn t) sin n * pi * x/a sin m * i * y/b where f(m, n) we write 0 to a, 0 to b f(x, y) sin m * pi * x/a sin n * pi * y/b dx dy where lambda square is given by this. Lambda square mn = pi square(m square/b square + n square/a square).

So here we have shown that with the help of variable separable method we can solve the heat equation in a rectangular domain and our solution is given by the following representation. So similarly we can discuss several kind of domain. Here we have taken only say rectangular domain we can take this circular domain or in 3 dimension we can do cylinder, we can take the sphere so all this thing can be dealt in a similar way.

But due to time consideration constraint we are not considering all these thing, but we say that in a similar way we can handle heat equation in these kind of domain as well. So with these remarks we will end our discussion here and next lecture we will continue our discussion. So with this I end. Thank you very much for listening us. Thank you.