Ordinary and Partial Differential Equations and Applications
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Lecture - 56
2 Dimensional Wave Equation and its Solution

Hello friends, welcome to this lecture. In this lecture we will continue our study of wave equation, in particularly wave equation. So, if you recall in previous lecture we have discussed the wave equations in 2 dimension particularly in rectangular domain and in cylindrical domain and in today's lecture we will discuss some more type of such equation in particularly we are looking as a nonhomogeneous wave equation.

So, nonhomogeneous wave equation means your vibration is happening under the say some kind of external driving force and because of external driving force your natural waves and natural frequencies are say perturbed and we try to see that how we can find out the solution under the presence of external driving force. So, let us consider one dimensional case and we consider the force vibration solution of nonhomogeneous equation.

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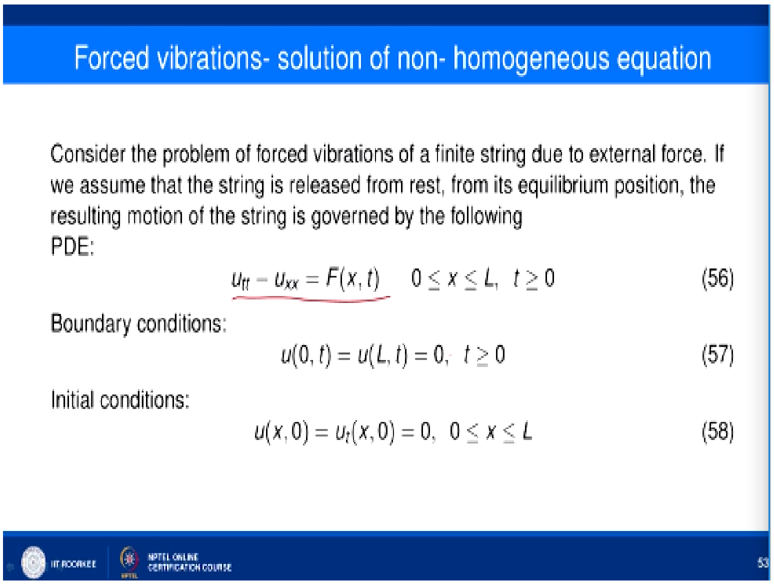

So, here we have to consider the problem of force vibration of a finite string due to external force. If we assume that the string is released from rest, from its equilibrium position the resulting motion of the string is governed by the following equation Utt – Uxx = F(x, t). So in the beginning it is released from rest and now from t > = 0 onward we are putting a function F(x, t) as a say pressure on this.

Say as a function which govern this entire partial differential equation along with the boundary condition u(0, t) = u(L, t) = 0 here and initial condition as u(x, 0) = ut(x, 0) = 0. So, here we are taking both boundary condition as well as initial condition as 0.

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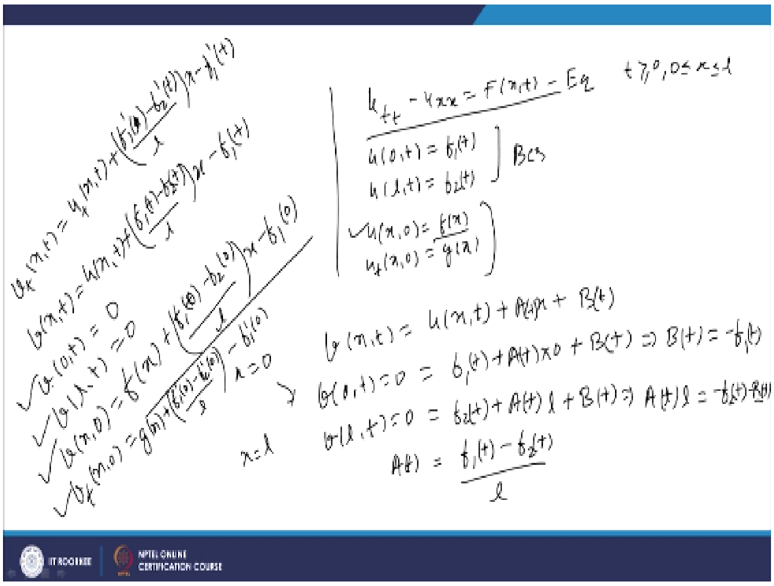

If you here we can take more general problem. Here we can consider this Utt, Utt – say Uxx = F(x, t) and your initial U(0, t) = say f1(t), U(l, t) = f2(t), these are boundary conditions. So, this is equation and equation is defined for t > or = 0 and x is lying between 0 to l here. So, here it is boundary conditions and u(x, 0) = say f(x) here and ut(x, 0) = g (x) here.

So, here our problem may be like this that your boundary conditions also nonhomogeneous and initial condition is also nonhomogeneous. So, here we can consider the falling way by which we can say make our problem in a homogeneous boundary condition rather than nonhomogeneous boundary condition. So, for that you consider the falling transformation, you consider another function say V(x, t) = U(x, t) + some function Ax + B here.

So, here I am assuming that A is a function of t and B as a function of t. So, we have taken a linear function and we now with the help of this new transform function we try to say change of a problem in a way say that our boundary condition is become homogeneous. Because so far the problem which we have consider is only for homogeneous boundary conditions. We have not considered the nonhomogeneous boundary conditions.

So, let us say that here let us utilize that we want that V(0, t) must be 0. But if you look at U(x, t) is what? U of 0, so at say x = 0 we want to see. So, x = 0 we want that V(0, t) has to be 0. But U(0, t) is f1(t). So, f1(t) + A(t) * 0 + B(t). So, this will give you that your B(t) must be = – f1(t), right. Now if we look at x = l then your Vl(t) if you want 0 but it is given as f2(t) + A(t) * l + B(t) here.

So, in this way you can find out your A(t) * l = - f2(t) – B(t). But B(t) we how already obtained. So, we can write our A(t) = - B(t) is what? – B(t) is f1(t), so we have f1(t) – f2(t)/l. So, you have A(t) as f1(t) – f2(t)/l here. Now, it means that if we assume V(x, t) as U(x, t) + f1(t) – f2(t)/l times x – f1(t) then this V(0, t) must be 0 and that you can check here. So, here if you took f1(t) and you can check that it is coming out to be 0 and V(l, t) is coming out to be 0 as well.

Because if we put V of here then we will get that it is coming out to be 0. But so in this way we can make our boundary conditions as a homogeneous boundary condition. But when you take V as a function of U then of course your equation will also change because right now U satisfy this equation as well as these boundary conditions. So, what we have seen that boundary condition can be made homogeneous.

But what about this initial condition and the equation itself? So, look at here, so here you need to find out what is U(x, 0)? So, let us find out what of V(x, 0). So, V(x, 0) is basically what U(x, 0), U(x, 0) is what? f(x), so we have f(x) + now it means f1(0) – f2(0)/l * x – f1(0). So, this in place of this V(x, 0), V(x, 0) will take this falling value. So, U(x, 0) if U(x, 0) take this f(x) value then V(x, 0) will take f(x) + f1(0) – f2(0)/l * x – f1(0).

Similarly you can calculate, see Vt(x, 0). So Vt(x, 0) will be what? Here you can calculate the Vt(x, t) = Ut(x, t) + f1 dash(t) – f2 dash(t)/l times x – f1 dash(t). Now, put 0 here, so you can find out let me do it here in, okay let me do it here. So, Ut(x, t) is your g(x) here + f1 dash 0. So, f1 dash (0) – f2 dash (0)/l – f1 dash 0 here. So, this is V(x, 0), this is Vt(x, 0), V0(t, 0), Vl(t, 0). Now, look at the equation itself. So, equation is what?

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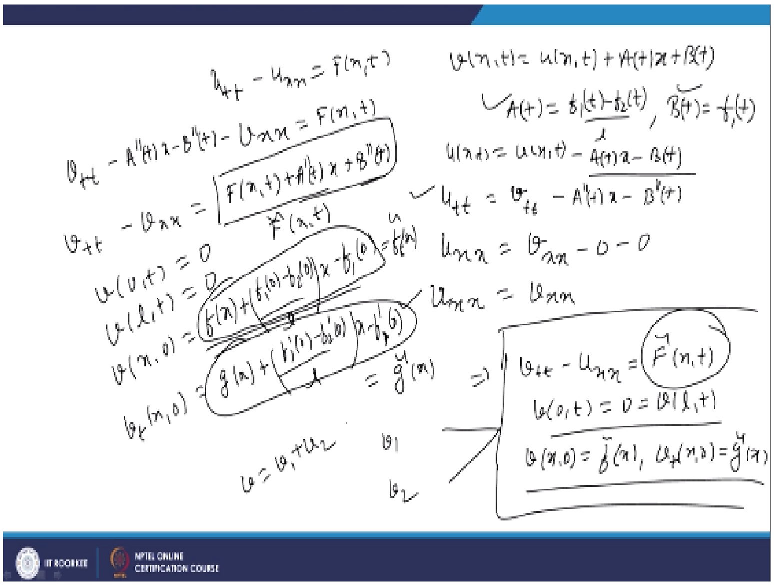

We have equation is say Utt – Uxx = f(x, t). So, here I have taken say, I am not taking c square but you can take c square. So now, here we have V(x, t) = U (x, t) + A(t, x) + B(t). Where A(t) is given as f1(t) – f2(t)/l and B(t) is now –f1(t), right. So, now let us find out what is the value of Utt. So, here if you differentiate let me write it U(x, t) here. So, U(x, t) is basically what? V(x, t) – A(t) x – B(t).

So, let us find out Utt. So, Utt will be Vtt – A double dash(t) x – b double dash(t). So, here I am assuming that your initial condition, say boundary condition B f1 and f2 are twice differentiable. Otherwise we cannot find out this. So, here your Utt is given as this. Now, look at Uxx, so Uxx will be Vxx – here if you differentiate A(t) x then it will, since it is a linear function A(t) x and B(t).

So, here if you take twice differentiable with this then you will get 0. So, here Uxx = Vxx. So, using this and this we can write it Utt as Vtt – a double dash(t) x – B double dash(t) – now Uxx is written as Vxx, so Vxx = F(x, t). So, we can write Vtt – Vxx = f(x, t) + A double dash(t) x + B double dash(t) and we already have values of A(t) and B(t) here. So now your equation is now reduced to Vtt – Vxx = F(x, t) + A double dash(t) x + B double dash(t).

Now, V(0, t) = 0, V(l, t) = 0 and V(x, 0) is this condition. So, previous condition which we have denoted as f(x) + let me write it here, f(x) + f1(0) – f2(0)/l * this x – f1(0) here. I think I have written correctly, yes and Vt(x, 0) is basically what? It is g(x) + f1 dash 0 – f2 dash 0/l times x – f2 dash, f1 dash 0 here. So, let us denote this entire thing as new say f ~ (x, t) this you denote as f ~ say x and this you denote as t ~ x, so this entire thing.

So, this entire thing let us denote as z ~ x and this entire thing you denote as f ~ x. Then this problem is now reduced to Vtt – Vxx = f ~(x, t), V(0, t) = 0 = V(l, t) and V(x, 0) = f ~ (x) and Vt(x, 0) g ~ x. Now problem is that okay we have solved, so far we have solved only this thing that our boundary conditions are homogeneous, initial condition was there. But we have not solved the nonhomogeneous problem.

So, how to solve this nonhomogeneous problem? So it means that we started with what? We started with this problem that we have Utt – Uxx = F(x, t). So, equation is also nonhomogeneous U(0, t) = f1(t), U(l, t) = f2(t). So, your boundary conditions are also nonhomogeneous and initial condition is already given that U(x, 0) = F(x), Ut(x, 0) = g(x). So first thing is what?

First thing is to convert our problem with homogeneous boundary condition. So, to convert our problem into homogeneous boundary condition we take this kind of linear transformation. So, V(x, t) is written as U(x, t) + A(t, x) + B(t) and we tried to find out A(t) and B(t) says that our boundary conditions are become homogeneous condition. But when we do this then your equation is also change and your initial condition will also change a bit.

So, now our problem is now reduced to this that your equation is your Vtt – Vxx = f ~ xt, V(0, t) = 0, V(l, t) = 0, V(x, 0) is f ~ x, Vt(x, 0) = g ~ x. Now, we need to solve this nonhomogeneous problem. So what to do here because so far we have not done, let us solve this problem into by splitting your V as to V1 and V2. Say as we are doing for your nonhomogeneous ordinary differential equation.

In nonhomogeneous ordinary differential equation, we write down the solution as solution of homogeneous problem + solution of a nonhomogeneous problem. So, here also we let us look at V1 and V2 are 2 such functions say that I can write V as V1 + V2. Now what V1 and V2 will satisfy let us say that here your V1 will satisfy the following thing.

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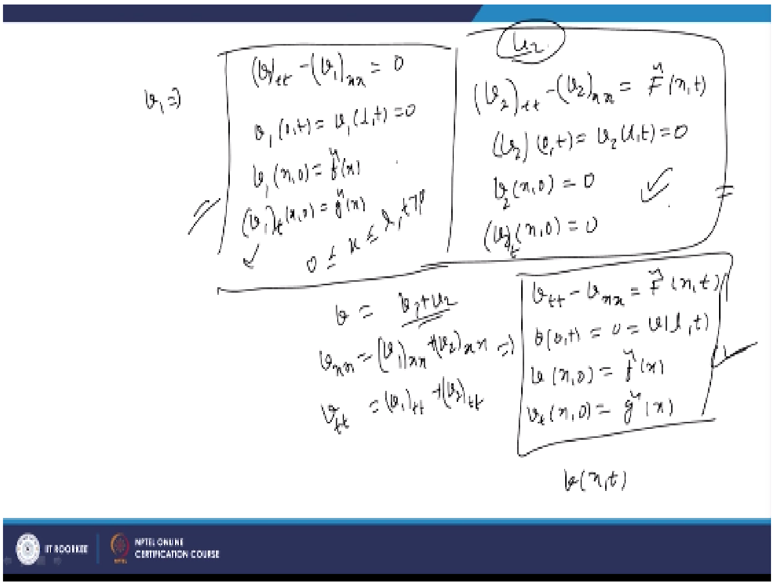

So, here V1 will satisfy what? V1 will satisfy this equation that (V1)tt let me use this – (V1)xx = 0. V1(0, t) = V1(l, t) = 0 and V1(x, 0) = f ~ x and (V1) t(x, 0) = g ~ x. So, V1 will satisfy this and V2 will satisfy what? (V2)tt – (V2)xx = f ~ (x, t), V2(0, t) = V2(l, t) = 0 and V2(x, 0) = 0 and (V2) t(x, 0) = 0. So, now we write that V1 satisfy this equation. So, here what is V1?

V1 is the homogeneous, V1 satisfy the homogeneous wave equation with homogeneous boundary condition and nonhomogeneous initial condition and we already know how to find out this solution, right? This we have done and here your x is lying between 0 to l and yes, t > 0. So, we already handle this kind of situation. And if we can solve this, then our claim is that this V1 + V2 will solve, will satisfy this equation.

In fact you can try this that if we take V = V1 + V2, then V will satisfy the following equation, Vtt – Vxx = f ~ (x, t) and V (0, t) = 0 = V(l, t) and V(x, 0) = f ~ x and Vt (x, 0) = G ~ x. So, here you simply do what? You have V1, you already know what V1 satisfy and U2 satisfy then you find out that here your Vxx = (V1)xx + (V2)xx and Vtt = (V1)tt + (V2)tt and then use here the expression V1 and V2 and you can see that that V will satisfy this equation.

So, it means that our problem, this problem is now reduced to solving 2 simpler problem. So, one problem is this which we have already solved. Another problem is this problem which we are going to solve. So, it means that if we solve this problem, this problem is already solved, so we can solve this problem, right and once we know the solution of this problem it means once we know V(x, t) A(t) is already know to us V(t) is already know to us.

And with the help of that now you can find out the solution of Ut, which we have started with this that Ut is this. So, it means that then we can solve a bigger problem. So, what we have done here, we are trying to solve a bigger problem by decomposing into simpler problems. So, first thing is whenever we have any nonhomogeneous equation is given we have to look at the boundary condition.

A boundary condition is nonhomogeneous then we will adopt this procedure if boundary condition are homogeneous then we start directly from this, okay. So, how to start directly with this? From here we find out V1 and V2 satisfying the following 2 PD1 and PD2. Now this is already solved in previous lecture and this we are going to solve in this lecture and we will see that if we solve this then V1 + V2 will solve a nonhomogeneous problem with homogeneous boundary condition with initial conditions.

So let us, this is the motivation why we are considering this kind of problem, nonhomogeneous problem with homogeneous boundary condition as well as homogeneous initial condition as well. So, now we will focus on how to solve, how to obtain this V2 here. So look at this problem now that we are considering the problem Utt – Uxx = f(x, t). So, here we have a nonhomogeneous equation.

Nonhomogeneous partial differential equation is given. Your boundary condition is 0 as well as the initial condition is also coming out to be 0. Now, once we solve this then we know we have solved for V2 and we have already V1 so we can solve the bigger problem. So, now let us focus now how to solve this particular problem.

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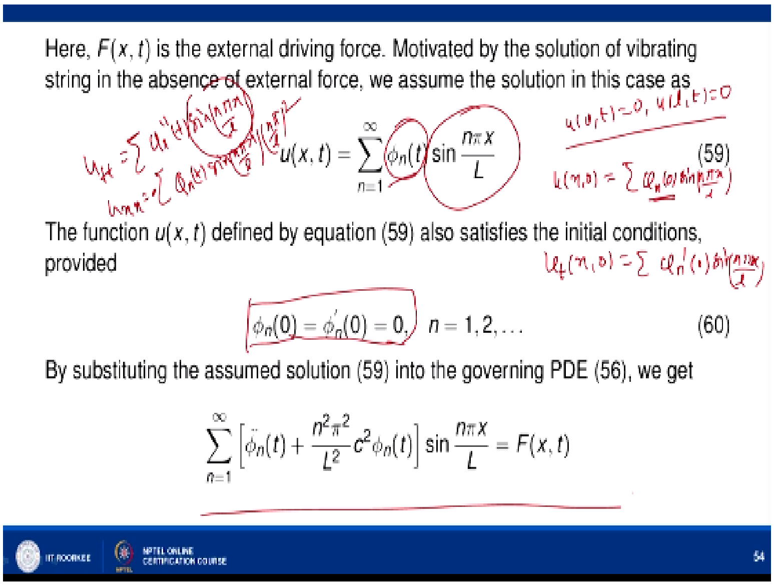

So here we already know that here f(x, t) is external driving force and motivating by the solution of vibrating string in the absence of external force we assume the solution in this case as this. If you recall in homogeneous problem we have written U(x, t) = this term sin n pi x/L will come because we have initial condition that U(0, t) = 0 and U(l, t) = 0. If you recall, we have solve our problem like this.

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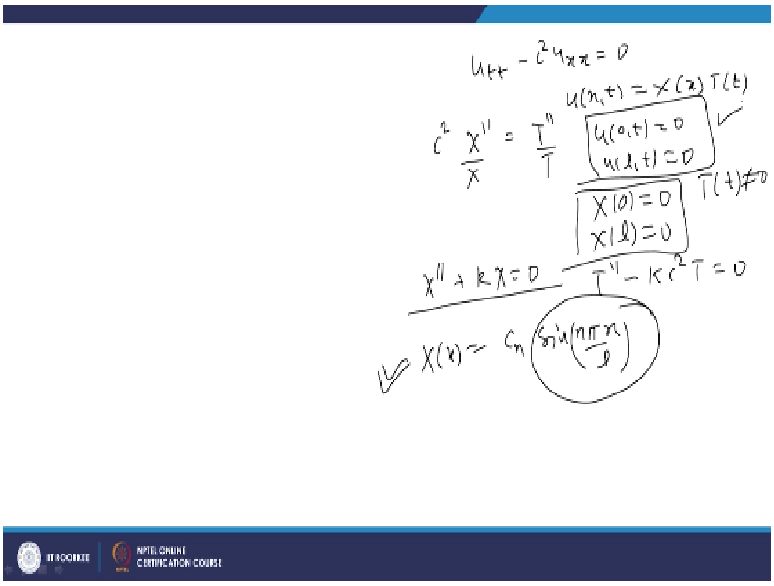

That we have our problem like Utt – c square Uxx = 0 and here you have to assume U(x, t) = X(x) and T(t). Then you solve this it is x double dash, c square x double dash = T double dash/X/T. And when you solve this then we have initial condition that is U(0, t) = 0 and U(l, t) = 0. So, it means that your X(0) = 0 and X(l) = 0 because we are assuming that T(t) is not and = 0.

So, we can say that we are solving our problem X double dash + some your separation constant KX = 0 and here similarly T double dash – Kc square T = 0. So, here we try to solve this equation using initial condition, boundary X(0) = 0 and X(l) = 0 and it is coming out to be that our solution is X = x = some Cn sin n pi x/l, right. So, this solution we obtain which satisfy our boundary condition that is x(0) = 0 or we can say that U(0, t) = 0 and U(l, t) = 0.

So, it means that this component is satisfying the boundary condition. So motivated by this fact, let us say that our solution for the nonhomogeneous problem is u(x, t) = n = 1 to infinity phi n(t) sin n pi x/L. Because this part is here for so that it satisfy the boundary condition given here and this phi n(t) is we obtain this involved in phi n(t) we obtained with the help of initial condition.

So, let us say that here phi n(t) is some unknown function and we try to find out phi n(t) which satisfy the given conditions. So, let us say that here we are assuming that our solution is written in this following form and here since for every n this is a solution of nonhomogeneous problems. So, let us use the principle of the position and we say that let us say that U(x, t) = n to 1 to infinity phi n(t) sin n pi x/L is a solution of nonhomogeneous problem or not?

So we have to find out phi n(t) so that it will solve our problem. So, now this will satisfy the initial condition. Initial condition what? U(x, 0) = 0 and U t(x, 0) = 0. So U(x, 0) is 0 means U(x, 0) is what? That is phi n(0) sin n pi x/L. Now, here we assume that your phi n(0) in a way says that it will satisfy this condition that phi n(0) = 0. Similarly if you look at Ut(x, 0) then it is what, summation phi n dash 0 sin n pi x/l.

So, here we put this additional condition that each phi n will satisfy these initial condition that phi n(0) = 0 and phi n dash(0) = 0. So, these are the condition which we are putting on this unknown function phi n(t). So, our aim is to find out this phi(n, t) so that it will solve our problem. So, sin pi x/L we have taken so that it satisfy the boundary condition and phi n(t) we have taken, we are going to find out a function which satisfy the initial condition that is phi n(0) = 0 and phi n dash (0) = 0 for each n.

Now, it will satisfy our problem, nonhomogeneous problem provided that it satisfy the partial differential equation. So, partial differential equation means Utt A – Uxx, here Utt – Uxx =

f(x, t) and please note down here I have taken c as 1. If c is here then you can put it no problem. It will not affect any of the procedure here the only thing is that presence of c will be recorded here.

So, now here we simply calculate Uxx, so Uxx in fact if you look at here u(x, t) = this then you can find out Utt summation phi n double dash(t) sin n pi x/L. No other chain will be there, right. But if you find out Uxx then you have to differentiate this sin n pi x/L twice. So, here we will get phi n(t) and when you differentiate this we would have cos n pi x/L * n pi/L and again if you differentiate then there is a – sin will appear.

And you will write it is sin n pi x/L * n pi/L whole square. So that is incorporating all these thing we have this n = 1 to infinity phi n double dash(t) + n square pi square upon L square c square phi n(t) * sin n pi x/L = F(x, t). Now, here let us assume that we have written here this c square. So, let us write here that phi n double dash (t) + n square pi square upon L square c square phi n(t) * sin n pi x/L = F(x, t).

So, here we have this equation n= 1 to infinity phi n double dash(t) + n square pi square c square upon L square phi n(t) sin n pi x/L = f(x, t) and we are focusing only on how to solve this phi n(t).

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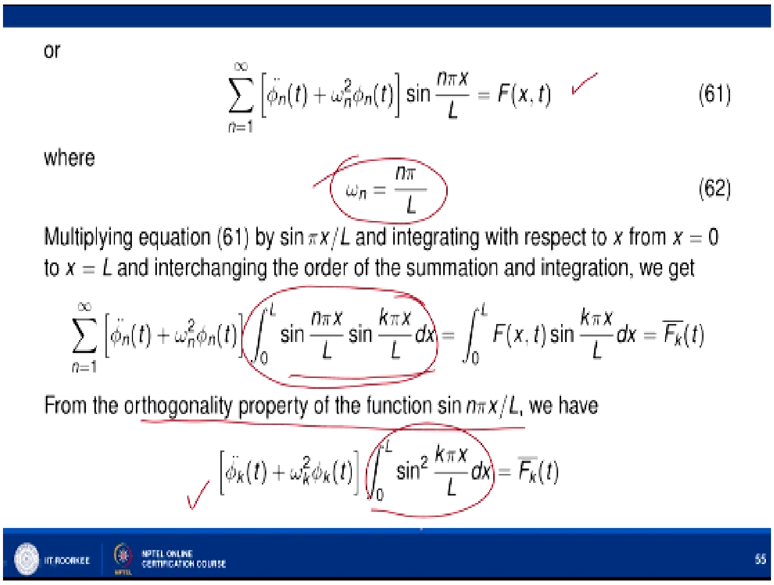

So, here we have write n = 1 to infinity phi n double dash t + omega n square phi n(t) sin n pi x/L = f(x, t). Here we simply simplify our notation n pi c/L/omega n and we multiply this equation by sin k pi x/L and integrating with respect to x from x = 0 to x = L and integrating interchanging the order of the summation and integration. So, here we have taken the liberty that here I can interchange the order of the integration and the summation.

So, here we are assuming that this series is uniformly convergent here. So, here if we can do this interchange then we can write that n = 1 to infinity pi n double dash(t) + omega n square phi n(t) * 0 to L sin n pi x/L * sin k pi x/L dx = 0 to L F(x, t) sin k pi x/L and denote this as F k bar(t). So, when we look at this using the orthogonality this value is non-zero only when k = n. Otherwise this is going to be 0.

So using the orthogonality property of the function sin n pi x/L we can say that if n=k then we have a non-zero value and value is given by L/2. So, it is now this series infinite series is now reduced to single term phi k double dash(t) + omega k square phi k(t) 0 to L sin square k pi x/L t(x) = 0 to L F(x, t) sin k pi x/L. Now using the value here 0 to L sin square k pi x/L dx.

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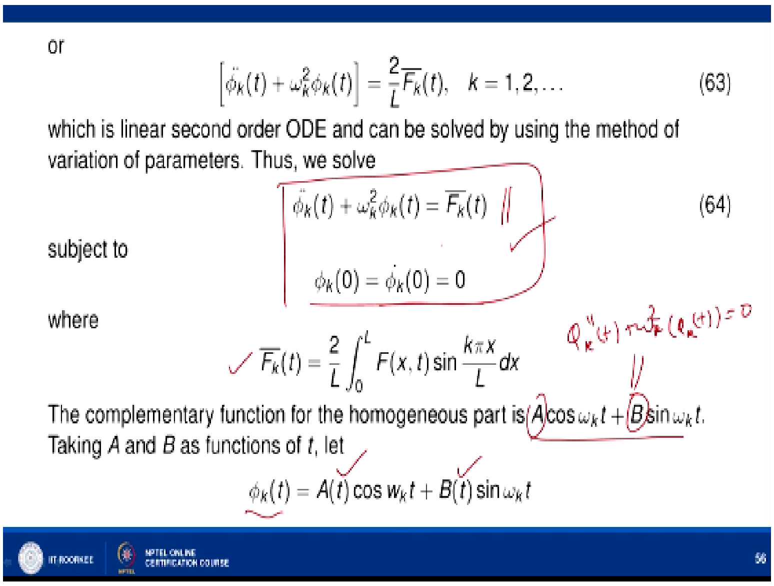

We can write down this equation as phi k double dash(t) + omega k square phi K(t) = 2/L Fk bar(t). So, now let us this is true for every k = 1 to n. Now which is a linear second order ODE and can be solve using the method of variation of parameter. So, now we want to solve phi k double dash(t) + omega k square phi k(t) = Fk bar t. Now, here 2/L is incorporative in definition of Fk bar(t) and we can write Fk bar(t) is 2/L 0 to L F(x, t) sin k pi x/L dx.

And here this second order ODE, Ordinary Differential Equation and satisfy the following initial condition that is phi k(0) = 0 and phi k dash(0) = 0. So, our aim is now solve this equation and which is a simple second order nonhomogeneous problem given in terms of t and we want to solve this. So, here we will let us assume that we have a variation of parameter method.

So, for that let us look at the complimentary function of the homogeneous problem that is phi k double dash(t) + omega k square phi k(t) = 0 and we already know the solution here is A cos omega k(t) + B sin omega k(t). So, here A and B are some constant. So, in a variation of parameter we simply assume that this A and B are not constant it is a parameter. So, let us say that this phi k(t) which is written as A(t) cos omega k(t) + B(t) sin omega k(t).

Now here A(t) and B(t) are parameter. Now we want to choose the parameter A(t) and B(t) in a way such that this phi k(t) satisfy the nonhomogeneous problem. So, if it satisfy in the nonhomogeneous problem then let us find A(t) B(t) by putting this phi k(t) * equation number 64.

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$$\dot{\phi}_k(t) = \dot{A}(t)\cos\omega_k t + \dot{B}(t)\sin\omega_k t - \omega_k A(t)\sin\omega_k t + \omega_k B(t)\cos\omega_k t$$

We choose $A(t)$ and $B(t)$ such that

$$\dot{A}(t)\cos\omega_k t + \dot{B}(t)\sin\omega_k t = 0 \qquad (65)$$

Therefore,

$$\ddot{\phi}_k(t) = -A(t)\omega_k^2\cos\omega_k t - B(t)\omega_k^2\sin\omega_k t - \dot{A}(t)\omega_k\sin\omega_k t + \dot{B}(t)\omega_k\cos\omega_k t$$

Substituting these expressions into equation (64), we get

$$\omega_k(\dot{B}(t)\cos\omega_k t - \dot{A}(t)\sin\omega_k t) = \overline{F}_k(t) \qquad (66)$$

Solving equations (65) and (66) for A and B, we obtain

$$\dot{A}(t) = \frac{\overline{F}_k(t)\sin\omega_k t}{\omega_k}$$

$$\dot{B}(t) = \frac{\overline{F}_k(t)\cos\omega_k t}{\omega_k}$$

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So, when you find out the first derivative we have phi k dash(t) = A dash(t) cos omega k(t) + B dash(t) sin omega k(t) – omega k A(t) sin omega k(t) + omega k B(t) cos of omega. We simply differentiate once. Now, we choose our A(t) and B(t) since A(t), B(t) are any arbitrary functions. So now our, we can put some condition on A(t) and B(t). So, let us put the following condition then this part = 0.

So, we choose our A(t) B(t) such that this A dash(t) cos omega k(t) + B dash(t) sin omega k(t) is 0. Now using these condition now then phi k(t) is reduced to what phi k(t) is reduced to only this term. Now again differentiate one more time we have A dash(t) omega k square cos omega k(t) – B dash(t) omega k square sin omega k(t) – A dash(t) omega k sin omega k(t) + B(t) omega and omega k(t).

Now, here using phi k double dash(t) and phi(t) phi k(t) put it back to equation number 64. When you put it back what you will have is the following thing that rest will cancel out and we will have, if you look at it is what? If you simplify you will get omega k B dash(t) cos omega k(t) – A dash(t) sin omega k(t) = Fk bar(t). So, here what we have done? We have use the value of phi k double dash(t) and phi k(t) put it back to equation number 64 and what we have is this.

There is a –n here. So, here we solve equation number 65 and 66 because these are the only 2 condition we have put on A(t) and B(t). So, we have 2 equation, A dash(t) cos omega k(t) + B dash(t) sin omega k(t) = 0 and omega k B dash t cos omega k(t) – A dash(t) sin omega k(t) = Fk bar(t). So, here we can solve this by say system of linear equation or you just simply solve, right.

So, here we multiply something and we simplify and we have A dash(t) = so let me do it this thing. This is a A dash(t) B dash(t) = 0 Fk bar(t) and here we have relation cos of omega k(t) sin omega k(t) and here we have this – sin omega k(t) * omega k. Here we have one more that omega k cos of omega k(t). So, we can solve this system of linear equation to find out A dash(t) and B dash(t).

So, let us say that we have solved we have A dash(t) = Fk bar(t) sin omega k(t) omega k and B dash(t) = Fk bar(t) cos omega (t)/omega k.

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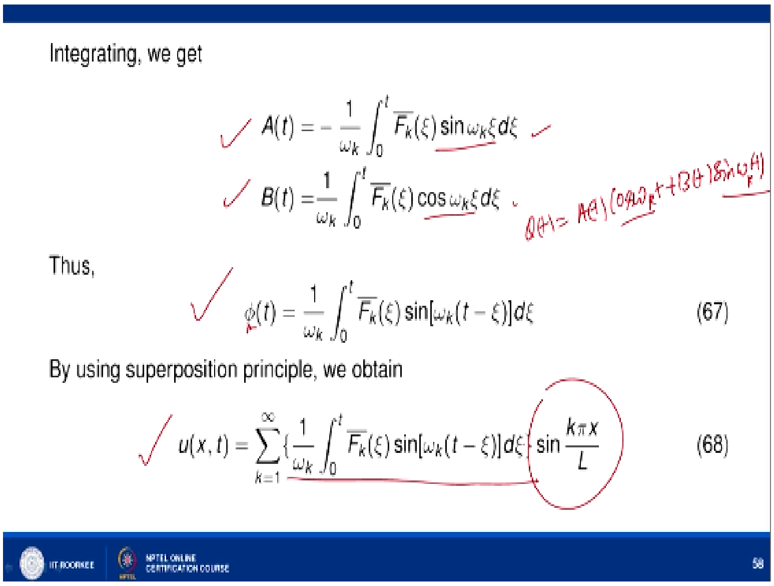

Now, once we have A dash(t) B dash(t) we can find out A(t) B(t) by integrating and we have A(t) as -1 upon omega k 0 to t Fk bar Xi sin omega k Xi d Xi and B(t) as 1 upon omega k 0 to t Fk bar Xi cos omega k Xi d Xi. So A(t) B(t) is given to us then put it back to phi(t). So, phi(t) will be what? A(t), I think A(t) cos of here we have assumed A(t) cos omega k(t), so let me write it here.

A(t) cos of omega k(t) + B(t) sin of omega k(t). So, here we just put it here the value of A(t) and B(t) and when you say cos omega k * sin omega k Xi and cos omega k * sin omega Xi. So, here using the sin A – B formula we can write phi k(t) = 1 upon omega k(0, t) F kbar Xi sin omega k(t) – Xi d Xi. So, once we have phi k(t) then our solution it is written as U(x, t) = k to, 1 to infinity.

Omega phi k(t) * sin k pi x/L. Now, omega k(t) is value one upon omega k 0 to t Fk bar Xi sin omega k(t) – Xi d Xi. So and once we have this so this is a solution of a particular problem.

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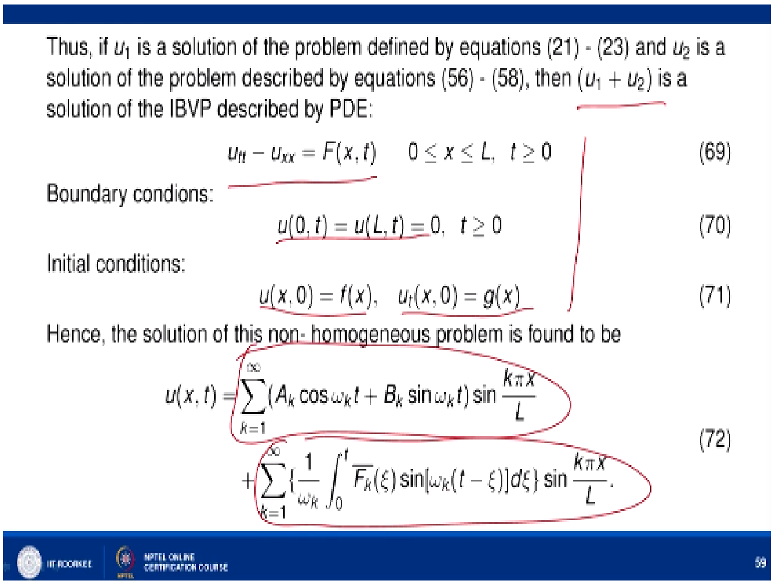

Then if U1 is a solution of the problem define means define for 21 and 21 that is homogeneous boundary value. Here I can say like this that here if B1 is the solution of this problem then and U(t) is the solution of the problem which we have just discussed then U1 + U2 is a solution of the following problem. That is Utt – Uxx = F(x, t), u0(t) = 0 = u(L, t) = 0 and u(x, 0) = f(x), Ut(x, 0) = g(x).

So, U1 is this part that k = 1 to infinity Ak cos omega kt + Bk sin omega kt sin k pi x/L and this is the part U2 which we have just obtained that it is k = 1 to infinity 1 upon omega k 0 to t. Fk bar Xi, sin omega k(t) – Xi d Xi sin k pi X by L.

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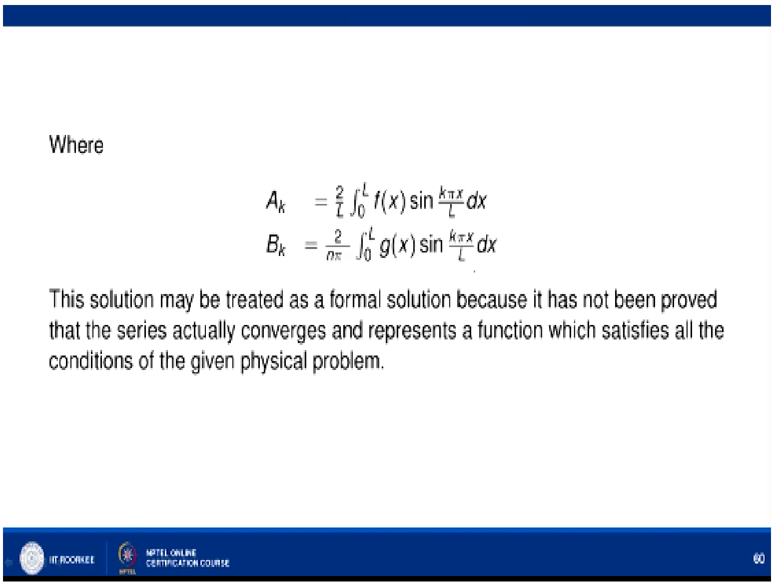

And here you can find out the value of Ak and Bk using the initial condition that is Ak = 2/L 0 to L f(x) sin k pi x/L dx and Bk = 2/n pi/c 0 to L g(x) sin k pi x/L dx. Now, here again if you look at this series solution 72 here we have not discuss any stability convergence criteria. So, right now I can say that this is just a formal series solution because we have not discuss any convergence criteria and we have utilized one thing that here your integration and summation can be interchange.

We have already utilized that. So, right now I can say that this equation number 72 will solve a nonhomogeneous problem, equation number 69 along with homogeneous boundary condition and nonhomogeneous initial condition. So, in this lecture we have seen how to solve a nonhomogeneous problem with nonhomogeneous boundary condition with nonhomogeneous initial condition.

If you have nonhomogeneous equation with nonhomogeneous boundary condition with nonhomogeneous initial condition then also how we can solve this problem, okay. By converting into a simpler problem, okay. So here we will stop, will continue in next lecture. Thank you very much for listening, thank you.