Ordinary and Partial Differential Equations and Applications
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Lecture – 55
One Dimensional Wave Equations and its Solutions- III

Hello friends, welcome to this lecture and in this lecture, we will continue our study of wave equation and if you recall in previous lecture, we have discussed the wave equation in infinite string, semi-infinite string, and finite string and also we have given the variable separable method for finding the solution in finite string and we have shown that by uniqueness theorem that this solution in the case of one dimensional wave equation with finite domain.

Your solution, if it exist has to be unique, so it means that whether you apply the D'Alembert's wave solution method or say, variable separable method, your solution come to be same and this will help us a lot in a sense that variable separable method is generally is easy to apply, so when we have uniqueness in our hand, we try to solve our problem using variable separable method. Now, in this lecture we will discuss the problem wave equation in 2 dimensions.

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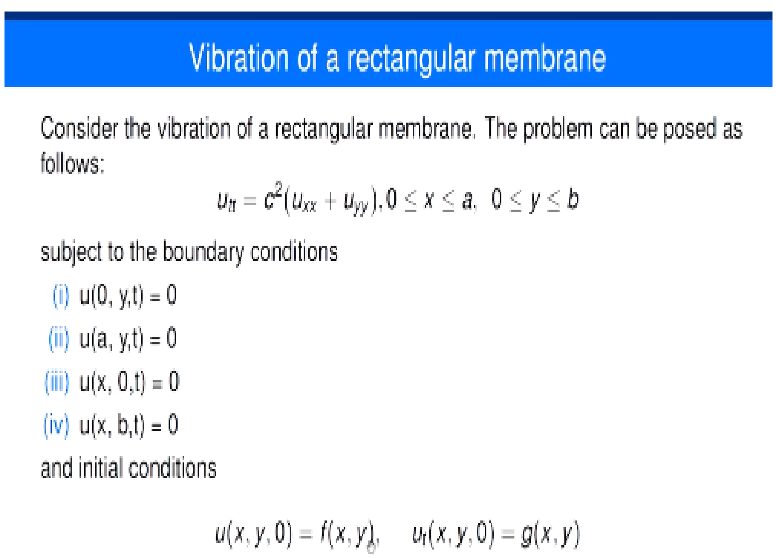

So, here let us look at here, consider the vibration of a rectangular membrane and the problem can be pose like this; utt = c square uxx + uyy 0 < or = x < = a 0 < = y < or = b here and subject to the boundary condition that u0y, t = 0, u of ay, t = 0, u of x, 0, t = 0, u of x, v, t = 0 and initial condition are given as uxy = uxy0 at f xy and ut xy0 = g of fxy.

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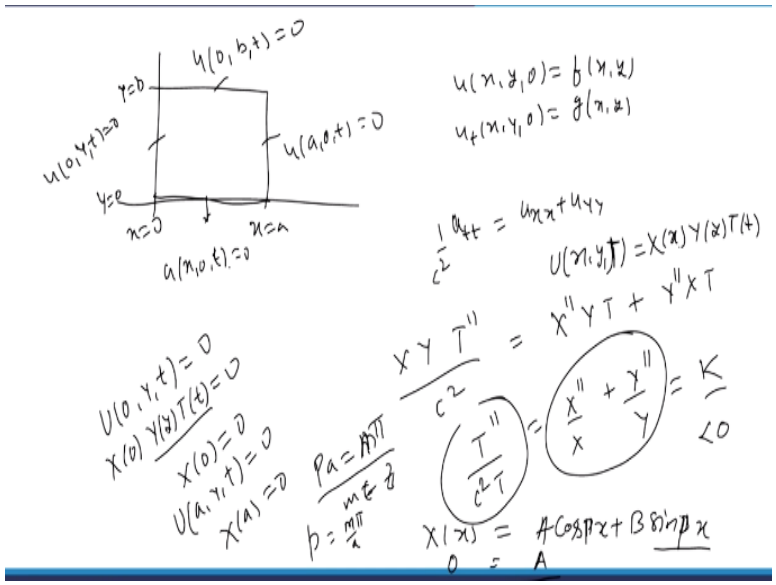

So here, the thing is that here we are considering this kind of domain, so here we have rectangular domain and here we have x = a, it is x = 0 and it is y = 0 and y = b, so your domain is a rectangular domain and here we assume that here your solution is what; u of x0, here solution in this is basically 0, so it means that u of x0 is coming out to be 0, so here is boundary condition is coming to be 0.

So, that is what we have written here that u of x, 0, t = 0, u of x, b, t = 0, u of x, b, t = 0 here, so it means that here u of x0 t0, here u of a, 0, t = 0, here u, 0, b, t = 0, here u of 0, sorry, u of 0, y, t = 0. So, here we have these boundary condition, initial condition is already given that u of xy,0 = f of x, y and ut of x, y, 0 = g of xy, so these are the initial condition given and now we want to find out the solution of this wave question.

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We look for a separable solution of the form

$$U(x, y, t) = X(x)Y(y)T(t)$$

Substituting into the given PDE, we get $\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2 \quad (say)$
Then

$$T'' + c^2\lambda^2 T = 0,$$
$$\frac{X''}{X} + \lambda^2 = -\frac{Y''}{Y} = \mu^2 \quad (say).$$

Thus yielding

$$Y'' + \mu^2 Y = 0, \quad X'' + (\lambda^2 - \mu^2)X = 0.$$

Let $\lambda^2 - \mu^2 = p^2$, $\mu^2 = q^2$. Then

$$\lambda^2 = p^2 + q^2 = r^2.$$

Now, here x is lying between 0 to a and y is lying between 0 to b here, now we look for a separable solution of the form, so here we apply the variable separable method and write down the solution ux y t as Xx, Y of y and T of t and when you put this solution into your wave equation, your equation is now reduced to 1 upon c square t double dash upon t = x double dash upon x + y double dash upon y.

Now, let us assume that it is a some –lambda square, as we have already seen in the case of one-dimension wave equation, we have shown that corresponding to the separation constant as – of lambda square, your solution is coming out to be nontrivial, so here we are using our experience and we say that the separation constant is coming to be -lambda square and then we try to solve this to 2-dimension wave equation.

So, here let me do it here, it is what; you have utt 1 upon c square utt = uxx + uyy, so here when you write u of xyt = X of x, Y of y and T of t, when you put it here, it is t double dash xy divided by c square = it is x double dash yt + y double dash xt, now divided by xyt, you will get t double dash upon c square T = x double dash upon x + y double dash upon y, now this is a separable constant, why we are separating this?

Because this is a function of T only and these are the function of X and Y, so it means that this will be equal when each quantity has to be a constant value, so it means that t double upon c square t is constant, x double dash + y double dash upon y = constant. Now, here we can again consider several cases that is k is positive, negative and 0 and we can verify that the corresponding value for k which is a negative value will have a nontrivial solution.

So, I am leaving the part that you verify that for corresponding k = say positive and 0 we will have only a trivial solution. So, now let us say that k = - lambda square is a separation constant, so when we write it then this 1 upon c square t double dash upon t = - lambda square is written as t double dash + c square lambda square t = 0 and using this I can write x double dash upon x + lambda square = - of y double dash upon y square = mu square some other constant.

Because this is a function of X only and this is a function of Y only and these to be equal when each one is separately equal to some constant value and let us call this as separation value as mu square. So, it means that now, if we consider the last 2 equations, it is given as y double dash + mu square y = 0 +; and x double dash + lambda square - mu square x = 0. Now, to simplify these 3 equations; 1 equation is this.

T double dash + c square lambda square t = 0 and y double dash + mu square y = 0 and x double dash + lambda square – mu square x =0, so to simplify, let us assume that let lambda square – Mu square = p square, you call mu square is q square, it is up to us, then I can write lambda square as what; lambda square has p square + mu square or we can say lambda square can be written p square + q square.

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Therefore, we have

$X'' + p^2X = 0,\ Y'' + q^2Y = 0,\ T'' + r^2c^2T = 0.$

The possible separable solution is

$$u(x,y,t) = (A\cos px + B\sin px)(C\cos qy + D\sin qy)(E\cos(rct) + F\sin(rct))$$

Using the boundary conditions: $u(0,y,t) = 0$ gives $A = 0$

$u(x,0,t) = 0$ gives $C = 0$

$u(a,y,t) = 0$ gives $p = m\pi/a,\ m = 1,2\ldots$

$u(x,b,t) = 0$ gives $q = n\pi/a,\ n = 1,2\ldots$

Now, denote this value as r square, so we just write down a simplified expression for these 3 constants, so in terms of p, q and r, now your equation is written as x double dash + p square x = 0, y double dash + q square r =0 t double dash + r square c square t = 0 here. Now, we already know how to find out the solution here, it is simple second order linear equation with the constant coefficient.

And we can find out the possible solution as ux yt as A cos px + B sin px corresponding to this and y double dash + q square y, we have C cos qy + D sin qy and corresponding to t double dash + r square c square t, we have E cos rct + F sin rct here, now here we have several coefficients here basically A, B, C, D, E, F are all constant, so now we just try to find out the value of these constant, so here we try to use our initial condition.

Now, initial condition is what; u0 y t = 0, so when you put u0 yt = 0 means here, u0 yt = 0 means X of 0, Y of y, T of t = 0, now these cannot be 0, so X of 0 has to be 0, similarly you have u of a y t = 0, so here again in a similar region, you have X of a = 0, so X of 0 = 0 means, your c is = 0, so here you simply put that since Acos px B sin px is = 0, so here we simply say, no, sorry, here A cos px B sin px at x =0 has to be value 0, so it means that A has to be 0, right, so that is let us write it here.

We have what; we have Acos px + B sin px and that is your X of x = 0, so let us say X of 0 is 0, so 0 = you will put A, so here A is coming out to be 0 here. Similarly, you can write it that X of A = 0, then this implies that this p of A has to be n pi kind of thing, so that we can get it here that u of a y t is 0, it means that pa = m pi that is what we wanted to write that pa is = m pi, where m is your integer value, m is coming from integer.

So, it means that p is now written as m pi/a, so which is what we have written, p = m pi/a, where m is from 1, 2 and so on. Now, similarly if you look at the other condition that is this condition that here we have assumed this condition also that u of 0yt = 0, so here if you look at; sorry here if you look at this condition, u of x0 t = 0.

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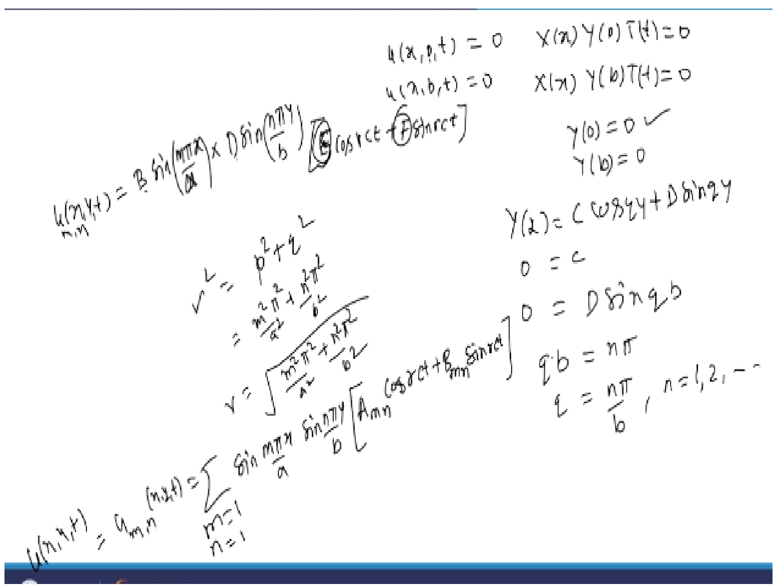

So, here we have u of x0t = 0 and u of x bt 0 0 here, so here in a similar way X of x, Y of 0, T of t = 0 and X of x, Y of b, T of t = 0, so this implies that Y of 0 is = 0 and Y of b = 0. Now, what is your Y of y here? It is we have written here, c cos qy + D sin qy; c cos qy + D sin qy, right, now put take this condition 0 = c, so it means that c is coming to be 0, now if we take yb = 0, it means 0 = D sin qb.

Now, if so it means that for non-trivial solution, you have to take qb as say, n pi, so it means that q is = ni pi/ b where n is say, 1, 2 and so on, we are leaving value 0 because if you take n = 0 since it is sin function, you will get a trivial solution here, so it means that here we will get q = ni pi/b, n = 1, 2 and so on, so on so it means that here we got A = 0, C = 0, p = mi pi/ a, q = n pi/ b here.

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By using principle of superposition, we get

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [A_{mn} \cos(rct) + B_{mn} \sin(rct)] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (39)$$

where

$$r^2 = p^2 + q^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$$

By applying the boundary condition: $u(x, y, 0) = f(x, y)$, equation (39) gives

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dxdy \qquad (40)$$

So, by using; so it means that corresponding to each m and n, we have a solution here and let us call this solution as u mn xy t, so here using principle of super position as we have done for first case, we can write here solution as m = 1 to infinity n = 1 to infinity, let us rename these constant as; let me write it here solution to make it more clear, so here we have written what; so here u of x y t is given as your S sin of m pi x/l; sorry, m pi x/ a.

And if you look at the y component, you will get a constant here as D sin n pi y/b and the component corresponding to Tt that is some c; E cos of r pi ct/l, so here we have the corresponding say, rct/l, so here we have E cos rct; so here E cos rct + F sin of rct now, what is the value of r? Here, r square is = p square + q square, so we can write this since p square and q square we already know, so it is m square pi square/ a square.

And q square is = n square pi square upon b square, so it means the value of r is known to us and we write m square pi square upon a square + n square pi square upon b square. Now, this b and d will be merged with this E and F and I can write this for each mn, let me write it, it is u mn and for each mn, we have a solution. So, let me write it, say solution as u mn, x yt as follows, you simply write sin m pi x/a sin n pi y/b.

And in a bracket, you write E as some mn; Amn cos of rct + Bmn sin of rct and since it is a true for each mn, so using principle of super position, if you sum over m = 1 to infinity and n = 1 to infinity, then the solution let us denote as u of xyt, so using the principle of superposition, your solution is given as this that uxyt = m = 1 to infinity Amn cos of rct + Bmn sin of rct * sin m pi x/ a and sin n pi y/b as we have pointed out.

Now, here the value of r square is p square + q square that is pi square m square upon a square + n square upon b square but is still in equation number 39; we have 2 unknown that is Amn and Bmn. Now, to find out Amn and Bmn, we will use the initial condition, the initial condition is that u of xy 0 = f of xy, now if f of xy when you put at t = 0, then when you put t = 0, this part is gone and we will have only Amn, so it means that Amn * sin m pi x/a sin n pi y/b, e = f of xy.

Now, using this I want to find out the value of Amn, now if you look at this can be easily obtain as we have obtained for first order, so here you multiply both side by say, sin k pi x/a * sin n pi y/b and integrate between 0 to a and 0 to b and using orthogonality, we say that you can obtain our solution as Amn = 2 times 2/a * 2/b, 0 to a 0 to b fx y sin m pi x/a, sin n pi y/b dx dy, so here this you can obtain using orthogonality of function sin m pi x/ a and sin n pi y/b here.

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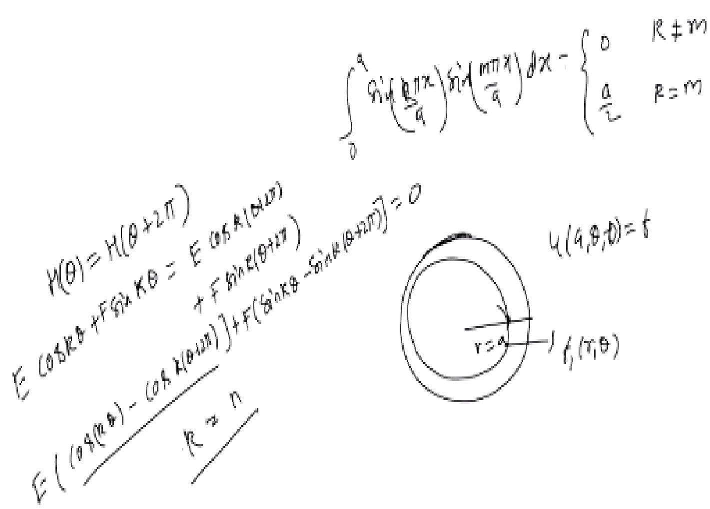

So, here we are just using the orthogonality condition that is 0 to a sin n pi x/a, sorry, this is ki pi x/a and sin mi pi x/a d of x is = 0, when k is != m and this a/2, when k = m, here so using this orthogonality condition, for both in terms of x, a in terms of y, we have obtain our coefficient Amn as 4 upon ab 0 to a, 0 to b, fxy sin m pi x/ a sin n pi y/ b dx dy. Now to find out Bmn, you use the condition that utx, y, 0 = g xy.

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Finally, by applying the boundary condition: $u_t(x, y, 0) = g(x, y)$, equation (39) gives

$$g(x, y) = cr \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where

$$B_{mn} = \frac{4}{abcr} \int_0^a \int_0^b g(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dxdy \qquad (41)$$

Hence, the required series solution is given by (39), where A_{mn} and B_{mn} are given by equation (40) and (41) respectively.

So, differentiate with respect to t, we will and using utxy; x, y, 0 = g xy, we can write gxy = c times r m = 1 to infinity, n = 1 to infinity, Bmn sin m pi x/a sin n pi y/b, again you multiply by sin k pi x/a, sin l pi y/ b and using orthogonality, you can find out Bmn as follows; Bmn = 4 upon ab and this cr is already there, so cr 0 to a, 0 to b g xy sin m pi x/a sin n pi y/b dx dy and hence it requires series solution is given by 39 that is this equation.

And the coefficient Amn and Bmn, you can obtain by equation number 40 and equation number 41 and with this, the solution of 2 dimension wave equation, we have obtain using variable and separable method. Now, we will move to next problem that is vibration of a circular membrane.

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Vibration of a circular membrane

To find the solution of the wave equation representing the vibration of a circular membrane, we use the polar coordinate systems (r, θ), $0 \le r \le a$, $0 \le \theta \le 2\pi$. Thus, the governing two- dimensional wave equation is as followes

PDE:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad (42)$$

and the boundary and initial conditions are given by

$$u(a, \theta, t) = 0, \quad t \ge 0 \qquad (43)$$

$$u(r, \theta, 0) = f_1(r, \theta), \quad \frac{\partial u}{\partial t}(r, \theta, 0) = f_2(r, \theta) \qquad (44)$$

respectively.

And it means that here, we want to find out the solution of the wave equation representing the vibration of a circular membrane, we use a polar coordinate system, say r theta, where r is lying between 0 to a and theta is lying between 0 to 2pi, so it means a problem is what; you have a circular domain like this and here we have a membrane and this is vibrating, so this; here this is a membrane and radius we are taking as r = a here.

And we have this, now to define this problem in an accurate manner, we use the cylindrical coordinate to write down our wave equation. So, in wave equation, we have written 1 upon c the square and dou2 u/dou u square and here uxx + uyy is now written in terms of polar coordinate that is dou2/dou 2u/ dou r square + 1 upon r dou u/ dou r + 1 upon r square dou 2u/ dou theta square, so that is a value of uxx + uyy written in terms of polar coordinate system.

Now, initial and boundary conditions are what; here boundary condition is that on the boundary that is ua theta t is coming out to be sorry, 0 is coming out to be let us assume that it is coming out to be f1 theta; f a r, so here we have assumed that u a theta is coming out to be 0, so boundary here; since here boundary is kind of fixed, so we can say that on the boundary to displacement is 0.

And inside, your r theta 0, so here inside at some lesser value say, here it is given as f1 r theta, so it is f1 r theta and in velocity in t is also given as f2 r theta, so here your u r theta, 0 is = f1 r theta and dou u/ dou t r theta 0 is given as f2 r theta and we try to find out the solution here. Now, again here you will try to use a separation and variable method, here the idea comes because the equation is separable and the boundary conditions are also separable.

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For the solution of equation (42), we use the following variable separable form:

$$u = R(r)H(\theta)T(t) \qquad (45)$$

Substituting into equation (42), we obtain

$$\frac{RHT''}{c^2} = R''HT + \frac{1}{r}R'HT + \frac{1}{r^2}RH''T$$

Dividing through by RHT, we get

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} = \frac{T''}{c^2T}(t) = -\mu^2 \quad (say)$$

Then

$$T'' + \mu^2 c^2 T = 0 \qquad (46)$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \mu^2 r^2 = -\frac{H''}{H} = k^2 \quad (say)$$

So, using this information in hand, we apply variable and separable method to solve this problem, so here let us apply for the solution of the equation 42, we use the following variable separable form that is u r theta t can be written as R of r H theta and T of t and now when you substitute your Rr H Tt, then you will get RHT double dash divided by c square = R double dash HT +1 upon Rr dash HT + 1 upon R square RH double dash T,

Then you divide by RHT, so you will get R double dash divided by R + 1 upon R R dash upon R + 1 upon R square is double dash upon H = T double dash upon c square T = -mu square. Now, since, these left hand side is a function of R and theta and the right hand side is a function of T only, so this can be true when it is separated by some constant, so let us say that the constant is given by – of mu square.

And this again I am leaving it to you that if you take positive constant of 0 constant, then we have only a trivial solution, so here this equation is not reduced to T double dash + mu square c square T = 0 and R square R double dash divided by R + Rr dash upon R + mu square R square =

now here also we separate R and theta = - H double dash upon H = k square. Now, here we use the separation variable that is say, k square.

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i.e.

$$r^2R'' + rR' + \left(\mu^2 r^2 - k^2\right)R = 0 \qquad (47)$$

$$H'' + k^2H = 0 \qquad (48)$$

Here, μ^2 and k^2 are arbitrary separation constants. The general solution of equations (46)- (48) respectively are

$$T(t) = A\cos\mu c t + B\sin\mu c t$$
$$R(r) = PJ_k\left(\mu r\right) + QY_k\left(\mu r\right) \qquad (49)$$
$$H(\theta) = E\cos k\theta + F\sin k\theta$$

where J_k, Y_k are Bessel functions of first and second kind respectively of order k. Thus the general solution of the wave equation (42) is

$$u\left(r,\theta,t\right) = \left(A\cos\mu c t + B\sin\mu c t\right)\left\{PJ_k\left(\mu r\right) + QY_k\left(\mu r\right)\right\}\left(E\cos k\theta + F\sin k\theta\right) \quad (50)$$

So, here now our equation is now reduced to this last equation is now reduced to R square r double dash + Rr dash + mu square R square – k square r = and H double dash + k square H = 0, if you look at here, your k and mu are arbitrary separation constant. Now, the general solution of these equations; equation T double dash + Mu square c square T = 0, r square r double dash + Rr dash + mu square R square - k square R = 0.

And H double dash + k square H = 0, if you look at these are simple second order differential equation and if you look equation number 47, it is a Bessel kind of Bessel's equation. Now, here we can find out the gender solution of these equation as follows; Tt = Acos mu ct + B sin mu ct, R is given as P times jk mu r + QYk mu r, here jk and yk are Bessel function of first and second kind respectively of order k.

And H of theta is given as E cos k theta + F sin k theta, now using the solution listed in equation number 49, now I can write, u r theta t is = A cos mu ct + B sin mu ct * PJK mu r + QYk mu r * E cos k theta + F sin k theta.

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Since $u(r,\theta,t)$ is a single- valued periodic function in θ of period 2π, k has to be and integral, say $k = n$. Also, since $Y_k(\mu r) \to -\infty$ as $r \to 0$, In order to avoid unboundedness of u at the center $r = 0$, take $Q = 0$.
Boundary condition (43) implies that the deflection u is zero on the boundary of the circular membrane, we obtain

$$J_n\left(\mu a\right) = 0 \tag{51}$$

which has an infinite number of positive zeros.
After using the solution of the circular membrane in the form

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(A\cos\mu_{nm}ct + B\sin\mu_{nm}ct)(E\cos n\theta + F\sin n\theta)PJ_n\left(\mu_{nm}r\right)$$

So, far we have not utilise any initial condition or any boundary condition, so now let us find out these constant A, B, P, Q, E, F and here we used that since u r theta t is a single valued periodic function of theta of period 2 pi, k has to be an integral value, so it means that here E cos k theta and F sin theta is a periodic with period 2 pi provided your k has to be integer that how you can check here, you can check like this that since H theta must be = H of theta +2 pi.

So, it means that E sin; E of cos k theta; E of cos of k theta + F of sin of k theta = E of cos of k, in place of theta, let us write theta 2 pi + f sin of k theta + 2 pi, now when you separate this out, you have E of cos of k theta - cos of k theta +2 pi + F sin of k theta - sin of k theta +2 pi= 0, then you can use the formula of cos A – cos B and sin A – sin B and if you simplify, you simply say that you will get that k theta has to be n pi n theta.

So here, sorry k has to be an integer value, so this I am leaving it to you that I am giving you the hint that using this and the formula of cos A - cos B sin A – sin B, you show that k has to be integer value, if k is not integer, then this implies that E and F are some 0 values, so here you have to show that k has to be n. So, it means that here to have a periodicity in our problem, we have to assume that k has to be an integral value say, let us say k = n value.

And also we assume that that in a domain in a bounded domain, your solution will be bonded all the time in fact, it means that if you have a cylindrical domain and if we have a member which is vibrating, this vibration should not be unbounded, so it means that inside the domain your solution will remain always finite and if you look at; look at the term, yk mu r and if you take r tending to 0, this y k mu r will be infinity, unbounded going to be – infinity.

So, it means that this if yk terms will be there, then we will have a unbounded solution, so to avoid this unboundedness of the solution r centre r = 0, let us take the coefficient of YK to be 0, so it means that your solution will not contain any Bessel's function of second order, so it means that your Q has to be 0. So, what we have achieved here; natural boundary conditions are--natural conditions are that your solution will be periodic.

Then this implies that k = n and solution will remain bounded inside your domain, so it means that your Q has to be 0, so it means that now if you look at the next boundary condition that at r = a, your solution is going to be 0, so it means that in a boundary, jn mu n of a a has to be 0, so it means that you have to choose Mu in a way such that jn of mu of a has to be 0, so it means that this must have; since jn of x =0 has infinitesimally many values.

So, it means us let us denote that mu as infinitesimally zeros of jn, so using all these information your solution of the circular membrane is given as mu of r theta t = m = 1 to infinity n = 1 to infinity A cos mu nm t. Here, mu nm is what; this mu nm represent the zeros of jn, so let us say that the zeros of jn is mu 1m, mu 11 mu, 1 2 and so on, so denote these zeros of jn as mu nm. So, let us write down our solution u r theta t = m = 1 to infinity, n = 1 to infinity A cos mu nm ct + B sin mu nm ct * E cos n theta + F sin n theta * PJn mu nm r.

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Alternatively,

$$u(r,\theta,t)=\sum_{m,n=1}^{\infty} J_n\left(\mu_{nm}r\right)\{[a_{nm}\cos n\theta+b_{nm}\sin n\theta]\cos\mu ct+[c_{nm}\cos n\theta+d_{nm}\sin n\theta]\sin\mu ct \tag{52}$$

Now, to determine the constants, we shall use the prescribed initial conditions which yield

$$\begin{aligned} f_1(r,\theta)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(a_{nm}\cos n\theta+b_{nm}\sin n\theta)J_n\left(\mu_{nm}r\right) \\ f_2(r,\theta)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mu_{nm}(c_{nm}\cos n\theta+d_{nm}\sin n\theta)J_n\left(\mu_{nm}r\right) \end{aligned} \tag{53}$$

So, here we are using principle of superposition to write down this general solution. Now, let us multiply this P inside and we try to have less constant, so now we write down our solution u r theta t = m, n = 1 to infinity Jn mu nm r * anm cos n theta + b nm sin n theta * cos of mu ct + cnm cos of n theta + d nm sin n theta * sin of mu ct. Now, we want to find out Anm, Bnm, Cnm and Dnm.

Now, determine the constant, we will use the prescribed initial condition that is f1 r theta = f1 r theta is basically u r theta, 0, so at t = 0, we have given 2 initial conditions, so let us utilise this, so f1 r theta = m = 1 to infinity n = 1 to infinity Anm cos of n theta + Bnm sin n theta * Jn * mu nmr, so next condition is f2 r theta = m = 1 to infinity n = 1 to infinity mu nm c nm cos n theta + D nm sin n theta * Jn mu nmr.

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Please note the following orthogonal property of Bessel's functions;

$$\int_0^1 xJ_p(\lambda_m x)J_p(\lambda_n x) = \begin{cases} 0, & \text{if } m \neq n; \\ \frac{1}{2}J_{p+1}^2(\lambda_n), & \text{if } m = n. \end{cases} \quad (54)$$

If $F(x) = \sum_{n=1}^{\infty} a_n\ J_p(\lambda_n x)$, then multiplying through by $xJ_p(\lambda_m x)$ and using (54), we have a_m

$$\int_0^1 xF(x)J_p(\lambda_m x)ds = \frac{a_m}{2}J_{p+1}^2(\lambda_m)\ , \text{ or}$$

$$a_m = \frac{2}{J_{p+1}(\lambda_m)^2}\int_0^1 xF(x)J_p(\lambda_m x)ds. \quad (55)$$

So, to prove; to find out Amn and Bmn, we use the orthogonality condition of Bessel's function and orthogonality condition of cosine and sin function. So here, please note the following orthogonal property of Bessel function, here it is 0 to 1x Jp lambda mx * Jp lambda nx, this is = 0, when m is != n, please note down here, here lambda n are 0's of Jp, so here orthogonal property is given in terms of zeros of Bessel's equation of pth order. So, let us say that it is 0 to 1x; x is the weight function, x Jp lambda mx, Jp lambda nx.

Here, lambda nm and lambda are zeros of Jp, it is given as zero, when m is != n and it is given as 1 upon 2 Jp +1 lambda n, J square P +1 lambda n, when m is = n, here, so here this square is the whole square; Jp + 1 lambda whole square. So, now using this orthogonal property, if I write any function f of x as summation n = 1 to infinity Jp lambda nx, then using orthogonality we can find out the coefficient here, so f of x = n = 1 to infinity An Jp lambda nx.

and we can find out the coefficient An are as follows, so what we do; we multiplied throughout by x Jp lambda mx and using the orthogonal property, we can write yours 0 to 1x fx Jp lambda mx dx = Am/2 Jp + 1 lambda whole square, now we can simplify this problem and we can write Am = 2 upon Jp + 1 lambda m whole square * 0 to 1x fx Jp lambda mx dx.

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Using (55) the solution of the circular membrane is given by equation (52), where

$$a_{nm} = \frac{2}{\pi a^2 [J_{n+1}(\mu_{nm})]^2} \int_0^{2\pi} \int_0^a f_1(r,\theta) J_n(\mu_{nm} r) \cos n\theta \, r dr d\theta$$

$$b_{nm} = \frac{2}{\pi a^2 [J_{n+1}(\mu_{nm})]^2} \int_0^{2\pi} \int_0^a f_1(r,\theta) J_n(\mu_{nm} r) \sin n\theta \, r dr d\theta$$

$$c_{nm} = \frac{2}{\pi a^2 [J_{n+1}(\mu_{nm})]^2} \int_0^{2\pi} \int_0^a f_2(r,\theta) J_n(\mu_{nm} r) \cos n\theta \, r dr d\theta$$

$$d_{nm} = \frac{2}{\pi a^2 [J_{n+1}(\mu_{nm})]^2} \int_0^{2\pi} \int_0^a f_2(r,\theta) J_n(\mu_{nm} r) \sin n\theta \, r dr d\theta.$$

So using this; using this observation we can find out our coefficient Amn, Bnm, Cnm and Dnm and here we can write down this is I am leaving it to you that Anm is written as 2/ pi of a square Jn +1 mu nm whole square * 0 to 2 pi, 0 to a f1 r theta Jn mu nm r cos nr cos n theta * rdr d theta and similarly, bnm, you can write, 2 upon pi a square Jn +1 mu nm whole square 0 to 2pi, 0 to a f1 r theta Jn mu nm r sin n theta * rdr d theta.

Similarly, you can find out cnm and dnm, so it means that with these coefficients our solution is given by this, u r theta t = m, n = 1 to infinity Jn mu nm r Anm cos n theta + B nm sin n theta cos of mu ct + cnm cos n theta + dnm sin n theta sin of mu ct and this will represent the solution of wave equation in a circular membrane; in vibration in circular membrane. So, with this I end our lecture, here in next lecture, we will discuss some more property of wave equation and heat equation, so with this I end this lecture, thank you very much for listening us, thank you very much.