Ordinary and Partial Differential Equations and Applications
Dr. D. N. Pandey
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 54
One Dimensional Wave Equations and its Solutions- II

Hello friends, welcome to this lecture, in this lecture, we will continue a study of wave equation and if you recall in previous lecture, we have discussed the one-dimension wave equation and we discussed the D'Alembert's solution for one-dimension wave equation and now we continue our study from that onward. So, we have discussed the string problem infinite length and the solution is known as D'Alembert's solution.

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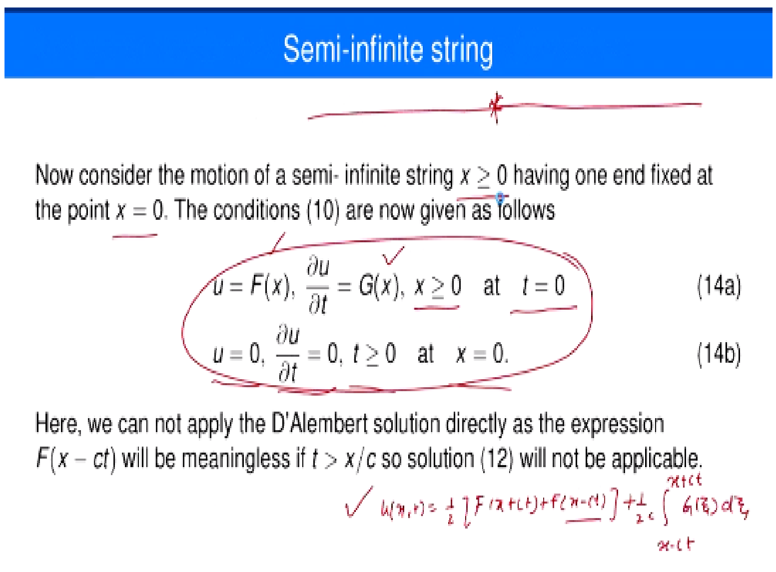

Now, let us move semi-infinite string, so it means that now consider the motion of a semi-infinite string, so here x is lying between 0 to infinity, so x is > = 0 having one end fixed at the point x = 0, so it is like this, you have this point and from here, it is already fixed, so at x = 0, your string is fixed at point x = 0. Now, the condition; initial and boundary condition are given like this that the initial shape is given as u = F of x and initial velocity dou u/ dou t given as G of x at x > = 0 at t = 0.

So, initial displacement and initial velocity is given by this and since one end is fixed at the x = 0, so here we will introduce boundary condition as well, so here at x = 0, your u is = 0 and dou u/

dou t is also 0, so at this fix point, the displacement as well as the velocity is 0, so at t > = 0 and x = 0, so here this is fixed, so your position as well as velocity will remain 0 for all time t > or = 0 and if you look at here, we cannot apply the D'Alembert's solution directly.

Because if you remember your solution will be what; uxt = 1/2 of F of x + ct + F of x - ct and +1 upon 2c x -ct 2x + ct G of xi d xi, and the problem is that here your F of x is defined for x > or = 0, so it means that argument of a F is positive only but and here if we take x – ct, now if x - ct is < 0, then this will not be defined, so it means that I cannot use the solution; D'Alembert's solution for this semi-infinite string right now.

So, what we trying to see here; so let us modify our problem in a bit such that I can apply this D'Alembert's solution for this semi-infinite string also, so what we try to do; we extend our this this part, the other half part in a way such that this problem; the consider problem is a particular part of this extended problem.

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Suppose, we consider an infinite string subject to the initial conditions

$$u = Y(x), \quad \frac{\partial u}{\partial t} = V(x) \quad \text{at} \quad t = 0,$$

where

$$Y(x) = \begin{cases} F(x), & x \geq 0; \\ -F(-x), & x < 0. \end{cases}$$

And

$$V(x) = \begin{cases} G(x), & x \geq 0; \\ -G(-x), & x < 0. \end{cases}$$

Then its displacement is given by

$$u = \frac{1}{2}\{Y(x+ct) + Y(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} V(\xi)d\xi \qquad (15)$$

So, here what we do, suppose we consider an infinite string subject to the initial condition. so, let us say that consider an infinite string where displacement is given as Y of x, initial velocity is given as V of x at t = 0, so here since it is an infinite string let us say that your Y of x is given as F of x as x > or = 0, so this is matching with your condition, which is already given as for this semi-infinite string problem.

So, Y of x is = F of x, when x >= 0, now we are extending finding these Y of x as a odd extension of this function F of x, so Y of x = - of F –x, when x is < 0, so it means that when is < 0, so –x is positive, so – of F of -x is defined here, so we can say that here Y of x is an odd extension of this function F of x. Similarly, we can extend our function g of x in odd manner for the entire –infinity to infinity and we defined V of x = G of x for x > 0 minus.

And – of G – of x, when x is < 0, so now once we define your yx and vx for the entire infinite region, so it means that then I can apply D'Alembert's solution for this infinite string and our solution is given as u = 1/2 y of x + ct + y of x -ct+1 upon 2c x – ct to x + ct V xi d xi, now our claim is that with the expression Y and V given as follows, this will act as a solution of the semi-infinite problem also.

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so that when $x = 0$

$$u = \frac{1}{2}\{Y(ct) + Y(-ct)\} + \frac{1}{2c}\int_{-ct}^{ct} V(\xi)d\xi \qquad (16)$$

and

$$\frac{\partial u}{\partial t} = \frac{1}{2}c\{Y'(ct) - Y'(-ct)\} + \frac{1}{2}\{V(ct) + V(-ct)\}$$

From the definitions of Y and V, both these functions are identically zero for all values of t and therefore the function (16) satisfies the condition (14b) as well as the differential equation (8) and also satisfies the condition (14a).

So, we need to see it will satisfy all the initial and boundary conditions that we have to look at here. So, here look at; at x = 0, so at x = 0, your uxt will be what; uxt = 1 upon 2 Y of ct + Y of –ct +1 upon 2c – ct to ct V xi d xi and dou u/ dou t you can calculate 1/2c y dash ct – y dash – ct +1/2 V ct + V of –ct. Now, as we have defined the function Y as odd function it means that Y of –x is - of Y of x, so this quality will be 0.

Similarly, V xi is also defined in an odd manner, so it means that this integral is going to be your 0, similarly you can say that this y dash ct – y dash – ct here, I am assuming that Y of - x = - of Y of x, so when you differentiated your, what you will get? Y dash of –x, -1 here = - of Y of x; Y dash x. So, using this, this quantity is 0 and this quantity is 0, so it means that u xt; u0; t is going to be 0 and dou u/ dou t at x = 0 is going to be 0.

And also that if you look at your solution, which is defined as 15, it will also satisfy the initial condition that is u of xt is = F of x; u of x0 = F of x for x > = 0, so it will also satisfy the initial condition as well as the initial velocity here. So, it means that the solution given by this equation number 15 will satisfy all the initial condition and the boundary condition given at x = 0 and so we can take the solution given in 15 as the solution of semi-infinite problem.

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In particular, if string is released from rest so that initial velocity G, and hence, V, is identically zero, we find the corresponding solution is

$$u = \begin{cases} \frac{1}{2}[F(x+ct) + F(x-ct)], & x \geq ct; \\ \frac{1}{2}[F(x+ct) - F(x-ct)], & x \leq ct. \end{cases}$$

And in this way our D'Alembert's solution is still applicable for semi-infinite problem as well, now in particular if a string is released from rest, so that initial velocity G is taken a 0 and if initial velocity G is 0, so V is also coming to be 0 and your solution u is defined as 1/2 F of x + ct + F of x – ct, when x is > or = ct, so argument x - ct is positive, x + ct is positive, so I can define uxt as this.

But when x is < or = ct then I can define as F of x + ct – F of x – ct, so here this is the small change you can say from the infinite string problem to semi-infinite problem. Now, let us move to finite string, so what we have done? We have considered at first the infinite string and we obtained a solution which is known as D'Alembert's solution and then we consider a semi-infinite problem.

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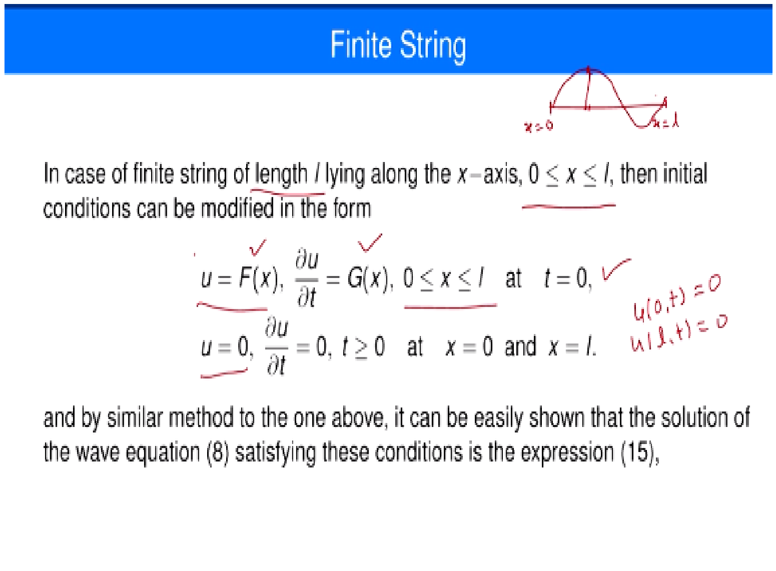

It means that a string is fixed at one end and then we modify our solution of infinite string to get a solution for semi-infinite string, now let us do the similar thing for finite string case, so in case of finite string of length l, lying along the x axis where x is between 0 to l, so here, your string is now clamped at 2 and say it is x = 0 and x = l. So, here now you we want to see the vibration in this string, so here suppose initial condition is given as u F of x.

So, it means that initial shape is something like this, so this will represent the at this point your displacement from the initial condition is F of x, so at any point if you look at you can find out the displacement, let us say that initial displacement is given by u of F of x and initial velocity is given by G of x and it is true for all x lying between 0 to l and t = 0, so initial displacement and initial velocity is given.

Now, since we assume that both the end are fixed line, right now for this particular problem, so it means that u0t is = 0 and dou u/ dou t at lt is = 0, so it means that u of 0t is = 0 and u of lt is = 0 here, so it means that because here we assume that your ends point are fixed here, now we want to find out the solution for this finite string problem and we try to see that the solution obtained for infinite string can be modified in a way that we can obtain a solution for this finite string as well.

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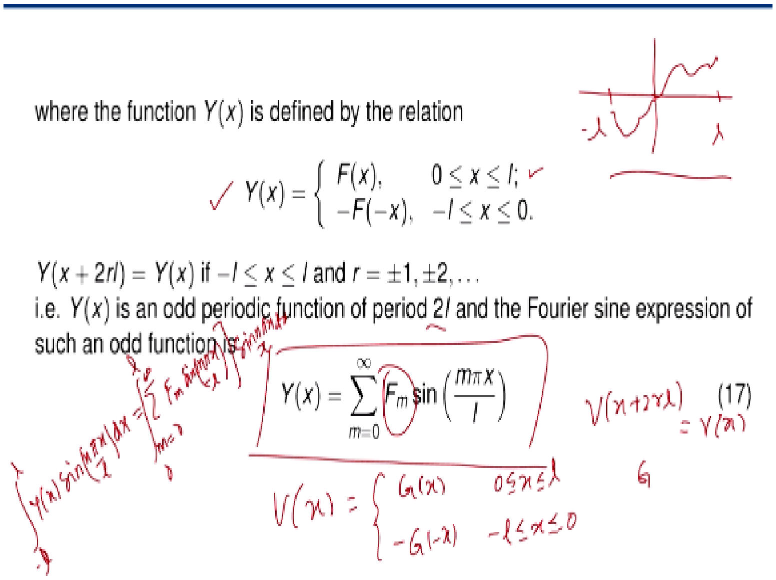

So, here we; as we discuss for semi-infinite string, let us modify our function Y of x as F of x lying between 0 to l and - of F of -x between –l to 0, so what we try to do here; we write it –l to this l, so here your function is something some function is given, so at 0, it is 0 only, so here some function is given. Now, you define in an odd manner, such that here you will get the similar thing here.

So, you define Y as a odd function; odd extension of F of x in –l2 0 also, so it means that now Yx is defined between –l to l and Y of x is = F of x in 0 to l and Y of x is = - of F of –x lying between –l to 0. Now then, you since it is given only for –l to l, then you use periodicity to; extend it to entire –infinity to infinity, so let us assume the periodicity as Y of x + 2rl = Y of x, where x is lying between –l to l and r is your integer values + -1 and + -2 and so on.

So, Yx is an odd periodic function of period 2l and the Fourier sin expression of such an odd function, you can find out like Y of x = m = 0 to infinity F of m sin m pi x/ l. Similarly, you can define your V of x as follows, let us say G of x, 0 < x < l and - of G of -x –lx < 0 and here also you can consider the corresponding Vx + 2rl = V of x, where x is lying between –l to l and r is + -1+ -2.

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where the coefficients F_m are given by the formula

$$F_m = \frac{2}{l}\int_0^l F(\xi)\sin\left(\frac{m\pi\xi}{l}\right)d\xi \quad (18)$$

Similarly,

$$V(x) = \sum_{m=1}^{\infty} G_m \sin\left(\frac{m\pi x}{l}\right) \quad (19)$$

where

$$G_m = \frac{2}{l}\int_0^l G(\xi)\sin\left(\frac{m\pi\xi}{l}\right)d\xi \quad (20)$$

And similarly, as Y is having a Fourier sine expression, similarly you can define the Fourier sine expression for V of x also and we can say that sorry; V of x can be written as m = 1 to infinity Gm sin m pi x/ l and here, the Fourier coefficient that is F of m and G of m, you can find out using orthogonality property of sin and cosine functions, so sin function here. So, here if you remember here, what is the solution here; yx is given by this.

So, if I want to find out Fm, then what you will do? You just multiply by sin m pi x/ l by, so you you have Y of x sin ni pi x /l d of x and you integrate between the 0 to; -l to l, right and see what you will get here, it is summation m = 0 to infinity F of m sin m pi x/l * 0 to l and then here you will get with in product sin of n pi x/l, d of x and then use of orthogonality property of sin n pi x/l and you will see that your F of m you can obtain by this.

(Refer Slide Time: 14:14)

Substituting the results, we get

$$\frac{1}{2}\{Y(x+ct)+Y(x-ct)\}=\sum_{m=0}^{\infty}F_m\sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{m\pi ct}{l}\right)$$

$$\frac{1}{2c}\int_{x-ct}^{x+ct}V(\xi)d\xi=\frac{l}{\pi c}\sum_{m=0}^{\infty}\frac{G_m}{m}\sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{m\pi ct}{l}\right)$$

which follow from these expressions, into the solution (15), the solution of the present problem is

$$u=\sum_{m=1}^{\infty}F_m\sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{m\pi ct}{l}\right)+\frac{l}{\pi c}\sum_{m=1}^{\infty}\frac{G_m}{m}\sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{m\pi ct}{l}\right)$$

where F_m and G_m are defined by equations (18) and (20) respectively.

It is 2/l 0 to l F of xi sin m pi xi/l d xi, similarly you can obtain Gm that is 2/l 0 to l G xi sin m pi xi/l d xi and if you look at both Y and V are similar to each other in fact, it is kind of symmetry, so once we have the expression for Y and V, we can write, we can substitute our relation, so here our solution 1/2 Y of x + ct + Y of x – ct, you can write it m = 0 to infinity F of m sin m pi x/ l sin m pi ct/ l.

So, here we simply use expression Y of x as this and when you have this then you can write Y of x as ct and 0 to infinity m = 0 to infinity F of m sin of m pi x – x + ct/ l, similarly you can write Y of x - ct and then you simplify, then you can use sin of a+ b + sin of a - b formula and you can get the formula; series expression like this, so we have this part, we have this part, so we can write down our solution uxt as m = 1to 1 to infinity Fm sin m pi x/ l cos of m pi ct/l.

This is corresponding to 1/2 Y of x + ct + Y of x - ct and then corresponding to integral part, 1 upon 2c x -ct x + ct V xi d xi, we can write down this expression and we have this, so we can write ux t as m = 1 to infinity Fm sin m x pi/ l cos m pi ct/l = l/ pi c m = 1 to infinity Gm/ m sin m pi x/ l sin m pi ct/ l and here the coefficient Fm and Gm, you can obtain using the expression 18 and this 20.

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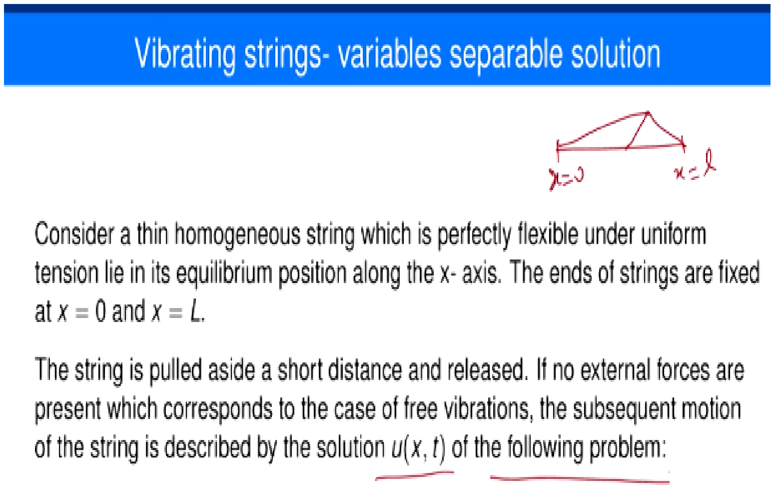

So, it means that solution is given as this, so now we have obtained this solution using D'Alembert's solution for infinite string and we modify our problem in a way such that we can utilise this D'Alembert's solution for finite string as well. Now, let us look at the same problem and now you try to find out the solution using variable and separable method. So, here we consider a thin homogeneous string which is perfectly flexible under uniform tension lie in the equilibrium position along the x axis.

The ends of a string are fix at x = 0 and x = L that lying in it at x =0 here and x = l here, the string is pulled aside a short distance, so once we have this thing then you simply pull along one side, then it is something like this kind of figure and released and if no external forces are present which correspond to the case of free vibrations, so it means that here once we put it like this and then we simply leave.

So, it means that no other force we; say, apply after that, so it means that at initial position, it is simply left, then we try to find out the vibration of this string and the subsequent motion of the string is described by the solution uxt of the following problem.

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PDE:

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 \leq x \leq L, \quad t > 0 \qquad (21)$$

Boundary conditions:

$$\begin{aligned} u(0,t) &= 0, t > 0 \\ u(L,t) &= 0, t > 0 \end{aligned} \qquad (22)$$

Initial conditions:

$$u(x,0) = f(x), u_t(x,0) = g(x) \qquad (23)$$

The following problem is this, it is PDE utt – c square uxx = 0, x is lying between 0 to L, t is > 0 and boundary conditions are what; if you look at we have taken the end fixed, so x = 0 u0t is 0 and ult is = 0 and initial condition is given as ux0 = F of x then utx0 is some G of x, so here we consider the same problem but now we try to solve in a different manner using separation and variable method.

(Refer Slide Time: 18:17)

To obtain the variables separable solution, we assume

$$u(x,t) = X(x)T(t) \qquad (24)$$

and substituting into equation (21), we obtain

$$X\frac{d^2T}{dt^2} = c^2 T\frac{d^2X}{dx^2}$$

i.e.

$$X\frac{d^2T/dt^2}{c^2T} = \frac{d^2X/dx^2}{X} = k \quad (a \ \ separation \ \ constant)$$

So, to obtain the variable separable solution, we assume that uxt is = Xx T of t, now here, please remember here that you cannot apply your variable separable method for any problem, here you need to have your problem separable as well as the boundary separable, so it means that you have to choose your coordinate system in a such that your equation is separable as well as the boundary is also separable.

We will see that here both the equation as well as the boundary condition are separable to each other, so let us see what I mean. So, here if you put uxt = Xx Tt, then your solution is what? It is your problem is what; utt = c square uxx then you will write T double dash x = c square x double dash T divide by xt, so you will get T double dash upon c square T = X double dash upon X that is what it is written here.

So, X d2T/dt square = C square T d2X/ dx square and then you divide by c square XT and we have this equation that d2T/ dt square divided by c square T = d2X/dx square divided by X = k here.

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Case I When $k > 0$, we have $k = \lambda^2$. Then

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0$$

$$\frac{d^2T}{dt^2} - c^2\lambda^2 T = 0$$

Their solution can be put in the form

$$X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} \tag{25}$$

$$X(x) = c_3 e^{c\lambda t} + c_4 e^{-c\lambda t} \tag{26}$$

Therefore,

$$u(x,t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}) \tag{27}$$

Now, here this k is a separation constant, so it means that we can write our equation like this, d2X/ dx square – lambda square X = 0 and d2T upon dt square – c square lambda square T = 0. Now, here let us consider 3 cases for these constants here; so, k let us assume that k is positive, negative and 0 and we try to find out the nontrivial solution in each cases, so here basically it is a Eigen value problem, system value problem.

And we have already discuss the similar system value problem, when we discuss the boundary value problem, so now we can find out the solution of these problems as X of x = c1 e to the power lambda x + c2 e2 to the power – lambda x and similarly, you can write down tt as c3 e to the power c lambda t + c4 e to the power c lambda t, so now our solution uxt is written as c1 e to the power lambda x + c2 e to the power –lambda x * c3 e to the power c lambda t + c4 e to the power –c lambda t here.

(Refer Slide Time: 21:02)

Now, using the boundary conditions, we have

$$u(0,t) = 0 = (c_1 + c_2)(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}) \quad (28)$$

which implies that $c_1 + c_2 = 0$. Also, $u(L,t) = 0$ gives

$$c_1 e^{\lambda L} + c_2 e^{-\lambda L} = 0 \quad (29)$$

Now, we have to obtain 4 constants; C1 C2 C3 C4, so here we will use the boundary condition, so boundary conditions are what? It is u0 t = 0 and ult = 0, so when you put your solution this X of 0, T of t = 0 and similarly X of l, T of t = 0 here, so here since T of t cannot be identically = 0, so this implies that X of 0 is = 0 and X of l = 0, so using this you can solve this problem, your X of x is what; X of x is = c1 e to the power lambda x + c2 e to the power –lambda x.

So, it is c1 e to the power lambda x + c2 e to the power –lambda of x, so use X0 = 0, so what you will get? It is 0 = c1 + c2 and xl0, it means this is 0 = c1 e to the power lambda l + c2 e to the power –lambda l. Then, you want to find outc1, c2, so c1 c2, you can find out using this matrix equation, this is 0 0 and you can get here 1 1 and e to the power lambda l e to the power -lambda l and if you see that this determinant is non-zero, so we can say that here I will get only a trivial solution that is c1 = c2 = 0.

(Refer Slide Time: 22:20)

Equations (28) and (29) possess a non- trivial solution iff

$$\delta = \begin{vmatrix} 1 & 1 \\ e^{\lambda L} & e^{-\lambda L} \end{vmatrix} = e^{-\lambda L} - e^{\lambda L} = 0$$

Since $L \neq 0$ so $e^{-\lambda L} - e^{\lambda L} = 0$ implies that $\lambda = 0$ which is against the assumption as in Case I. Hence, solution is not acceptable.

So, it means that in this case, we will have only a trivial solution, so if since this value is a nonzero implies that lambda has to be 0, so this value determinant has to have value 0 provided u lambda is = 0 but lambda; we have assumed that lambda is nonzero because here we have assumed that k is > 0 here, so it means that lambda cannot be 0 but this determinant is having 0 value when lambda is = 0.

So, it means that it is against our assumption in this case, so hence our solution is non-acceptable means, your; in this case when k is positive, we will not get any nontrivial solution, so here what we have seen here that in case when k is positive, we do not have any nontrivial solution.

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Case II Let $k = 0$. Then we have

$$\frac{d^2X}{dx^2} = 0, \frac{d^2T}{dt^2} = 0$$

Corresponding solutions are:

$$X(x) = Ax + B, T(t) = ct + D$$

Therefore, the required solution of the PDE (21) is

$$u(x, t) = (Ax + B)(ct + D)$$

Using the boundary conditions, we have $u(0, t) = 0 = B(ct + D)$ implying $B = 0$
$u(L, t) = 0 = AL(ct + D)$ implying $A = 0$
Hence, only trivial solution is possible.

So, now let us consider next case that is let k is = 0 here and in this case our equation is reduced to d2X/dx square = 0 and d2T/dt square = 0 and in this case you can find out your solution as X of x = Ax + B and Tt = ct + D, again as we have pointed out the solution uxt is written as Ax + B * ct + D here.

(Refer Slide Time: 23:58)

$$X(0) = 0 = X(l)$$

$$X(x) = Ax + B.$$

$$0 = B \qquad 0 = Al$$

$$A = 0$$

$$X(x) \equiv 0$$

$$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$

$$X(0) = 0 = c_1$$

$$X(l) = 0 = c_2 \sin \lambda l$$

$$\lambda l = n\pi$$

$$\lambda = \frac{n\pi}{l}, \quad n = \pm 1, \pm 2, --$$

Now, our initial condition is what; initial condition we have seen that it is same; x of 0 = 0 = x of l, so our solution is X of x = say, Ax + B here, so put x0 = 0, so this implies that B = 0 and then when you put xl = 0, then it is Al = 0 and since l is nonzero, so this implies that A is = 0, so here this X of x is coming out to be identically = 0. So, in this case using the boundary condition we have u0t = 0 * B ct + D implies that B = 0 and similarly, we can say that A = 0.

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Case III When $k < 0$, say $k = -\lambda^2$, the differential equations are

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0$$
$$\frac{d^2T}{dt^2} + c^2\lambda^2 T = 0$$

Their general solution give

$$u(x, t) = (c_1 \cos \lambda x + c_2 \sin \lambda x)(c_3 \cos c\lambda t + c_4 \sin c\lambda t) \qquad (30)$$

So, here also this part is completely 0, so it means that uxt is only a 0 solution, so it means that when k = 0 also, then we do not have any nontrivial solution, so this case is also, we will leave. Now, let us consider the third case that is k is < 0, so in this case let us assumed that k is = -lambda square and the corresponding differential equation is d2X/ dx square + lambda square X = 0 and d2T/dt square + c square lambda t = 0.

So, here general solution is given as corresponding to this general solution is C1 cos lambda x + c2 sin lambda x and corresponding to this, we have c3 cos c lambda t + c4 sin c lambda t is given, now you try to fix your C1 and C2 and you can see that how we can fix this C1 and C2, so here let us say we have a problem C1 cos of lambda x + C2 sin of lambda x and that is your X of x, so let us put term X of 0 = 0 that implies that your C1 has to be 0.

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Using the boundary conditions: $u(0, t) = 0$, we obtain $c_1 = 0$. Using the boundary conditions: $u(L, t) = 0$, we obtain $\sin \lambda L = 0$ implying that $\lambda_n = n\pi/L$, $n = 1, 2, \ldots$, which are the eigenvalues. Hence the possible solution is

$$u_n(x, t) = \sin \frac{n\pi x}{L}\left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L}\right), n = 1, 2, \ldots \quad (31)$$

Using the superposition principle, we have

$$u(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L}\left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L}\right) \quad (32)$$

Then X of l = 0 implies that C2 sin of lambda l = 0 and if we take this C2 is 0, then we will again get a trivial solution, so we want that this lambda l to be chosen like here n pi so, let us say lambda is = n pi/l, where n is taking as + -1+ -2 and so on, so it means that if we take lambda as n pi/l, then we will get a nontrivial solution, so it means that let us see this implies that here you will get a nontrivial solution provided this lambda is n pi/l, which is given here that using the boundary condition u0t = 0, we get c1 = 0 as we have pointed out.

And using the boundary condition, ult = 0, we have sin lambda l = 0, this implies that lambda is written as n pi/L, where n is 1, 2, 3 and any integer value, which are the eigenvalues corresponding to the Eigen function sin lambda. So, here you can write down the solution as u and xt = sin n pi x/L An cos n pi ct/ L + Bn sin n pi ct/L, here I am using the value lambda as n pi/L in this case here, you use C1 is 0, so this is gone.

And here sin n pi/Lx is there, you just replace the value of lambda and since this is true for every n, so we can write down for every n, we can write down the solution u and xt as sin n pi x/L An cos n pi ct/L + Bn sin n pi ct/L, now since for each n, we have a solution then of the homogeneous problem, then you should take this summation from n = 1 to infinity, then also it will remain the solution of this homogeneous problem.

So, using; this is known as superposition principle that if we have a homogeneous problem and we have v1 and v2 at 2 solutions, then linear combination of v1 and v2 will also be a solution of the homogeneous problem and this is known as a superposition principle, so using superposition principle we can write down our solution as uxt = n = 1 to infinity sin n pi x/ L An cos of n pi ct/L + Bn sin n pi ct/L.

(Refer Slide Time: 28:19)

The initial conditions give

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

which is a half- range Fourier sine series, where

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx \qquad (33)$$

Now, here your coefficient An and Bn is still unknown, so to find out the coefficient of An and Bn, we will use our initial condition that is ux0 = f of x, when you put t= 0, then this part is gone, so you will get An cos of n pi ct/L is going to be 1, so we will get ux0 = An sin n pi x/L, so that is what we have written; ux0 = n = 1 to infinity An sin n pi x/L and we already know the value of ux0 that is given as f of x.

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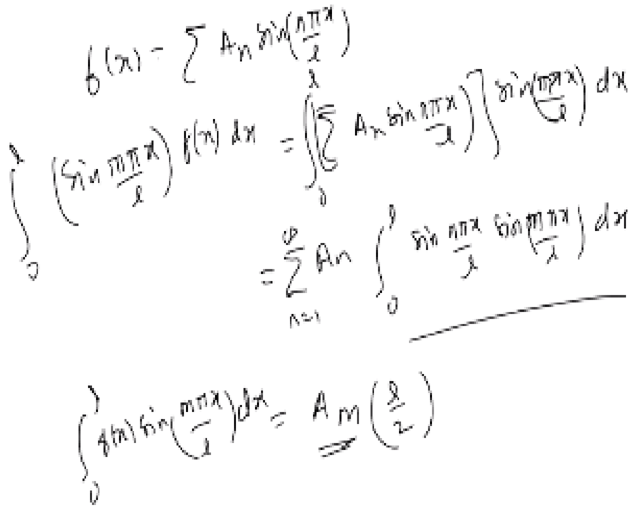

So, now with the help of this we try to find out the value of An, so how to find out value of An, so here you just multiply the sin m pi x/L, so let us consider this problem, so here we have your f of x = summation An sin n pi x/L, so multiply both side by say, sin sin m pi x/L * F of x dx and integrate between 0 to L here, which is given as summation integrate between 0 to L An sin n pi x/L * sin m pi x/ L and dx and we say that this I can write as summation An assuming that this infinite series is uniformly convergent.

So, we can write it here n = 1 to infinity An 0 to L sin n pi x/L sin m pi x/ L dx of here, so using the orthogonality, we say that fm is !=n, then this value is = 0 and when m is = n or you can say the value An, this n is = m then this value is coming out to be l/2, so we can write the value Am L/2 is = 0 to l f of x sin of m pi x/ l dx, so in this case we can find out the coefficient Am has 2/l 0 to l fx sin m pi x/ l, so that is what we have written here An = 2/ L 0 to L fx sin n pi x/ L dx.

(Refer Slide Time: 30:53)

Also,

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\left(\frac{n\pi c}{L}\right)$$

which is also a half- range Fourier sine series, where

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx \qquad (34)$$

Hence the required solution is obtained from equation (32), where A_n and B_n are given by equations (33) and (34). $u_n(x,t)$ given by equation (31) are called normal modes of vibration and $\omega_n = n\pi L/c$, $n = 1, 2, \ldots$ are called normal frequencies.

Similarly, using the next condition that is utx0 = g of x and that you can get it from n = 1 to infinity Bn sin n pi x/ L * n pi c/L, so that we have obtained from this equation, you differentiate with respect to t and put t = 0, you will get this relation that utx0 = n = 1 to integral Bn sin n x pi/L * n pi c/L, now this value is given as g of x, again using the orthogonality property we can write Bn as 2/ n pi c is = L gx sin n pi x/ L dx.

So, here we required the solution as obtained by say this equation number 32 that is uxt = n = 1 to infinity sin n pi x/L An cos n pi ct/L + Bn sin n pi ct/L and the coefficient An and Bn is obtained in the equation number 33 and equation number 34, so it means that once we have An, Bn given to us, then we can write down the solution for this finite string problem as equation number 32 I think, 32.

(Refer Slide Time: 32:11)

Uniqueness of the solution for the wave equation

As we have developed the variables separable method to find the solution to the wave equation with certain initial and boundary conditions. In this part, we shall show that the solution of one dimensional wave equation with finite domain is unique.

And here the un xt is given by equation 31 are called normal modes of vibration and the corresponding frequency omega n is = n pi L/c are called normal frequency. Then, the next result is uniqueness of the solution for the wave equation, so as we have discussed the variables separable method to find the solution to the wave equation with certain initial and boundary condition for say, finite boundary value problem.

So, it means that in a finite string which is say, fix at the end x = 0 and x = n, we have found out the solution using variable separable method but if you remember we have also discussed the same problem and we found the solution using D'Alembert's solution of string and we came to know that both are coming as same. So, now how it is coming and so here in this particular part, we prove that the solution of one-dimension wave equation with finite domain is a unique solution.

So, it means that whether I will use D'Alembert's solution or say, variable separable method, our solution is coming to be unique, so it is up to you whether you will choose D'Alembert's solution or say, this variable separable method, the solution is coming to be same here.

(Refer Slide Time: 33:35)

Uniqueness Theorem.

The solution to the wave equation

$$u_{tt} = c^2 u_{xx} + F(x,t), \quad 0 < x < L, \quad t > 0 \qquad (35)$$

satisfying the initial conditions:

$$u(x,0) = f(x), 0 \leq x \leq L$$
$$u_t(x,0) = g(x), 0 \leq x \leq L$$

and the boundary conditions:

$$u(0,t) = f_1(t), \; u(L,t) = f_2(t),$$

where $u(x,t)$ is twice continuously differentiable function with respect to x and t, is unique.

So, let us do this problem, so this is a uniqueness theorem; the solution to the wave equation, utt = c square uxx + f of xt, here x is lying between 0 to L t> 0 satisfying the initial condition that is ux0 = f of x; x is lying between 0 to L, utx0 = g of x; 0, x lying between 0 to L and boundary condition uot = f1t, uL t = f2t, where uxt is a twice continuously differential function with respect to x and t is unique.

Now, here you note down certain things that here we are writing this function F of xt, it means that we are also considering the external force, it means that it is not say, this string is not left; say, at initial position f of x having initial displacement F of x and initial velocity is g of x but for every x and t we are also applying an external force which is given as f of xt and also that here your end points are not fixed in fact, end points are given as f1t and f2t.

It means that the endpoints are not fixed in fact, that is also depending on some functions, say f1t, f2t so at t is increasing, your initial point; initial displacement is also the endpoint is also moving kind of thing, so it is a moving end point problem and we say that if it is a; if it has a solution that solution must be unique, now the case which we have discuss earlier is a special case of this by assuming that f of xt is 0 and f1t and f2t are 0.

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Proof. Suppose u_1 and u_2 are two solutions of the given wave equation (35) and let $v = u_1 - u_2$. Then $v(x,t)$ is the solution of the following problem:

$$v_{tt} = c^2 v_{xx}, \quad 0 < x < L, \quad t > 0 \tag{36}$$

$$v(x,0) = 0, 0 \leq x \leq L$$
$$v_t(x,0) = 0, 0 \leq x \leq L$$

and

$$v(0,t) = v(L,t) = 0,$$

We have to prove that $v(x,t) \equiv 0$. Consider the function

$$E(t) = \frac{1}{2}\int_0^L (c^2 v_x^2 + v_t^2)dx \tag{37}$$

which represents the total energy of the vibrating string at time t.

So, it means that it is a special case of this uniqueness theorem, so if this problem has a unique solution the special case will also have a unique solution, so now let us try to prove this result, so here, suppose u1 an u2 are 2 solution of the given wave equation 35 and satisfying these initial and boundary condition. Then suppose that we have 2 solutions and now we want to show that these 2 solutions are identically same.

So, let us assume that v = u1 – u2, then this function v is a solution of the following problem; vtt = c square vxx, x is lying between 0 to L, t is > 0, v x0 is 0, v tx0 = 0 and v0t and vLt is coming out to be 0, so it means that if you look at here our problem is what?

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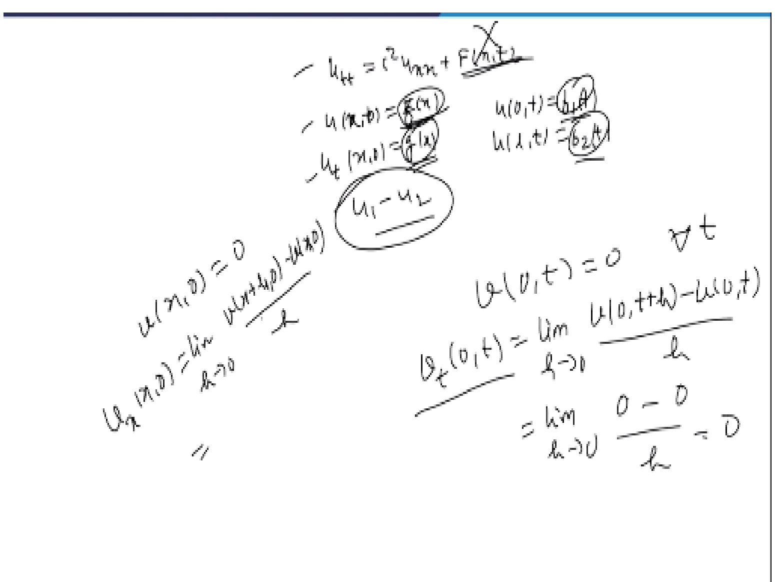

Our problem is utt = c square uxx + f of xt and u0t is going to f of x; u f of x and ut; sorry, ufx0 is = ut of x0 = g of x u of 0t = f1 t and u of lt = f2t. So, now you let us say, u1 and u2 are 2 solutions, so it means that you u1 will satisfy the same equation and u2 will also satisfy the same equation, then when you look at the u1 – u2, then this part will be cancel out, this part will also cancel out because it is independent of u1 * u2.

So, it means that now you can write it here that the corresponding solution satisfy by u1 – u2 will not have this part, this part, this part and this part, it means that the corresponding difference; u1 – u2 will satisfy the homogeneous problem like vtt = c square vxx and the homogeneous initial and boundary condition. Now, we want to show that here your vxt is identically = 0 and to prove that here, we consider the function ET = 1/2 0 to L c square vx square + vt square dx.

Now, we want to show that how we show that vxt is 0; by showing that its partial derivative r0 and how we can show that partial derivative r0, here we show that this function Et is coming, v0, so it means that if this function which is known as generalised energy function. If generalised energy function is 0, it means that your string is lying in an equilibrium position that is the idea which we have started.

That if we do not have energy, then we like to be in equilibrium position, so here let us say that Et is = 1/2 0 to L c square vx square + vt square dx and which represent the total energy of the vibrating string at time t.

(Refer Slide Time: 38:09)

Differentiating both sides of (37), we get

$$\frac{dE}{dt} = \int_0^L (c^2 v_x v_{xt} + v_t v_{tt})dx \tag{38}$$

Integrating by parts, the right- hand side of the above equation gives us

$$\int_0^L c^2 v_x v_{xt} dx = [c^2 v_x v_t]_0^L - \int_0^L c^2 v_t v_{xx} dx$$

Now, $v(0, t) = 0$ implies $v_t(0, t) = 0$ for $t \geq 0$ and $v(L, t) = 0$ implies $v_t(L, t) = 0$ for $t \geq 0$. Hence, equation (38) reduces to

$$\frac{dE}{dt} = \int_0^L v_t(v_{tt} - c^2 v_{xx})dx = 0$$

Now, differentiating both sides of 37, with respect to t, so dE/dt = 0 to L, now here when you differentiate c square vx square, so it is 2vx vxt and here it is 2vt vtt, so we will write it here as 0 to L c square vx vxt + vt vtt dx, now integrating by part; this part with respect to say, dx. So, when we have 0 to L C square vx vxt dx, this we write first indication of second, so we will get c Square vx vt between 0 to L – 0 to L c square vt and vxx.

So, here indicating this thing and differentiating this, so we will get 0 to L c Square vt vxx dx, now if you look at this boundary term; c square vx vt lying between 0 to L and our claim is that it is 0, how we can show, since v is 0 t is 0, so it means that if v is 0, t is 0, then vt0 t is also 0, why? So, here if you look at vot is 0, means what; v0 t is = 0 for all t, so it means that if you want to calculate vt0 t, then it is what limit?

Say, S tending to 0, v of 0, t + h – v of 0t divided by h, now since it is true for all t, so this implies that limit s tending to 0, this value is also 0, this is already 0, so it is nothing but 0, so vt0 t is coming out to be 0, so here vt0 t is 0 and since vlt is = 0, since in a same manner, we can say vt lt0 is also 0, so it means that this boundary term is gone, is that okay, so it means that our equation number 38 is reduced to d/dt = 0 to L.

In place of 1, we are writing here -0 to L c square vtt vxx, so if we take out this vt term out because here we have vt vx and here it is vt vtt, so we can write vt times vtt – c square vxx. Now, if you look at; v is a solution of homogeneous problem, it means that vtt – c square vxx has to be 0, so it means that since v is a solution of homogeneous problem, this dE/dt is coming out to be 0, so it means that E is constant with respect to time t.

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Hence,

$$E(t) = constant = c \ \ (say).$$

Since $v(x,0) = 0$ implies $v_x(x,0) = 0$ and $v_t(x,0) = 0$, we can evaluate c and find that

$$E(0) = c = \int_0^L \left[c^2 v_x^2 + v_t^2\right]\Big|_{t=0} dx = 0$$

which gives $E(t) = 0$, which is possible if and only if $v_x = 0$ and $v_t = 0$ for all $t > 0$, $0 \leq x \leq L$, which is possible if and only if $v(x,t) = constant$.

However, since $v(x,0) = 0$, we find $v(x,t) = 0$. Hence,

$$u_1(x,t) = u_2(x,t).$$

This shows that the solution of the wave equation is unique.

So, here Et is constant, let us say constant value is c, now if Et is constant and we already know that if vx0 is 0, so vxx0 is 0. in a similar manner we can prove that if vx0 is 0 for all x, then the derivative of vx0 with respect to x is again 0, so this also we can see like this. If vx0 is 0, this imply we want to find out vx of x is 0 that is limit say h tending to 0, v of x + h, 0 – v of x , 0 divided by h, now this is 0, this is 0.

So, it is coming out to be 0, so it means that vxx, 0 is 0, vt x, 0 is 0, so we can calculate e of 0; e of 0 will be what? E of 0 is = what; 0 to L c square vx0 0t and vt square 0t, so when you do this your this value is coming out to be 0, this value is coming out to be 0 and hence, E of 0 is coming out to be 0, but since Et is constant, so it means that this constant value is coming out to be 0.

So, it means that Et is 0, for all t, it means that this is possible only when the integrant; integrant is what? Integrant is c square vx and vt square, these are 0 because integrant is nonnegative value, so it means that vx is 0, vt is 0, so it means that vt is 0 and vx is 0 for all t > 0 and if partial derivative are 0, so it means that we say that it is possible only when vxt is constant with respect to x as well as with respect to t.

So, it means that vxt is constant but we already know that vx0 is 0, so it means that this constant value has to be 0, so it means that vxt is 0 for all x and for all t, so it means that if vxt is = 0 0 means, vxt is what; difference of 2 solutions; it means that difference of 2 solution is identically = 0. So, it means that u and xt is = u2 xt, so this shows that the solution of the wave equation for finite domain is unique.

So, whether you apply the separable variable method or say, D'Alembert's wave solution method both are coming out to be same, so with this I end our lecture and we will continue our lecture in next lecture, so thank you for listening us, thank you very much.