Ordinary and Partial Differential Equations and Applications
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Lecture - 51
Laplace Equation - III

Hello friends. Welcome to my lecture on Laplace equation. This is third lecture on Laplace equation. We will be considering Laplace equation in spherical coordinates in this lecture.

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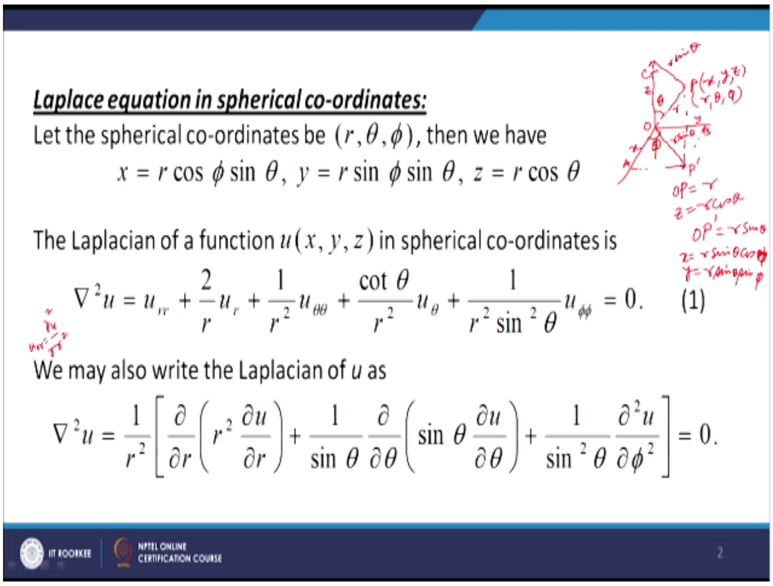

We know that the spherical coordinates r, theta, phi are related to the Cartesian coordinates x, y, z by the questions x=r cos phi sin theta, y=r sin phi sin theta, z=r cos theta where theta is the polar angle, phi the azimuthal angle. You can see suppose we take this point p here whose Cartesian coordinates are x, y, z and spherical polar coordinates are r, theta phi. Then, this is your orthogonal projection p dash, we join origin to p dash.

Then, this is r, draw lines through p dash parallel to x and y axis, then this is x, this is y, so oa is x, ob is y and oz this is c okay, so this is point c, this one okay so this is your oc is z and this angle is phi because phi is the azimuthal angle, this angle is theta okay. So then what we notice is that op is r okay so op is r therefore and this z axis makes angle theta with op so z=r cos theta.

Now when this is r cos theta, oc is r cos theta, this is r sin theta so this is also r sin theta. So op dash is r sin theta and therefore the components of op dash along x and y axis are r sin theta cos phi and r sin theta sin phi. So this is x is r sin theta cos phi and y is r sin theta sin phi and then z is r cos theta and when we find the Laplacian of a function u x, y, z in its spherical coordinates what we notice is that del square u is u rr, u rr is second order partial derivative of u with respect to r as usual okay.

This is first derivative of u with respect to r so 2/r ur 1/r square, then this is second order partial derivative of u with respect to theta. Then, cot theta/r square first order derivative of u with respect to theta, then 1/r square sin square theta u phi phi, u phi phi is second order partial of u with respect to phi. We are taking this without proof because its proof is very, very lengthy.

So we may also write the Laplacian of u. Now you can see the terms u rr+2/r ur can be expressed as 1/r square times partial derivative with respect to r of r square del u/del r and then here the 2 terms 1/r square u theta theta+cot theta/r square u theta is nothing but 1/r square times 1/sin theta partial derivative with respect to theta of sin theta first derivative of u with respect to theta.

And the last term is 1/r square sin square theta u phi phi, so this equation an also written in this form.

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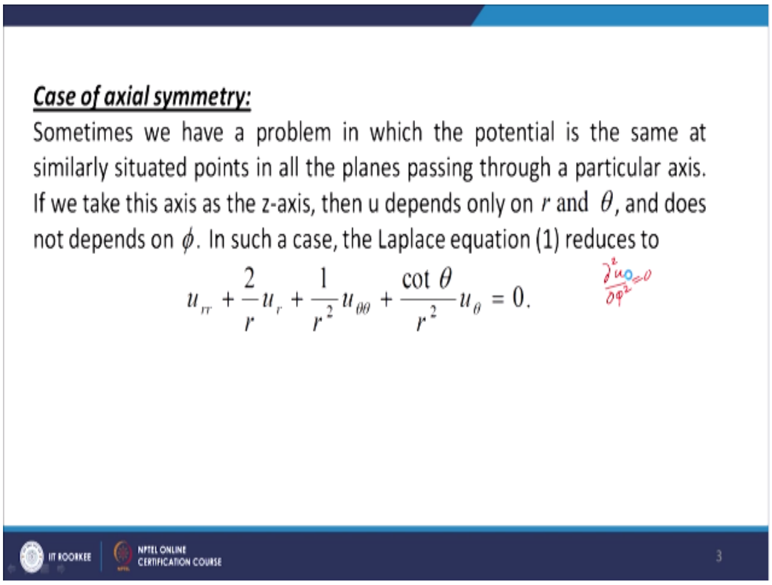

Now we will come across several problems where the potential will be same at similarly situated planes in all the planes passing through a particular axis. So if you take that axis as z axis then you will depend only on r and theta. It will not depend on phi, so in such a case the second order partial derivative of u with respect to phi will be 0 and therefore Laplace equation 1 will reduce to u rr+2/r ur+1/r square u theta theta+cot theta/r square u theta=0.

So we will have this equation, now let us study how to solve this Laplace equation by separation of variables method.

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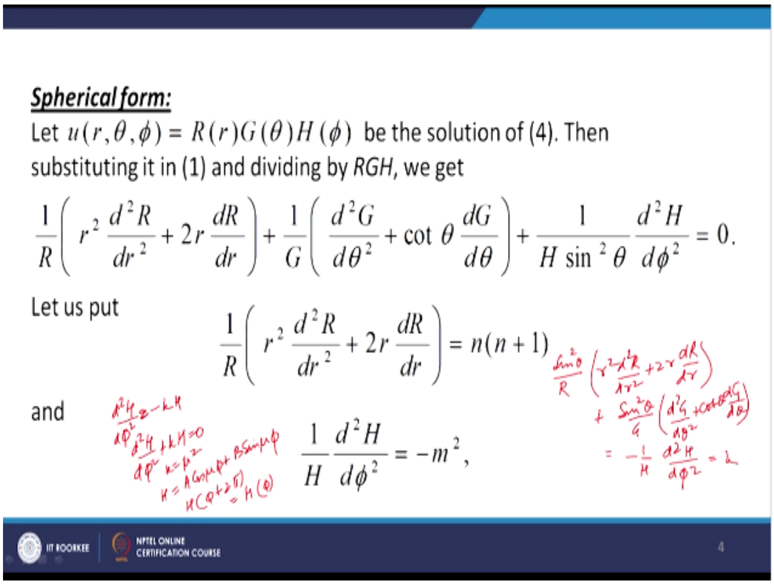

So let us assume the solution of the equation 1 to be product of R r G theta H phi. We are using separation of variable methods, so let us take R to be a function of r only, G to be a function of theta only, H to be a function of phi only and then substitute this u r, theta, phi=RGH into the equation 1 okay. So let this be the solution of equation 1, then substituting it into 1 and dividing by RGH.

You can easily verify that what we get is 1/R r square d square R/dr square+2r dR/dr+1/G d square G/d theta square+cot theta dG/d theta+1/H sin square theta d square H/d phi square. Now this equation can be written as you can multiply by sin square theta, so when you multiply by sin square theta what you get, sin square theta*1/R r square d square R/dr square+2r dR/dr+sin square theta/G*d square G/d theta square+cot theta dG/d theta=-1/H d square H/d phi square.

Now you can see left hand side is the function of r and theta only, right hand side is the function of only phi. So this can be possible only when both are equal to a constant. So let us take this equal to a constant say k then what we will have d square H/d phi square=-kH so we will have d square H/d phi square=-kH or we will have d square H/d phi square+kH=0.

Now the boundary conditions that are given in the case of a physical problem, we shall see that the case k=0 is not possible and the case k=mu square if you take k=mu square you get H to be let us take k=mu square then you will get H=A cos mu phi+B sin mu phi okay, so like what I have written here 1/H d square H/d phi square=-m square okay, so you take k to be mu square then you get the solution.

And for this solution what we will need that because the potential is a single valued function okay. So H phi+2 phi should be=H phi and therefore what we will have here I have written mu but here it is m, so this mu or you can say m must be an integer.

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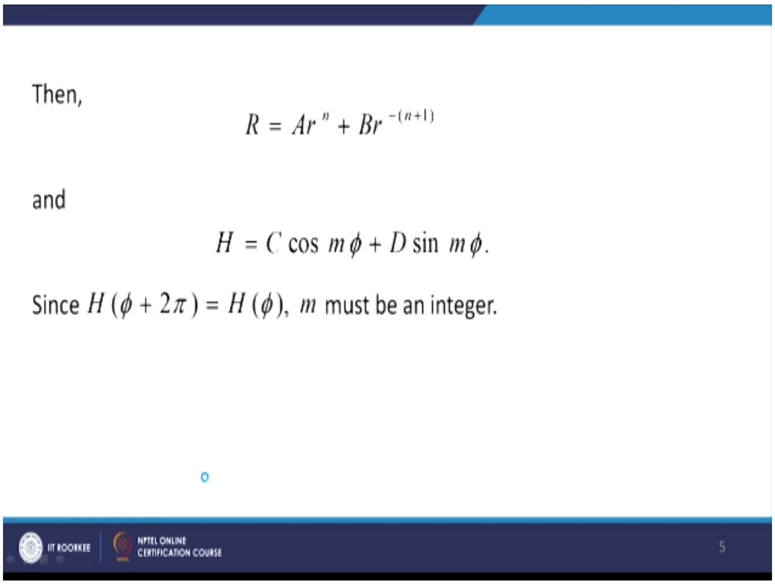

So let us see we will have H=some constant*cos m phi+D sin m phi, if we take this=-m square okay so then what we will have this because of H phi+2 phi=H phi, m must be taken as an integer. Now what happens when you take this as –m square okay 1/H d square H/d phi square as –m square you take 1/R r square d square R/dr square+2r dR/dr=n*n+1, why we take it as n*n+1 it will become clear later on.

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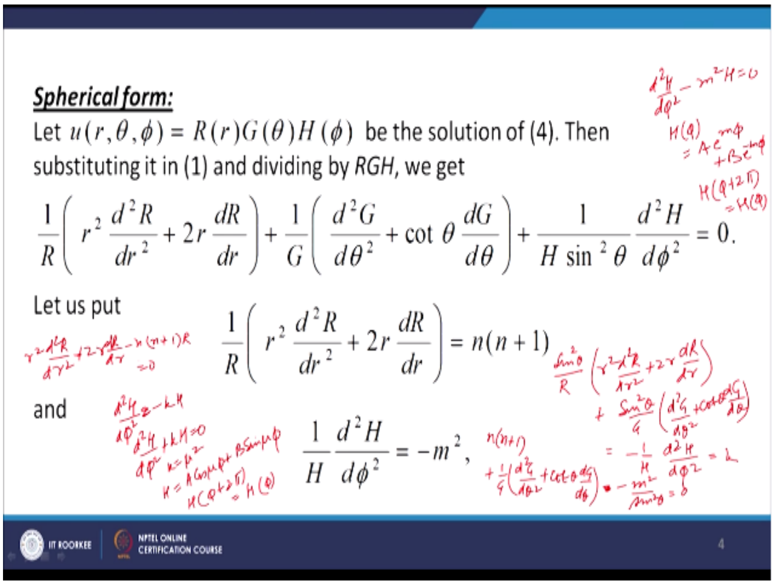

So what we will have, you will have 1/R r square /d square R/dr square+2r dR/dr is n*n+1, so we will get n*n+1 okay so n*n+1+1/G d square G/d theta square okay+cot theta dG/d theta, 1/H d square H/d phi square we have written –m square, so we will have –m square/sin square theta=0 okay. So this can be written as d square G/d theta square+cot theta dG/d theta+n*n+1-m square/sin square theta*G=0 which is a Legendre's equation.

So we will discuss that later on and so let us see when we write this equation=n*n+1 what we get, now here the point is why we are not taking it as +m square. If you take it as +m square then what you will get, d square H/d phi square –m square H=0, instead of –m square if you take +m square here then what do we will get, so the solution will be H phi=A e to the power m phi+B e to the power –m phi because the auxiliary equation will have 2 real distinct roots +-m.

Now here because we require that H phi+2 phi be=H phi okay, so this cannot be fulfilled here H phi+2 phi=H phi and therefore taking 1/H d square H/d phi square to be positive m square is not appropriate. Now let us look at this equation, 1/R r square R double dash 2r dR/dr so this can be expressed as r square=d square r/dr square+2r dr/dr-n*n+1 r=0 okay. So let us discuss how we solve this in the next slide.

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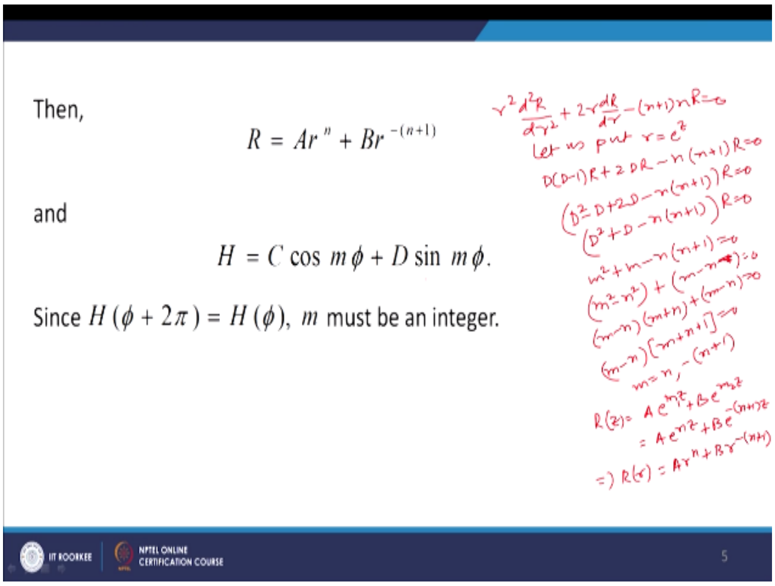

So r square d square R/dr square+we have 2r dR/dr okay, so let us put r=e to the power z, let us change the independent variable from r to z then what we will have r square because this is Cauchy-Euler equation. So r square d square R/dr square will become D*D-1 R+2 DR-n*n+1 R=0 or I can write it as D square-D+2D-n*n+1 operating on R=0 and what we get then D square+D-n*n+1 R=0 okay.

So in order to find the solution of this, now this is a linear differential equation with constant coefficients. So we write the auxiliary equation m square+m-n*n+1=0 and what we can do we can write it as m square-n square+m-n-1=0. Now m square-n square and then what we will get m-n, only m-n is left okay, so this is m-n*m+n and we have m-n here, so taking m-1 m-n common we get m+n+1 okay.

So there are 2 roots of the auxiliary equation m=n and –n+1, so the general solution of this equation is R z=some constant let us say A e to the power m1 z+B e to the power m2 z, m1 is say n so A e to the power nz+B e to the power –n+1*z. Now e to the power z=R okay so this implies replacing z/r we get A r to the power n+B r to the power –n+1. So this is how we get value of r, H we have already seen, H=C cos m phi+D sin m phi.

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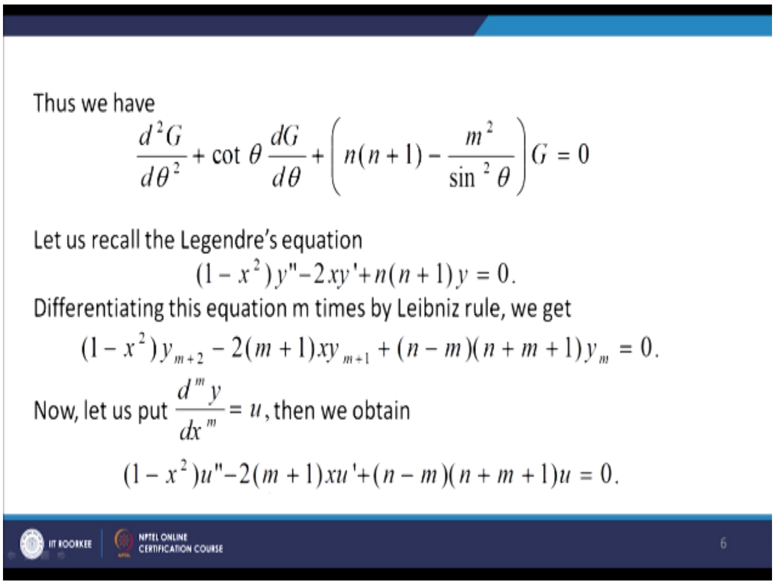

And then the remaining expression in theta gives us this differential equation. Now this differential equation is associated Legendre's equation, we can arrive at this equation using the Legendre's equation. Legendre's equation is given by 1-x square y double dash-2xy dash+n*n+1 y=0. Let us differentiate this equation m times by using the Leibniz rule. So when you use the Leibniz rule you get 1-x square ym+2-2 times m+1 x/m+1+n-m*n+m+1 ym=0.

And here when you put dm y/dx m that is the mth derivative of y=u but you get is 1-x square u double dash-2m+1 x u dash+n-m n+m+1 u=0.

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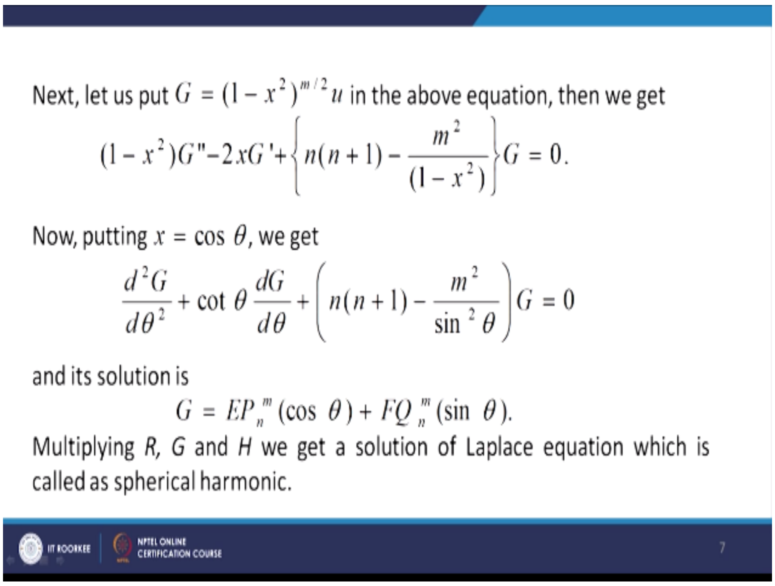

Next, let us put G=1-x square to the power m/2 u in the above equation. So we change from U to G okay by making the substitution in the above equation. Then, we get 1-x square okay

G double dash means d square G/dx square-2x dG/dx+n*n+1-m square/1-x square*G=0. Now we put x=cos theta, so we change the independent variable from x to theta here and when you change the independent variable from x to theta you can verify what we get is d square G/d theta square+cot theta*dG/d theta+n*n+1-m square/sin square theta*G=0.

And this is now associated Legendre's equation, its solution is G=EPnm cos theta+FQnm sin theta. So we have the values of R, G, and H all of them, let us multiple R, G, and H then we get a solution of Laplace equation which we call as spherical harmonic.

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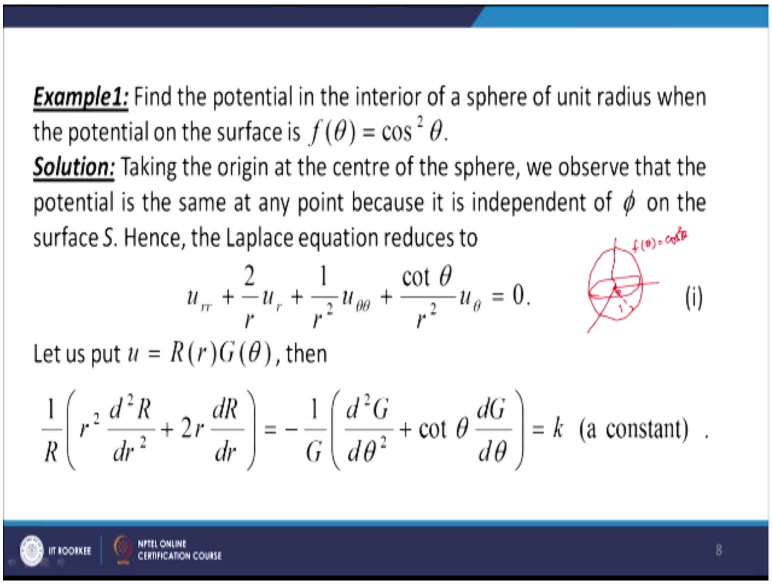

Now let us discuss a problem, find the potential in the interior of a sphere of unit radius when the potential on the surface is cos square theta. So if you take the surface the center of the surface at the origin okay let us take the center of the sphere as the origin okay we observe that here F theta=cos square theta. The radius is unity okay, so we observe that the potential is same at any point because it is independent of phi okay on the surface.

On the surface, F theta is cos square theta, the potential is cos square theta, so the potential u will not depend on phi, hence the Laplace equation reduces to u rr+2/r ur+1/r square u theta theta+cot theta/r square u theta=0. Now we have to find solution of this equation. So let us take u to be a function of r and theta, so u=R r G theta, you put it in this equation what we will get is 1/R r square d square R/dr square+2r dR/dr=-1/G d square G/d theta square+cot theta dG/d theta.

Now this quantity on the left depends on R the independent variable R only, the right hand side depends on theta only, both are equal so they must be equal to a constant. So they are equal to a constant let us say k okay.

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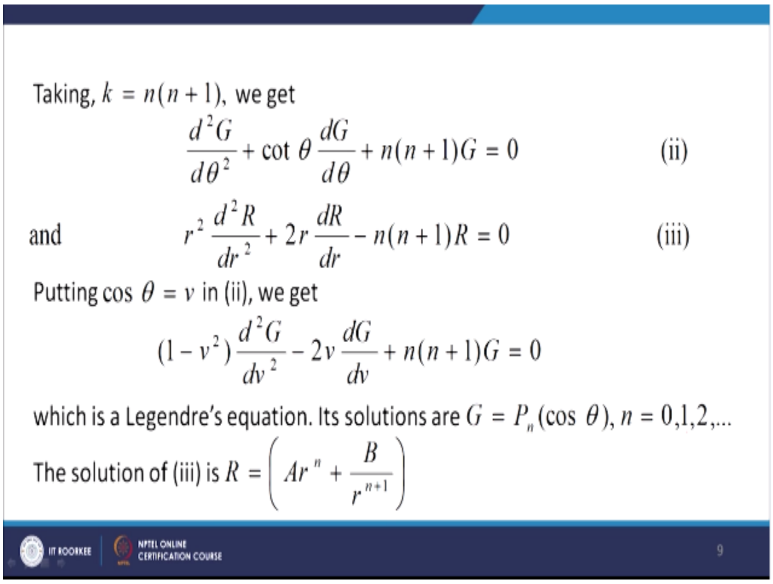

Let us take k=n*n+1, by taking k=n*n+1 we know the equation this one r square d square R/dr square+2r dR/dr-n*n+1*R is a Cauchy-Euler equation whose solution we know that is solution is Ar to the power n+B times r to the power –n-1. So that is the solution of this equation 3 we have already discussed and d square G/d theta square+cot theta dG/d theta+n*n+1 G=0 which we get the other equation that we get here okay.

That changes to this equation if you put cos theta=v in this okay. So taking cos theta=v this equation changes to so we are changing the independent variable from theta to v now. So 1-v square d square G/dv square-2v dG/dv+n*n+1 G which is=0 which is the Legendre equation okay. So which is the Legendre equation, its solutions are the Legendre polynomials Pn cos theta where n takes values 0, 1, 2 and so on.

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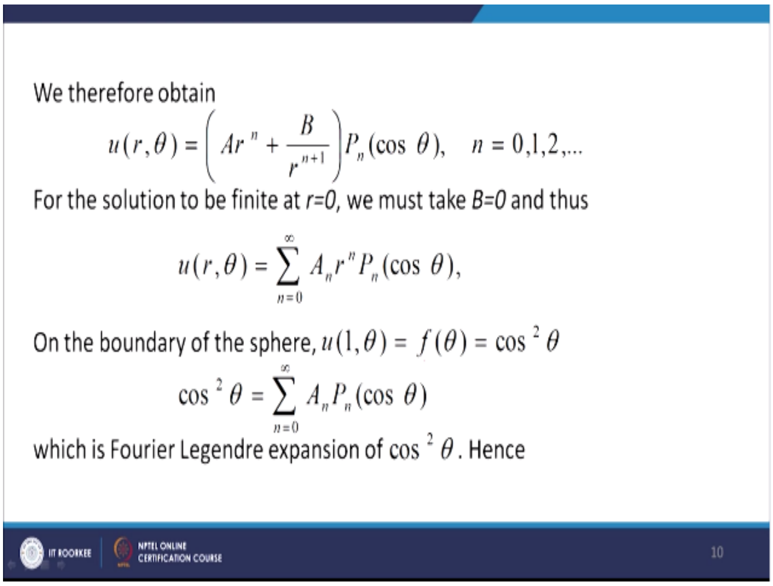

So we can obtain therefore the expression for u r, theta, u r, theta is rr*G theta so Ar to the power n+B/r to the power n+1 Pn cos theta where n=0, 1, 2 and so on. Now because our region includes the origin r=0, r is 0 at the center of the sphere, so we want the solution to be finite and when we want u to be finite at r=0 we must take B=0 otherwise B is not 0 this u cannot be finite at r=0.

So we must take B=0 and therefore we have u r, theta=sigma n=0 to infinity An r to the power n Pn cos theta. Now on the boundary of the sphere, we are given the potential to be = F theta that is cos square theta. So taking r=1 in this equation what we get, cos square theta=sigma n=0 to infinity An Pn cos theta, which is Fourier Legendre expansion of cos square theta.

And we know that in the case of a Fourier Legendre expansion of a function F theta what we get is this formula.

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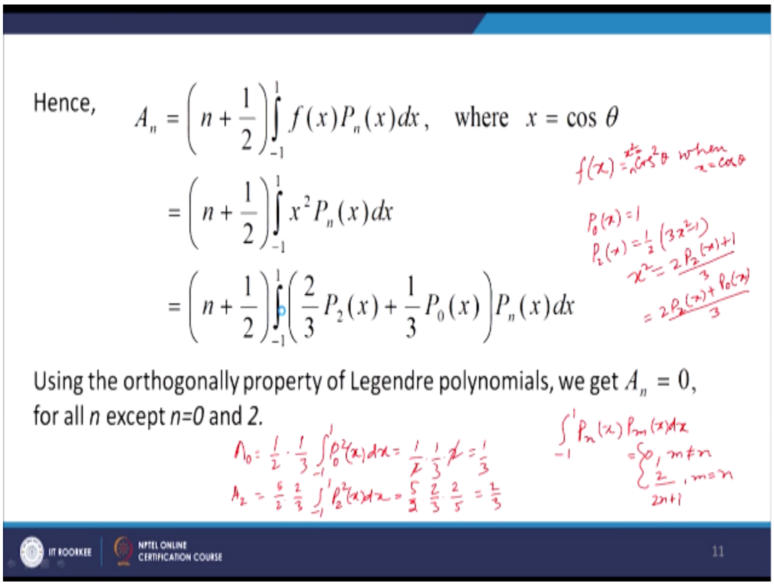

So constants An can be computed by the formula n+1/2-1 to 1 f x Pn x dx where x is cos theta. Now we are given that on the boundary of theta is cos square theta okay, so f x is equal to now here f x is what f x becomes cos square theta when x=cos theta okay. So I should write f x becomes cos square theta let me write f x=x square okay so what I mean is that on the boundary the temperature is cos square theta.

So f x becomes x square because when your f x is x square f cos theta becomes cos square theta. So f x gives cos square theta when x=cos so this is x square actually, f x=x square which is cos square theta okay. So f x=x square gives cos square theta when x is cos theta. So this formula actually is for here we have f x here okay and we have sigma n=0 to infinity An Pn x. So it is the formula for that, now here in place of x we have cos theta so we need to write the corresponding formula in terms of x.

So n+1/2-1 to 1 x square Pn x dx and now what we do we remember that in the case of Legendre polynomials p and x, p0 x=1 while p2 x is 1/2 *3x square-1. So we can express x square as 2 times p2 x+1 okay 2 times p2 x+1/3 or we can write it as 2 times p2 x+P0 x/3. So x square will be replaced by 2/3 P2 x+1/3 P0 x*Pn x. Now let us recall the orthogonality property of Legendre polynomials.

The orthogonality property is that -1 to 1 Pn x*Pm x dx=0 whenever m is!= n okay and this is 2/2n+1 when m=n okay. So applying that what will happen integral/-1 to 1 Pn x*P2 x will be 0 for all n other than 2 okay. So when n=2 integral over -1 to 1 P2 x*P2 x that is P2 square x, P2 x square can be computed from here, it will be 2/phi okay and similarly -1 to 1 P0 x*Pn x will be 0 for all n!=0.

And when n=0 we will have -1 to 1 P0 square x and P0 square x will be 2/1 so it will be 2 okay. So using the orthogonality property of Legendre polynomials we get An=0 for all and except n=0 and 2. So let us find A0 okay. We can find A0 here. A0=put n=0 so 1/2 we are taking n=0, n=0 means we will have 1/3 integral/--1 to 1 P0 square x dx. So this will be 1/2*1/3 integral/-1 to 1 P0 square x dx is 2 okay.

So we get 1/3 and A2 we can similarly find, A2 will be 2+1/2 so phi/2*2/3 integral/-1 to 1 P2 square x dx, so this will be phi/2*2/3*and when you take n=2 it is 2/5 so we get 2/3. So we know the values of A0 and A2 and so we can write the expression for ur theta.

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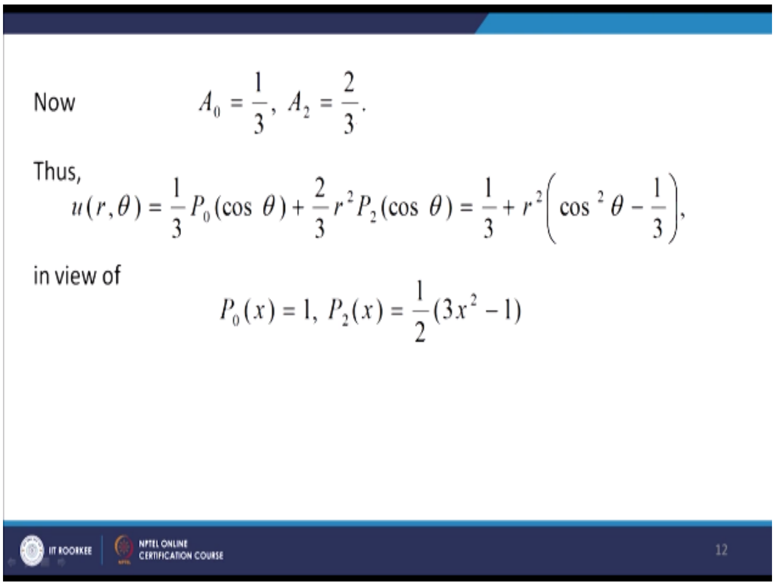

So what will happen, we have A0=1/3, A2=2/3, u r, theta will then be A0*r to the power 0 okay.

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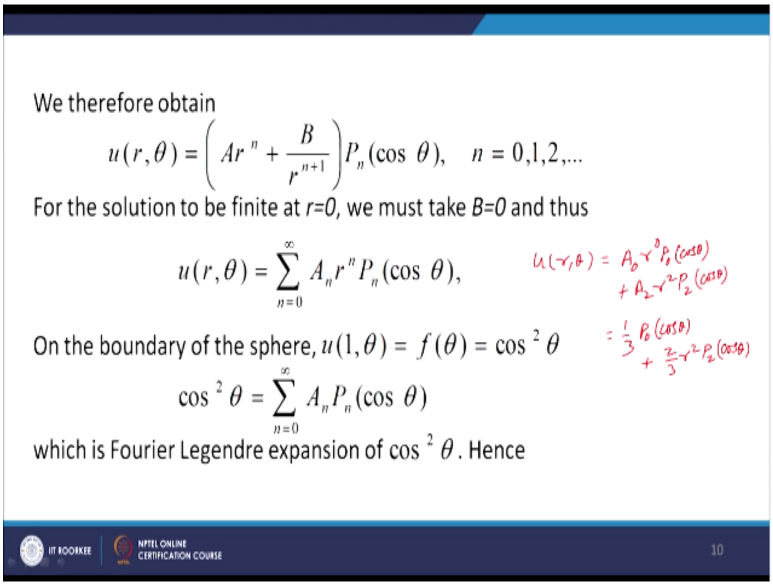

We will have u r, theta=An is 0 for all and except n=0 and 2 so we have A0 r to the power 0 P0 cos theta+A2 r to the power 2 P2 cos theta. We have found A0=1/3, A2=2/3. So this will be 1/3 P0 cos theta+2/3 r square P2 cos theta okay.

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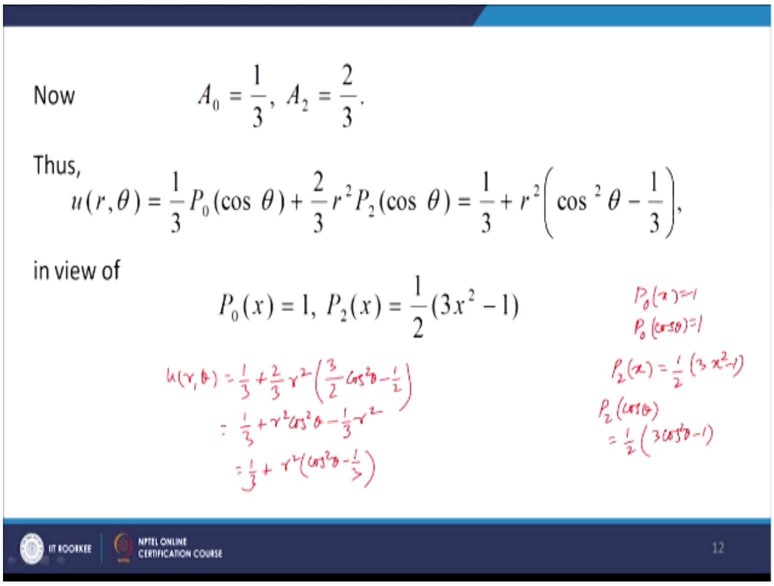

So let us go there we have 1/3 P0 cos theta+2/3 r square P2 cos theta okay. Now we know P0 x=1 okay, so P0 cos theta is also 1, P0 cos theta is 1 and then we know P2 x=1/2 3x square-1, so P2 cos theta will be=1/2 3 cos square theta-1. So let us put these values, so u r, theta will be=1/3 P0 cos theta is 1+2/3 r square and P2 cos theta is 3/2 cos square theta-1/2. So what do we get 1/3, 2/3 r square if we multiple 2/3 will cancel with 3/2 we get r square cos square theta.

And then 2/3*r square we multiply to -1/2 so what do we get 2 2 will cancel will get -1/3 r square. So we get 1/3 r square times cos square theta-1/3 okay in view of P0 x=1, P2 x=1/2 3x square-1. So this is the solution to the given problem.

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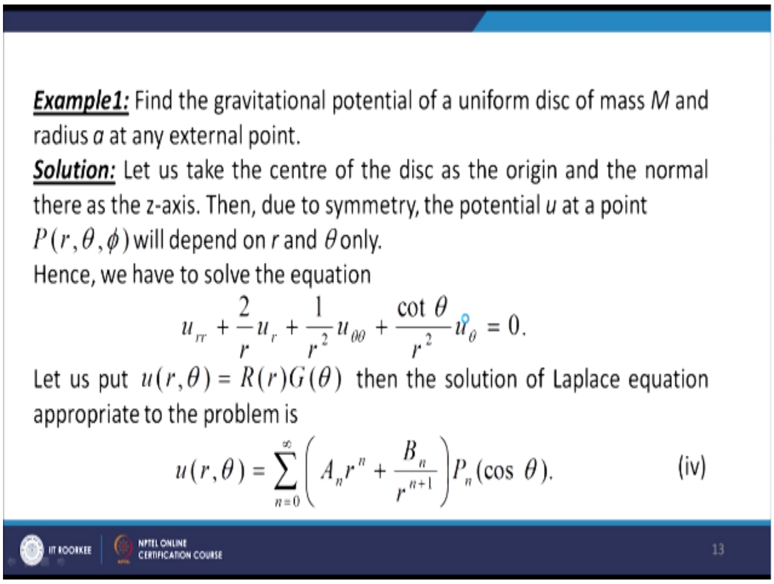

Now let us find the gravitational potential of a uniform disc of mass M and radius a at any external point. So let us take the center of the disc as the origin and the normal there as the z-axis.

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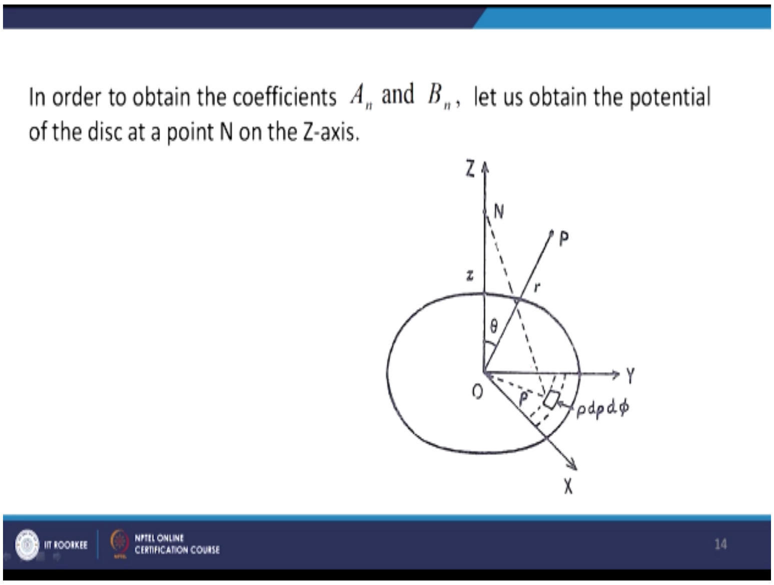

Let us look at this is the figure okay, so this is the disc and this is the center of the disc and this x is normal x line at the point is the z-axis. By symmetry the potential u at any point P r, theta, phi will depend on r and theta only because of the symmetry of the disc. So hence we have to solve the equation u rr+2/r ur+1/r square u theta theta+cot theta/r square u theta=0. Again let us put u r, theta=R r*G theta.

Then, the solution of the Laplace equation appropriate to the problem is given by u r, theta=sigma n= 0 to infinity An r to the power n+Bn/r to the power n+1 Pn cos theta. Now in order to obtain the values of An and Bn let us obtain the potential of the disc at a point n on the z-axis. So for this what we do is we take a small element here at a distance rho from the center and making this angle is let us say this element rho d rho d phi okay.

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From the figure, the potential of an element $\rho\, d\rho\, d\phi$ of the disc at N

$$= \frac{GM}{\pi a^2}(\rho\, d\rho\, d\phi)\frac{1}{\sqrt{(z^2+\rho^2)}}$$

Hence, the potential of the whole disc

$$= \frac{GM}{\pi a^2}\int_0^{2\pi}\int_0^{a}\frac{\rho\, d\rho\, d\phi}{\sqrt{(z^2+\rho^2)}} = \frac{2GM}{a^2}\left[\sqrt{(z^2+a^2)} - z\right]$$

Rho d rho d phi we take and then for this element what we will have the potential of an element rho, d rho, d phi of the disc at n at the point n okay will be=GM/pi a square rho d rho d phi/1/root z square+rho square, root z square+rho square is the distance of the point okay this distance okay this distance is root z square+rho square so we have 1/root z square because we know that potential is GMm/r okay.

So this is actually r or r is under root z square+rho square and this gives you GMm/r so hence the potential of the whole disc is GM/pi a square integral over 0 to 2 pi integral over 0 to a rho d rho d phi/under root z square+rho square. Now this we can easily integrate and when you integrate what we get is 2GM/a square under root z square+a square-z okay.

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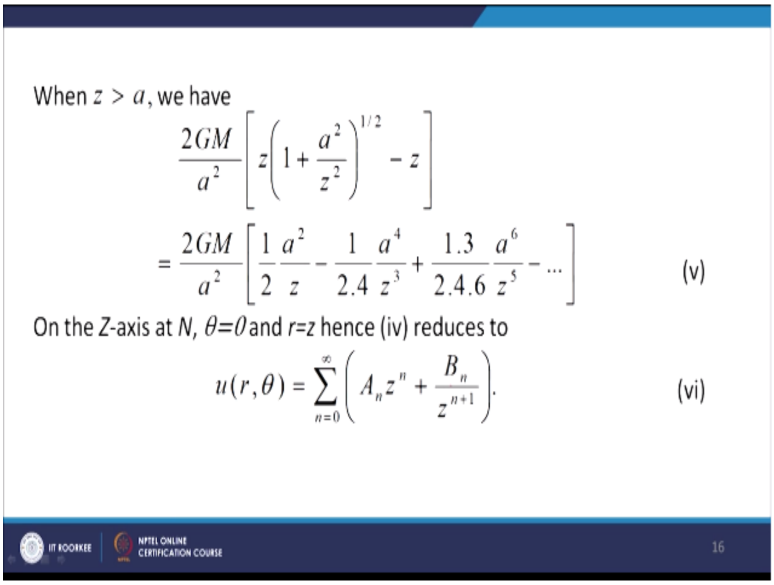

Now what we have when z is>a, we have 2GM/a square z*1+a square/z square to the power 1/2 z okay. Here we are integrating with respect to rho okay yeah. So what we have let us look at this. This expression under root z square+a square if z is>a can be written as a times under root 1+a square/z square and 1+a square/z square to the power 1/2 then we can write its binomial expansion and we get 2GM/a square 1/2 a square/z-1/2*4 a4/z cube 1*3/2*4*6 a to the power 6/z to the power 5 and so on.

On the z-axis at the point n okay let us look at the figure at the point n theta=0 and z=r so r=z okay this distance this is r okay which is z. So what we will have, so this solution reduces to An z to the power n becomes r becomes z, An z to the power n Bn z to the power n+1 and theta is 0 so pn0 pn0 is always 1 and so u r, theta becomes this. Now comparing 5th and 6th, let us compare 5th and 6th, when you compare you see that Bn=0 when n is odd okay.

Because we have only odd powers of 1/z here in this expression okay. In this infinite series, only odd powers of z are present so when n is odd okay then n+1 will be even. So Bn=0 when n is odd okay, An is 0 for all n because there is no term in positive integral powers of z, so An=0 for all n and Bn=0 when n is odd okay.

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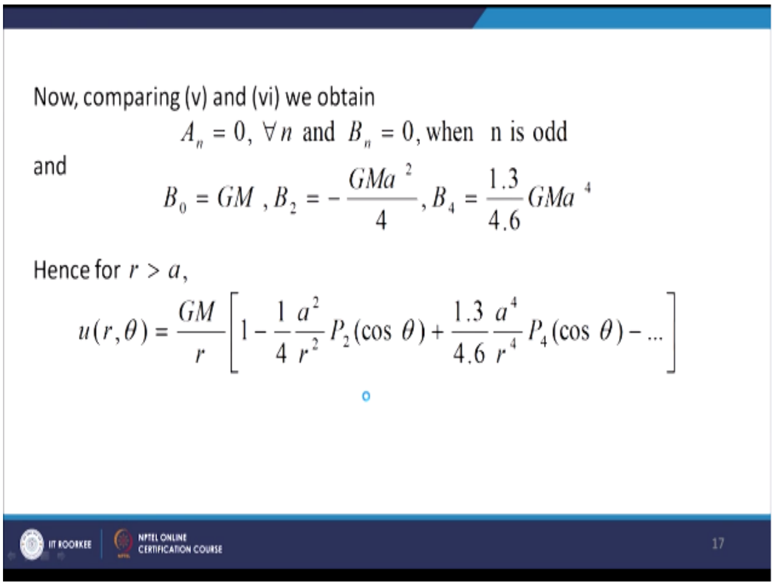

And we can write the values of B0, B2, B4 from here by comparison okay. So we have B0 this, B2 this, B4 this and therefore for r>a okay r>a u r, theta is=this one u r, theta=GM/r 1-1/4 a square/ r square P2 cos theta we can put the values of B0, B2, B4 in this expression okay.

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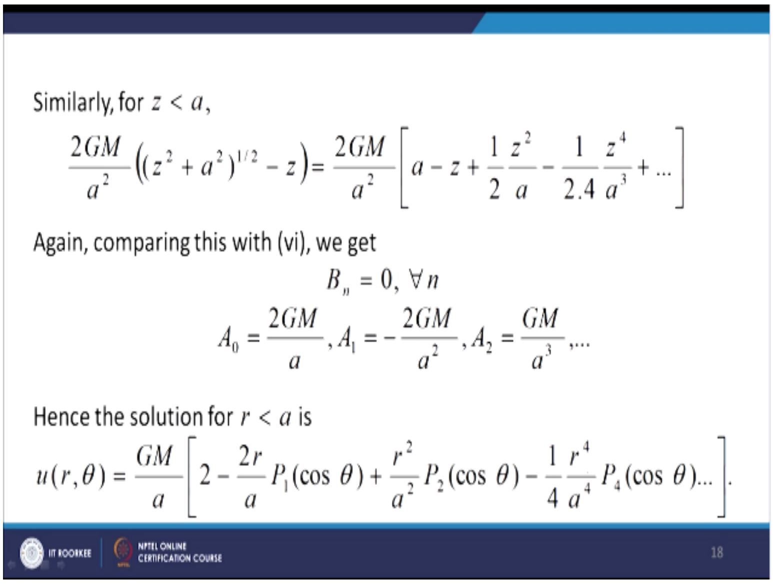

Now when we have z<a okay, when we have z<a that is n is at a distance less than a from the center of this sphere then we will have 2GM/a square. Again look at this expression, 2GM/a square under root z square+a square-z so this time z<a so we shall write a we will take outside the square root so a under root 1+z square/a square we shall consider. So a times under root 1+z square/a square, z/a is<1 so we can again expand it to binomial expansion arrive here.

Again compare this with the equation 5 this equation okay because we are getting this expression at the point n. Now here what is happening is you can see we have A0=A0 is the coefficient of z to the power 0. So A0 is present there but A1, A2, A3 they are all 0s because the positive integral powers of z are absent in that expression okay. So we have positive integral powers are there, negative integral powers are not there.

So Bn=0 for all n, A0 is 2GM/a A1 is -2GM/a square, A2 is GM/a cube and so on so the solution for r<a is u r, theta=this. So with this I would like to end my lecture. Thank you very much for your attention.