Ordinary and Partial Differential Equations and Applications
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Lecture - 47
Review of Integral Transform- II

Hello Friends, Welcome to my lecture on review of integral transforms II, first we shall look at an application on Laplace transform to heat conduction equation.

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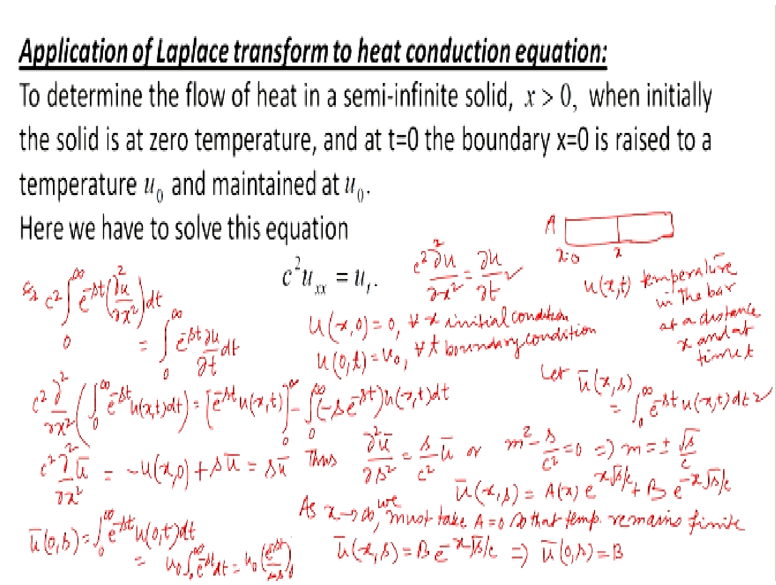

In the last lecture, we had discussed Laplace transforms. Here is the example on the application of Laplace transform to heat conduction equation. We are going to determine the flow of heat in a semi-infinite and solid when x > o, when initially the solid is at 0 temperature in at t0 and the boundary x is 0 and is raised to a temperature at u0 and u0 maintained at u0. So, let us assume that uxt denotes the semi-infinite solid at a distance x from one end.

Suppose this is one dimensional bar so here x=0. We have an element here x at a distance x, x=0 at time t, so we want to determine uxt at the temperature in the bar at a distance x, let me say this is end a and time t. So, we have to solve the first order or say second order differential equation c square uxx that is c square ux square=del square u/del x square=delta u/delta t. So, we have to determine the temperature distribution uxt in the bar.

Here we are giving the initial condition at t=0, the solid at 0 temperature so we have given ux=0 for all x. okay At time t=0, temperature is 0 and the boundary condition is that x=0 is raised to a temperature at u0 and maintained at u0 so at u 0t=u0 for all the time t okay. So, these are initial condition and we have this as a boundary condition. So, in order to solve this differential equation what we do is we multiply by e to the power st.

And both the sides of the equation and integrated over the interval 0 to infinity. So, what we have c square integral 0 to infinity e to the power-st=this okay. Now since we are integrating x, x and t are independent variables so I can also write it as. Now, here this is a product of 2 functions. Let us integrate the by parts so e to the power–st * into the integral of delta u/delta t which is u x,t – integral 0 to infinity derivative of e to the power-st which is-s.

Now, let us denote, let us say that Laplace transform of uxt is urxs. okay, so let us say that the this is the Laplace transform of ux, t so we will have here c square this u r=ur this ur as t goes to infinity uxt e to the power – st when t goes to infinity goes to 0. Okay this is positive uxt e to the power-st remains goes to 0 because uxt remains I mean finite. So, so this goes to 0 then when t=0.

What we get this is e to the power 1 this is ux0 so-ux0+s times integral 0 to infinity e to the power-st nuxt dt which is the function urxs we get sur. So, so you are given ux0=0 so this=s u r so what do we have or thus we have this one okay. Or we can say it can be regarded as an ordinary differential equation of second order where the variable is s and we are keeping x as constant.

So, this is of the form ordinary equation will be m square – s/c square=0 if we regard it as ordinary differential equation of second order and n=+-root of s/c. Okay so we have urxs urxs will be some constant which will depend on s x e to the power x root s/c +B times – x root s /c now then x goes to infinity. We must take A=0 in order that u r does not become infinite okay so to keep ur finite okay, x tends to infinity.

We must take A=0 so that temperature remains finite. Okay So what we have urxs=B times e to the power –x root of s/c. Now let us put x=0 means putting x=0, we get ur0s+b. Now let us find ur0s so we have u0B=U0 here you put this is your Laplace transform urxs=integral 0 to infinity e to the power – st uxt dt put x=0 in this so we get ur0s=integral 0 to infinity e to the power-st u0t dt okay u0t is u0.

So, this is u 0 times u0 is constant so integral 0 to infinity e to the power-st dt okay and this is u0 times e to the power– st/-s dt. 0 to infinity. So, if s is positive then st goes to infinity this will go to 0 so you will get u0/s so we get B=u0/s and therefore what we have so B=u0/s and hence urxs=u0/s * e to the power– x root s/c okay so we get this.

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$$\bar{u}(x,s) = \frac{u_0}{s} e^{-x\sqrt{\frac{s}{c}}}$$

$$L^{-1}\left(\bar{u}(x,s)\right) = u_0 L^{-1}\left(\frac{e^{-x\sqrt{s}/c}}{s}\right)$$

$$\Rightarrow u(x,t) = u_0\left(1 - erf\frac{x}{2c\sqrt{t}}\right)$$

$$\bar{u}(x,s) = L\left(u(x,t)\right)$$

So

$$L^{-1}\left(\bar{u}(x,s)\right) = u(x,t)$$

Using $L^{-1}\left(\frac{e^{-b\sqrt{s}}}{s}\right) = 1 - erf\frac{b}{2\sqrt{x}}$, we have

$$u = u_0\left(1 - erf\frac{x}{2c\sqrt{t}}\right).$$

So we get urxs=u 0/s*e to the power – x root s/c. Now let us take inverse Laplace transform of this so L inverse of urxs=uo*l inverse of u 3 power-x root s/c/s. Now in the last lecture we have found inverse Laplace transform of e to the power-B root. x/s=1-error function of v/2 times root x. So, let us apply this result this result we have seen we have proved using the inversion formula in the Laplace transform.

And so this we can use and this will give you and let us note that urxs=Laplace transform of uxt. So, L inverse of=urxs=uxt and therefore this will give you u x,t=u 0. So, now let us compare here we have root s*-B _ B is replaced by x/c here we can put t instead of x because here we had the function of x so we had this is=1– error function of in place of B we will put x/c.

So, we will have x/2 root c x over 2c root t. Inverse Laplace formula we take uxt in this formula it will give us this. So, this is the solution to the heat equation by using the Laplace transform.

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Fourier Series: Let f be a periodic function of period 2π, which can be represented by a trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (1)$$

If term by term integration of the series is allowed then we obtain the so called Euler formulas

1, cos x, sin x, cos 2x, sin 2x, - -

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx, \quad k = 0,1,2,.....$$

and

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx, \quad k = 1,2,3,..... \qquad (2)$$

Now let us consider Fourier series, let f be a period function of period 2pi which can be represented/a trigonometric series fx=a0/2+sigma infinity to n=1 an cos nx+ bn sin nx. If term/term of this series is allowed, term/term integration is allowed, then we obtain the so called Euler formula. We can see that because we have 1, cosx, sinx, cos2x, sin2x they are all orthogonal functions/the interval-pi pi to pi pi, okay.

That means if you take any pair of any 2 functions/the interval of-pi to pi their product you take and integrate/-pi to pi, then this integral is always 0. So, what we have from this equation, when you multiply this equation/cos nx when you multiply this equation/cos nx and integrate/the interval-pi to pi, then what value will you get an will survive where n will be=m, then only we will get non 0 value here, otherwise it is all 0.

So, we get am is arbitrary, so we get ak=1/pi,-pi to pi fx cos kx dx. This includes this a0 also. So we have k=0,1,2. So bns are given/the formula b k=1/pi,-pi to pi fx sinx kx dx.

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Note that because of the periodicity of the integrands, the interval of integration in (1) can be replaced by any other interval of length 2π, for instance, by the interval $0 \le x \le 2\pi$.

If f is continuous or merely piecewise continuous (continuous except for finitely many finite jumps in the interval of integration), the integrals in (2) exist and hence we may compute $a_k, k = 0,1,2,...$ and $b_k, k = 1,2,3,...$ and form the trigonometric series

$$\frac{a_0}{2} + a_1 \cos x + b_1 \sin x + ... + a_n \cos nx + b_n \sin nx + ... \quad (3)$$

This series is then called the Fourier series of f and its coefficients $a_k, k = 0,1,2,...$ and $b_k, k = 1,2,3,...$ are called the Fourier coefficients of $f(x)$.

Now, note that because the periodicity of the integrands, you can see fx is 2pi periodic, cos kx is 2 pi periodic fx cos kx is 2pi periodic and here fx* sin kx is 2pi periodic. So, the interval of the integration can be changed from o-pi to pi to any interval of length 2pi. For example, we can take the interval 0 to 2pi. So, because of the periodicity the interval and integration can be replaced/any other interval of length 2pi. For instance, 0<=x <2pi <=2pi.

If f is continuous or merely piecewise continuous, piecewise continuity meaning continuous except for finitely many jumps in the interval of integration, the integral of 2 will exist and hence can be evaluated. Hence we compute the values of ak and bk. And then form this trigonometric series a0/2+a1cosx+ b1sinx and so on. This is called the Fourier series of f and the coefficients of ak and bk are called the Fourier coefficients of f.

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The following theorem gives sufficient conditions for the representation of a function by a Fourier series.

Theorem: If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \le x \le \pi$ and has a left and right hand derivative at each point of that interval, then the corresponding Fourier series (3) with coefficients (2) is convergent. Its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left and right hand limits of $f(x)$ at x_0.

Now the following theorem gives sufficient conditions for the representation of a function/a Fourier series. Now let us look at the question of the convergence of this Fourier series. When the Fourier series converge and if it converges whether it represents the function fx or x0. So, this theorem gives us the sufficient conditions for the convergence of the Fourier series. If a periodic function fx with a period 2pi is piecewise continuous.

In the interval-pi to pi and has a left and right hand derivative at each point of the interval, then the corresponding Fourier series 3 with coefficients 2 is convergent. This series is convergent and its coefficients are given/these formula. These formulas are called as Eulers formulas. Its sum is fx. So, the sum of the series will be fx at each point of continuity of f and at the point of discontinuity.

Its sum will be equal to the average of left hand and the right hand limits, so at the point of x0. So fx0+and fx0-/2. So, these conditions are known as (()) (15:55) conditions and almost every function fx of period 2pi can be represented or Fourier series that we come across applications.

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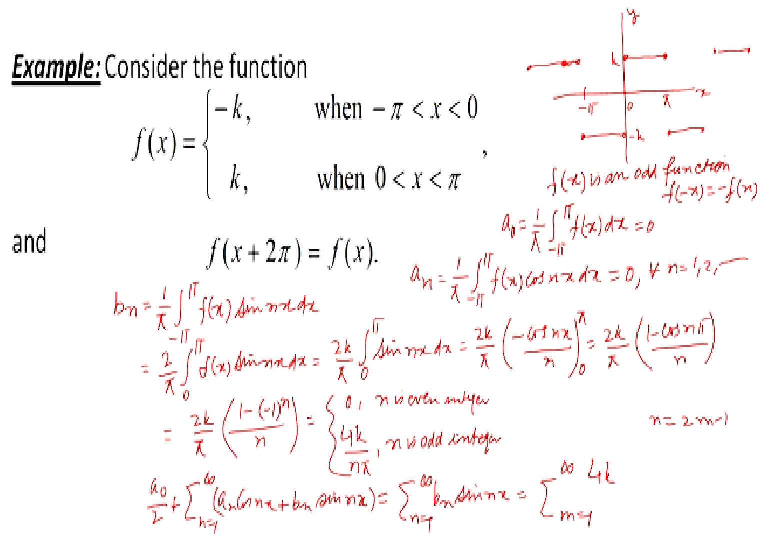

So consider the function fx=-k when -pi < x < 0 and fx=k where 0<x<pi x is < 0 < pi. Now let us see the graph of this function. See-pi to 0 it is-k. Let us take k positive and draw the graph like this. And then, from 0 to pi it is k. Okay. And then it is extended to pi periodicity like this. So, you can see that, this is an odd function okay fx is an odd function here that is f-x=-fx, we can see that. If we look at the Eulers formula, Eulers formula for ak, a0=1/pi-pi to pi fx dx.

Since fx is an odd function, integral/-pi to pi fxdx will be 0. So a0 is=0, and then the value of an=1/pi-pi to pi fx cos nx dx. Since f is an odd function cos nx is even function, their product is an odd function, and therefore this integral will be 0 and this is 0 for all n=1,2,3 and so on. So now, let us look at the formula for bn. bn=1/pi, integral/-pi to pi fx sinx dx. So here fx is odd function and sinx dx is odd function.

So, their product is even. So I can write 2/pi/0 to pifx*sin nx dx. Now/interval 0 to pi fx=k. So we can write it as 2k/pi, k is a constant 0 to pi sin nx dx. And this will be=2k/pi integral of sin nx will be-cos nx divided/n, so 0 to pi, we will get-2k/pi 1 – cos n pi/n/. So this is therefore = 2k/pi, 1--1 to the power n/n/. So this is equal to if n is even, this will be 1-1 so 0, if n is even integer and this will be=2 if n is odd. So it is 4k/npi when n is odd integer.

So, we can now write the Fourier series for the function f. Our Fourier series is, a0/2, +sigma n=1 to infinity an cos nx+bn sin nx. So this is given/a0 is 0, an s are 0, so sigma n=1 to infinity bn sin nx. Bn is=4k/npi when is odd integer. So, replacing n/2m-1, where m will take values 0,1,2 and so on. So, we will get sigma m=1 to infinity m-1 that is 4k/2m-1*pi sin2m-1*x.

Or I can write it as 4k/pi, as 4k/pi is a constant sigma n=1 to infinity sin2n-1*x/2n-1. This is the Fourier series for the function f. And its sum is=fx over the points of continuities and the points of discontinuities at the average of left hand and right hand limits.

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Hence the Fourier series of f is given by

$$f(x) = \frac{4k}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}, \quad \text{for } -\pi < x < 0 \text{ and } 0 < x < \pi. \qquad (4)$$

At $x = 0, \pm\pi$, the sum of the series is the average of the left and right hand limits of $f(x)$.

If we take $x = \frac{\pi}{2}$, then from (4)

$$f\left(\frac{\pi}{2}\right) = \frac{4k}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)\frac{\pi}{2}}{2k-1}$$

and so $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}.$$

So the Fourier series is given by-pi < x < 0 because f is continuous here and 0 < x< pi where also the function is continuous. The discontinuity points you can see in the interval-pi to pi the discontinuities occur at-pi 0 and pi. So, the sum of the series here is the average of left hand and the right limits of fx. Now let us look at the particular case suppose if I take x=pi /2. At x=pi/2 what do we notice at x=pi/2.

Function is continuous and it takes value k okay and so since function is continuous fx will be=this series okay. So, we will have f Pi /2 will be=4k/pi sigma k=1 to infinity Sin this will be sigma instead of k here it should be n so n it should be so sigma k n=1 to infinity Sin 2n-1×pi /2 / 2n-1. So, then what do you see is that f pi /2=k=4k/pi sigma n=1 to infinity Sin 2n-1×pi /2/2n-1.

This k will cancel with k and we will have sigma n=1 to infinity Sin 2n-1×pi /2/2n-1=4 pi /4, Okay and sin 2n-1×pi /2=Sigma n=1 to infinity-1 to the power n-1/2n-1. So this is=pi/4 then we get 1-1/3 +1/5-1/7 and so on=pi/4. So, by using the Fourier series we can also find the sum of series of cosine terms in some particular cases. Now let us discuss Fourier series of functions having arbitrary period.

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Fourier series of functions having arbitrary period:

In applications, periodic functions rarely have period 2π. Therefore in order to find the Fourier series of a periodic function *f(t)* with period, say T, we use the change of scale. We introduce a new variable x such that *f(t)* as a function of x, has period 2π. Let us define $t = \frac{T}{2\pi}x$ then $x = \pm\pi$ corresponds to $t = \pm\frac{T}{2}$ and $f(t) = f\left(\frac{T}{2\pi}x\right) = \phi(x)$, say.

Then $\phi(x+2\pi) = \phi(x)$. So we can find the Fourier coefficients for the function ϕ by the Euler's formulas

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\cos kx\,dx,\; k = 0,1,2,\ldots$$

In applications periodic functions rarely have period 2pi therefore in order to find the Fourier series of a periodic function ft with period say T, we use the change of the scale. We introduce a new variable x such that ft as a function of x has period 2 pi. So, let us define T=T/2 pi*x, then x will take value +-pi T will take values +-T/2. So ft=ft/2pi×x which is the function of x let us call it as pi x.

Then becomes the 2 pi periodic function, you can easily check that pi x+2pi+ f of T/2 pi, x2pi and this is=ftx upon 2pi+T.Since f is periodic with period T this will be=f of Tx upon 2pi and which is=Pi x. So, Pi is the periodic function with period 2pi and so we can find the Eulers formula for the function pi by writing Eulers formula for the function Pi. We can determine the coefficients of the Fourier series for the function pi ak=1/Pi-Pi to Pi 5x cos kx dx k=0, 1, 2 and so on.

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$$\text{and} \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\sin kx\, dx, k = 1, 2, \ldots$$

$$\text{or} \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{T}{2\pi}x\right)\cos kx\, dx, k = 0, 1, 2, \ldots$$

$$\text{and} \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{T}{2\pi}x\right)\sin kx\, dx, k = 1, 2, \ldots$$

Now, since $x = \frac{2\pi}{T}t$, we have $dx = \frac{2\pi}{T}dt$ hence

And When bk=1/pi-pi to pi 5x sin kx dx, k=1, 2 and so on and now here ak=let us put the value of

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$$\text{and} \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\sin kx\, dx, k = 1, 2, \ldots$$

$$\text{or} \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{T}{2\pi}x\right)\cos kx\, dx, k = 0, 1, 2, \ldots$$

$$\text{and} \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{T}{2\pi}x\right)\sin kx\, dx, k = 1, 2, \ldots$$

Now, since $x = \frac{2\pi}{T}t$, we have $dx = \frac{2\pi}{T}dt$ hence

Let us replace pi x/fT/2pix then we will get ak=1/pi-5t/2pi x cos kx dx=to 0,1,2 and so on and here bk=1/pi-pi2/ft/2pi sin kx dx, k=1,2 and so on. Now since x=2pi/t let us the value of xs 2 pi/t then this f becomes ft. So, and sin kx and k will be t will be=x=2 pi/t. so dx=2pi/2dt and the limits of integration will change from –pi to-T/2 to T/2 and we will have these formulas.

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$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi kt}{T}\,dt, k = 0,1,2,... \qquad (5)$$

and
$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2\pi kt}{T}\,dt, k = 1,2,...$$

The Fourier series of *f(t)* becomes

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi nt}{T} + b_n\sin\frac{2\pi nt}{T}\right)$$

The interval of integration in (5) may be replaced by any interval of length *T* for instance $0 \le t \le T$.

ak=2/T-T/2 to T/2 ft cos/2 pi kt/T, k=0,1,2 and so on bk=2/T-T/2-T/2 ft sin 2 pi kt/T k=1.2 and so on. The Fourier series of ft now becomes a0/2 sigma k=1 to infinity an cos 2 pi nt/T+bn sin 2 pi nt/T, which was Fourier series from the function pi.

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and
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\sin kx\,dx, k = 1,2,...$$

or
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{T}{2\pi}x\right)\cos kx\,dx, k = 0,1,2,...$$

and
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{T}{2\pi}x\right)\sin kx\,dx, k = 1,2,...$$

Now, since $x = \frac{2\pi}{T}t$, we have $dx = \frac{2\pi}{T}dt$ hence

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos kx + b_k\sin kx\right)$$

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos\frac{2\pi kt}{T} + b_k\sin\frac{2\pi kt}{T}\right)$$

Is a0/2+sigma, k=1 to infinity ak cos nx+bk sin these are Fourier functions pi of the series. So, let us put x=2pi T/t to get the Fourier function of the s. So, a0/2+sigma a/k+ 1to infinity ak cos 2 pi here kx and here also kx.so2 pi kt/T+bk sin 2 pi kt/t.

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$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi kt}{T}\,dt, k = 0,1,2,\ldots \qquad (5)$$

and

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2\pi kt}{T}\,dt, k = 1,2,\ldots$$

The Fourier series of *f(t)* becomes

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi nt}{T} + b_n\sin\frac{2\pi nt}{T}\right)$$

The interval of integration in (5) may be replaced by any interval of length T for instance $0 \le t \le T$.

We can find the periodic function for any Fourier series with period T. so this is the Fourier series the interval of integration here again since a period T cos 2pi kt/T is also period T. So, the integrant of periodic function with period T so we can change the interval of integration from-T/2 to T/2 any interval of length T for example 0 to T. So, in place of T/2 to-T/2 we can also write integral over 0 to T in the Eulers formula for ft.

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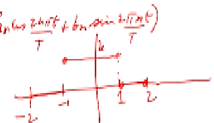
Example: Let

$$f(t) = \begin{cases} 0, & \text{when } -2 < t < -1 \\ k, & \text{when } -1 < t < 1 \\ 0, & \text{when } 1 < t < 2 \end{cases}$$



and $T = 4$, then the Fourier series of f is given by

$$f(t) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \frac{1}{5}\cos\frac{5\pi t}{2} - \cdots\right)$$

Now let us say for example function for 0-2<t<-1< k 1-t<1<t<t let us draw the graph of this s. so here this is-1,-2. So, from-2 to-1 the function is 0 and when-1 to 1, it takes the value of k and from 1 to 2 also it is 0 this one okay. You can see here and then T=4 okay so it is the periodic

function with period 4 ft+4=ft that is given to us. So, if you look at the graph of this function over– 2 to 2 we see that the function f is an even function.

When f is an even function if we look at the Eulers formula for bn=1/pi integral of –pi to pi this is in the case of 2pi periodic function but we have a T periodic function then bn we have to write as 2/T-T/2 to-T/2 ft sin 2 pi kt/T dt. Let us use this formula, let us use bn=2/T-T/2 to T/2 okay 2/T-T/2 to T/2 ft sin2 pi nt/T dt. Let us use this formula for bn why I am writing this formula for bn first because you will see that bn=0 for all n.

Since f is an even function and sin nt/T is an odd function. So, the product in an odd function is therefore T/2 – T/2 will be 0, these 0 for all n. so we need to calculate only a0 and an. a0=2/T-T/2 to T/2 ft dt. So T=4 2/4 – T/2 to T/2 this is-2 to 2 ft dt. Since f is an even function I can write it as 2 times 0 to 2. so that means 4/4, 0 to 2 ft dt and then this is=4/4 is 1 and so now we can get integral 0 to 1 ft dt+ integral/1 to 2 ft dt, integral/0 to 1 f=k.

So, we have integral over 0 to 1 k dt and so this is k and interval/1 to 2ft is 0.so we have 0 So it is=to k. Similarly, we can find a n=2/T – T/2 to T/2 f t cos 2 pi nt/ T dt this is again T=4 so 2/4 and integral is over – 2 to 2 ft cos 2 pi nt/ 4 dt F is an even function sin is also an even function. So, we get integral over – 2 to 2 is 2 times 0 to 2 so we get 4/4 integral 0 to 2 ft cos 2 pi nt /4 dt Over 0 to 1 function is k and over 1 to 2 function is 0.

So, we can write integral over 0 to 1 k times cos pi nt/2 dt and this is k times sin pi nt/ 2 upon n pi/2 and their limits are 0 to 1, So we get T=0 sin pi nt/y=0 but when t=0 we get sin pi n/2 so we get 2k/n pi sin n pi/ 2. So when n=1 we get sin pi/2 so this is=to 2k/pi. When n=2 sin pi 0 so this is 0 when n is even when n is odd say 1 3 5 7 we will get sin 3 pi/2 is – 1 so-1 to the power n – 1 when n is odd this is multiplied by 2k/pi that is 2k/npi.

So, what do we get here f t is a 0/2+sigma n=infinity an cos 2 pi nt/ T+bn sin 2 pi nt/T. So, bn=0 a 0=k and we get k/2 when we put the value of an from here to get this cosine series. So, ft because f is a living function so f t the Fourier series of f is even.

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Fourier series of even and odd functions : The Fourier series of an even function *f(t)* of period *T* is a "Fourier cosine series"

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi nt}{T}\right)$$

$$\text{with} \quad a_k=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\cos\frac{2\pi kt}{T}dt, k=0,1,2,\ldots$$

The Fourier series of an odd function of an odd function *f(t)* of period *T* is a "Fourier sine series"

$$\sum_{n=1}^{\infty}b_n\sin\frac{2\pi nt}{T}$$

Now let us consider Fourier series of even and odd functions which we have already seen Fourier series of even function f of t is a Fourier Cosine Series a0/2+sigma n=1 infinity * a n cos 2 pi nt /T with a n is given by 2/T _ t/2 to T/2 f t cos 2 pi k t /T dt. And taking values 0 1 2 etc the Fourier series of an odd function f t of period T is Fourier sine series Sigma n=1 infinity bn sin 2pi nt/ T as we have seen earlier.

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$$\text{with} \quad b_k=\frac{2}{T}\int_{-T/2}^{T/2}f(t)\sin\frac{2\pi kt}{T}dt, k=1,2,\ldots$$

Theorem: The Fourier coefficients of a sum f_1+f_2 are the sums of the Fourier coefficients of f_1 and f_2.

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi nt}{T}+b_n\sin\frac{2\pi nt}{T}\right)$$

$$a_0=\frac{2}{T}\int_{-T/2}^{T/2}(f_1+f_2)dt=\frac{2}{T}\int_{-T/2}^{T/2}f_1(t)dt+\frac{2}{T}\int_{-T/2}^{T/2}f_2(t)dt$$

$$a_n=\frac{2}{T}\int_{-T/2}^{T/2}(f_1+f_2)\cos\frac{2\pi nt}{T}dt=\frac{2}{T}\int_{-T/2}^{T/2}f_1(t)\cos\frac{2\pi nt}{T}dt+\frac{2}{T}\int_{-T/2}^{T/2}f_2(t)\cos\frac{2\pi nt}{T}dt$$

Where bn are given by 2/T – T/2 and T/2 f t sin 2 pi kt /T dt and taking values 1 2 and so on the Fourier coefficients of sum f1+f2 are the sums of the Fourier coefficients of f1 and f2. It is very easy to see Suppose the Fourier sums of f1 and f2 are a 0/2+sigma n=1 infinity an cos 2 pi nt /T.

Let us assume that f1+f2 are periodic function with period T+bn sin 2 pi nt /T. So, here let us begin with a 0 A 0 is 2/T –T/2 and T/2 and then f1+ f2 dt.

So, we can write it as 2 /T integral over – T/2 to T/2 f1 t dt+2/ T integral – T/2 to T/2 f2 2 dt. So, the Fourier coefficient a 0 form the function f1+f2 are the sum of coefficients of the function f2 and f2. Similarly, we can write an =2/T – T/2 to T/2 f1+f2 Cos 2 pi nt/T dt This becomes 2/T – T/2 to T/2 f1 t cos 2 pi nt/T dt+2/T – t/2 to T/2 ft sin 2 pi nt/Tdt. I will have formulae for f1 coefficient of cos 2 pi nt /T in the Fourier series of F and t.

And this is the coefficient of cos f in the coefficient of cosine terms in the series of the function f2. So, an is given by the sum of coefficients of f1T and f2t they are the 2 functions for cosine terms Similarly we can show for bn bn are also sums of the corresponding Fourier coefficients of sin terms in f1 and f2.

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Half range expansions: In various physical and engineering problems we need to find the Fourier series of a function *f(t)* which is defined on some finite interval, say $0 \le t \le l$. As it is immaterial whatever the function may be outside the range $0 \le t \le l$ so we extend the function to cover the range $-l \le t \le l$ so that the new function may be odd or even. The Fourier expansion of such a function of half the period, therefore, consists of sine or cosine terms only. Let $f_1(t)$ denote an odd periodic extension of f. Since $T = 2l$, the Fourier series of $f_1(t)$ is given by

$$\sum_{n=1}^{\infty} b_n \sin \frac{2\pi nt}{T}$$

or

$$\sum_{n=1}^{\infty} b_n \sin \frac{\pi nt}{l} \qquad (6)$$

Now we see half range expansions in various physical and engineering problems we need to find the Fourier series of function f t which is defined on some finite interval say 0 < =to t <=to l as it is immaterial. Whatever the function may be outside the range 0 <=to t < =to l so we extend the function to cover the interval – l to l so that the new function may be odd or even. The Fourier expansion of such a function of half the period therefore consists of sin or cosine terms only.

Because if we extend it by taking odd function then we will get the Fourier series of function f1 consisting of sin terms If we consider the even extension then the function will consist of cosine terms. So, let f and t to denote an odd periodic extension of f since T=2 l the Fourier series of f1 t is given as sigma n=1 infinity bn sin 2 pi nt /T now here T =2l so we get sigma n=1 infinity bn sin 2 pi nt/ l.

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where
$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_1(t)\sin\frac{2\pi nt}{T}dt$$
$$= \frac{2}{l}\int_0^l f(t)\sin\frac{\pi nt}{l}dt \quad n = 1,2,3,.....$$

In the case of an even periodic extension $f_1(t)$ of $f(t)$, we have

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_1(t)\cos\frac{2\pi nt}{T}dt$$
$$= \frac{2}{l}\int_0^l f(t)\cos\frac{\pi nt}{l}dt.$$

Bn =2/l f t sin pi nt /l dt so we can put the values of t we get 2/ l integral over. Now f1t is an odd function sine is an odd function product is even so we can write it as sin pi nt/ l and this will be 0 to l here we will have 4/ l and 0 to l ft by sin nt/l dt. Here in the case of living periodic extension f1 t of ft f1 t is even cos sin is even so this has been 4/t and this means t=12 so 2/l 0 to T ft cos pi nt/ l dt.

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The Fourier series is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{l} \qquad (7)$$

The series (6) and (7) are called the half range sine and cosine series of f respectively.

Example: Let $f(t) = t,\ 0 < t < 2.$

Then the half range sine series of f is given by

$$f(t) = \frac{4}{\pi}\left(\sin\frac{\pi t}{2} - \frac{1}{2}\sin\frac{2\pi t}{2} + \frac{1}{3}\sin\frac{3\pi t}{2} - \cdots \right) \text{ for } 0 < t < 2.$$

Fourier series is given by a l=0 /2+sigma n=1 infinity a n cos n pi t / l the series 6 and 7 are called the half range sine and cosine series of f respectively. For example, if we take ft =t 0< t < 2 then the half range sin series of f is given we consider an odd periodic extension of ft and then we consider an odd periodic function of t then the Fourier series of ft will be considered and consist of only sin terms.

We can evaluate the values of bn by this formulae coefficients of sin pi nt/ l l =2 here so sin pi nt/2 okay. So, the co-efficient of sin pi nt/2 can be evaluated by this formula 2/l 0 to l ft sin pi nt/l dt. And we can easily show that the half range sin series of f is given by this.

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and the half range cosine series of f is given by

$$1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi t}{2}, \qquad 0 < t < 2.$$

And Similarly we can find the half range cosine series of f to get the half range cosine series of f we consider even periodic extension of f. And when we consider even periodic extension of f we apply this formula an=2 over l 0 to l ft cos and pi t over l dt So let us see how we get this we can show one of the 2

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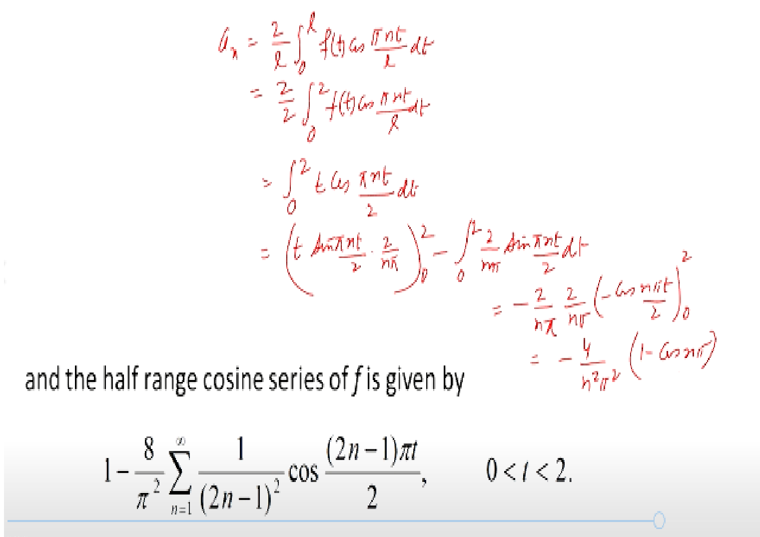

So an=2 over l 0 to l ft cos pi nt/l dt with this function defined about 0 to 2 so l=2 so we get 2 integral over 2 ft cos pi nt/l dt. And this is therefore 0 to 2 and the interval 0 to 2 ft is defined as t so we can write it as t cos pi and t/2 dt. Now we can integrate it by parts okay so we get t times sin pi nt /2*2 over n pi 0 to 2 and then-0 to 2 and we get the derivative of t is 1 so 2 over n pi sin pi nt/2 dt.

Okay so when you put T=2 we get sin n pi so that will be 0 when we put 0 that is 0and we get-2 over n pi and integral of sin n pi T/2=2 over n pi again and –cos and pi t/2 from 0 to 2. Okay So we get-4 upon n square pi square and then we get 1-cos n pi when n is even this will be 0 and when n is odd it will be 2 okay So we can calculate the Fourier sin series for f.

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Complex from of Fourier series: The Fourier series of a periodic function $f(t)$ of period T is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}\right). \qquad (8)$$

Since $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

then (8) may be written as

$$f(t) = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{2in\pi t/T} + c_{-n} e^{-2in\pi t/T}\right) = \sum_{n=-\infty}^{\infty} c_n e^{2in\pi t/T}, \qquad (9)$$

This complex of the Fourier series of the function f: The Fourier series of a periodic function of f period t is given by a0/2+sigma n=1 to infinity n cos 2 pi n pi t /t+bn sin n pi t bt t. Let us put cos theta=e to the power I theta+ e to the power-I theta /2 and sin theta=e to the power theta – e to the power –I theta /2I. Then the quotient 8 can be written like this so let us see how we get this.

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$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\left(\frac{e^{i\left(\frac{2\pi nt}{T}\right)} + e^{-i\left(\frac{2\pi nt}{T}\right)}}{2}\right) + b_n\left(\frac{e^{i\left(\frac{2\pi nt}{T}\right)} - e^{-i\left(\frac{2\pi nt}{T}\right)}}{2i}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - ib_n)\, e^{2\pi i nt/T} + \frac{(a_n + ib_n)}{2} e^{-2\pi i nt/T}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(c_n e^{2\pi i nt/T} + c_n e^{-2\pi i nt/T}\right)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{2\pi i nt/T}$$

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}$$

We write ft=a0/2+sigma n=1 to infinity and we get here an cos 2pi ntn is cos 2pi nt/2. So, an e to the power I theta theta is 2pi nt/T so 2pi nt/T+e raised to the power –I 2pi and nt /t/2 And then we have bn e raised to the power I times 2pi nt/T+– e raised to the power –I 2pi nt/T over 2i okay. So, this is what we have and then we collect the coefficient of e to the power 2i n pi t/T+e to the power of 2i and pi t/T okay, We will get this form this will be =a0/2+sigma n=1 to infinity 1 /2 an now 1 over I is –I so-ibn e to the power 2pi i nt/T okay. And then here whenever I is –I we will get an+ibn/2 e to the power 2pi I nt /t with the negative sign okay. So if I write this as cn and this as –cn so we get a0 /2+ sigma n to infinity okay So let me write c0=a0 /2cn=an+ibn/2 and c-n=An-ibn/2 okay. Let us see what we have defined.

c0=a0 /2 cn=An – Ibn /2 and cn=an+ibn /2 Let us choice like this this is an-ibn and this is an+ibn Then we will get here cn e to the power-2pi I nt /T and this will be c–n e raised to the power-2pi I nt /t and we able to write it as sigma n=-infinity to infinity cn e raised to power 2pi I nt/T with this choice where cns are given by this expression.

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where,

$$c_0 = \frac{a_0}{2}, c_n = \frac{(a_n - ib_n)}{2} \text{ and } c_{-n} = \frac{(a_n + ib_n)}{2}$$

and hence

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-2\pi i n t/T}, n = 0, \pm 1, \pm 2, ... \qquad (10)$$

(9) is unknown as the complex form of the Fourier series of f and its coefficients are given by (10).

The complex form of Fourier series is useful in problems on electrical circuits having impressed periodic voltage.

N takes value o+-1+-2 and so on now this is known as the complex form all the Fourier series of f okay and its Fourier coefficients are given by 9. So, this is known as the complex form of the Fourier series of f where cns are given by this expression This is useful in problems on electricals circuits having impressed periodic voltage.

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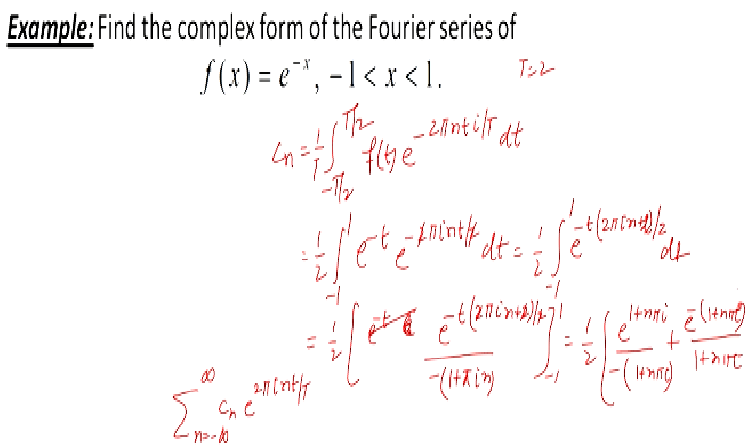

This complex form of the Fourier series of f so here we need to find cns cn=as we have seen earlier 1/T – T/2 to T/2 and we have ft e to the power-2pi nt/T okay. So,-2pi I nt /T okay let us put together now ft=e to the power –t T=2. So, we get 1 over 2-1 to 1 T=2 here and ft=e to the power –t we get e to the power-2pi I nt/2 dt okay. So, we get here is 1 over 2 then we integrate this by parts and when you integrate and evaluate this you will get the value of cn.

The complex form of the Fourier series is this one n=-infinity to infinity cn e to the power of 2in pi t /T so what we do is this is n=-infinity to infinity cn e to the power 2pi I nt /T. So, when you integrate /parts we will get e to the power –t and e to the power-2pi I. We can write it like this instead of we can join this like this 2 integral over –pi to-1 to 1 e to the power – t times 2pi I n+1/So we get 2 here /2 okay dt.

And then we can integrate so we get here e to the power –t times 2pi I n+2/2 okay I can /2 so this is/-1+pi I n and we put the limits-1 to 1 so this is 1/2 okay when we put the limits e to the power 1+n pi I and over-1 +n pi I and then we get+ e to the power-1+n pi I/– and this becomes+ okay so 1+n pi i and then we simplify this and when you simplify this you get

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Then

$$e^{-x} = \sum_{n=-\infty}^{\infty} (-1)^n \frac{(1 - in\pi)\sinh 1}{(1 + n^2\pi^2)} e^{n\pi i x}.$$

The representation of e to the power of –x in the complex form of the Fourier series. With this I would like to end my lecture. Thank you very much for your attention.