Ordinary and Partial Differential Equations and Applications
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Lecture - 46
Review of Integral Transform- I

Hello friends welcome to my lecture on review of integral transforms. This is our first lecture on this topic. Let us review the 2 most powerful methods to solve the differential equations. The methods are Fourier transforms and Laplace transforms. These transforms allow us to turn a complicated problem into a simpler one. The idea behind these transforms is very simple. Suppose we want to solve a differential equation with the unknown function f.

Then we apply the transform to the differential equation to turn it into an equation which we can solve easily. This equation is an algebraic equation for the transform of F of f.

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Now, we shall review two extremely powerful methods to solve the differential equations: the Fourier and Laplace transforms.
These transforms allow us to turn a complicated problem into a simpler one. The idea behind these transforms is very simple.
Suppose we want to solve a differential equation with unknown function f. Then we apply the transform to the differential equation to turn it into an equation which we can solve easily. This equation is an algebraic equation for the transform F of f. We solve this equation for F and then apply the inverse transform to find f. Diagrammatically, this idea can be represented as

We solve this equation of F and then apply the inverse transform to find f diagrammatically this idea can be represented as

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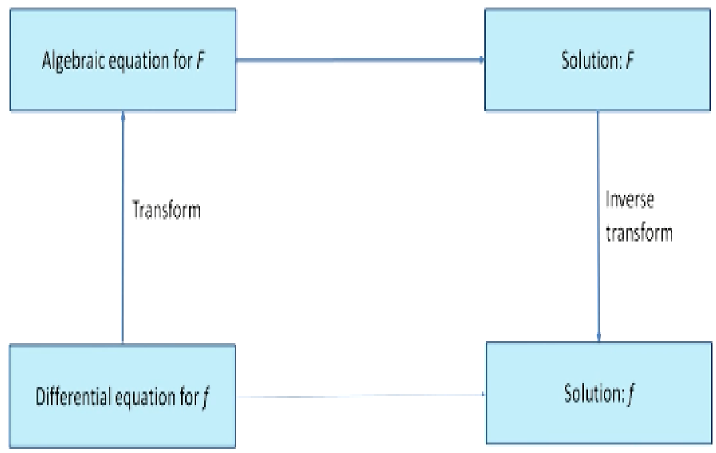

See the diagram; we are given this differential equation if you want to find the solution of the differential equation directly, then most of the times its difficult, so we apply this transform technique. We apply the transform to the differential equation for the function F and obtain an algebraic equation. This algebraic equation is solved to get the function F and the function F is then inverted to get the solution of the differential equation that is the function F.

This is a simple idea. Let us see what is the use of Fourier and Laplace transforms and where they are used.

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The Fourier transform is a tool to find the steady state solutions to differential equation. These solutions may be thought of as the response of a system which has been driven for such a long time that any transient solutions have died out. But in many systems, we also need the transient solutions then the Laplace transform is used. Further, this transform takes into account the initial conditions in a natural way.

We start by reviewing the Laplace transform.

Laplace transform: Let $f(t)$ be defined for all $t \geq 0$, then the Laplace transform of f is a function of s, say $F(s)$ defined as

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt, \qquad (1)$$

The Fourier transforms is a tool to find the steady state solutions to differential equation. These solutions may be thought of as the response of a system which has been driven for such a long time that any transient solutions have died out. But in many systems, we also need the transient solutions then the Laplace transforms is used. Further this transform takes into account the initial conditions in a natural way. Let us review the Laplace transform.

Let us first define a Laplace transform let ft be a function defined for all t >=0, then the Laplace transform of f is a function of s, say Fs defined as Fs=0 to infinity e to the power–st*ft dt.

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provided the integral in (1) exists. This might restrict the values of s for which the transform exists.

Existence theorem for Laplace transforms: Let $f(t)$ be a function which is piecewise continuous on every finite interval in the range $t \geq 0$ and satisfies

$$|f(t)| \leq Me^{\alpha t}, \qquad \text{for all } t \geq 0 \qquad (2)$$

and for some constant α and M.

Then the Laplace transform of $f(t)$ exists for all $\mathrm{Re}\,(s) > \alpha$.

Provided the integral on the right side of equation 1 exists. This might restrict the values of s for which the transform exists. Existence theorem; let us see what are the sufficient conditions on the function F so that its Laplace transform exist. Let Ft be a function which is piecewise continuous on every finite interval in the range t >=0 and satisfies mod of ft <= M times e to the power alpha t for all t >=0 and for some constants alpha and M.

Then the Laplace transform of ft exists for all real s >alpha. s could be a complex number. So this is the existence theorem for Laplace transform. The conditions in the above theorem are sufficient rather than necessary, and it is easy to check whether a given function satisfies n in equality of the form 2. The function f is in exponential order or not.

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The conditions in the above theorem are sufficient in most applications and it is easy to check whether a given function satisfies an equality of the form (2).

For example, $\cosh t < e^t, t^n < n! e^t, \quad n = 0,1,2..... \; \forall t > 0$ and any bounded function $\forall t > 0$, such as $\sin t$ and $\cos t$.

If $f(t) = e^{t^2}$ then it does not satisfy (2) because however large M and α we choose

$$e^{t^2} > Me^{\alpha t}, \quad \forall t > t_0$$

where t_0 is a sufficiently large number, depending on M and α.

$f(t) = \cosh t = \frac{e^t + e^{-t}}{2}$

$e^{-t} \le e^t, t \ge 0$

$e^{-t} < e^t, t > 0$

$|f(t)| \le K, \forall t > 0$

$|f(t)| \le Me^{\alpha t}$

$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$

$\cosh t = \frac{e^t + e^{-t}}{2} < \frac{e^t + e^t}{2} = \frac{2e^t}{2} = e^t$

For example, consider, cos hyperbolic t=you can see cos ft=cos hyperbolic t=e to the power t+e to the power – t/2. We know that e to the power –t is <= e to the power t if t is >=0. If t=0 then Ft=1+1/2 that is 2/2 that is 1. Otherwise if t is >0, then e to the power – t is strictly < e to the power t if t is >0. So, e to the power t+e to the power –t /2 is < e to the power t+e to the power t/2 which means that we have, 2 e to the power t/2 which is= e to the power t.

So, cos hyperbolic t is < e to the power t for all t better than 0. Similarly e to the power t we know that it is sigma n=0 to infinity t to the power n/n factorial. If t is positive, then t to the power n /n factorial will always be < e to the power t. so cos hyperbolic t is <=t, t to power n is <n factorial e to power t for all t >0 their both functions ft= cos hyperbolic t and ft= t to the power n satisfy the condition mod of ft is <= m times e to the power alpha t.

If you take any bounded function, mod of ft <=some constant k for all t > 0, then it obviously satisfies the conditions mod of ft <=m times e to the power alpha t. For example, you can take ft=sin t and cos t they are bounded by 1. So, all bounded functions satisfy the condition of existence given in the existence theorem. If ft= e to the power t square then we see that the condition 2 is not satisfied.

Because however large M and Alpha are chosen, e to the power m square will be > M times e to the power Alpha t, for all t > t0, where t0 is a sufficiently large number and of course will be depending on M and Alpha.

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Note that the conditions in the existence theorem are sufficient rather than necessary. For example, the function $f(t) = \frac{1}{\sqrt{t}}$ is infinite at t=0,

But its transform exists. We have $L\left(\frac{1}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}}$.

Further, if the Laplace transform of a function exists, it is unique. Conversely, if two functions have the same Laplace transform then they can not differ over an interval of positive length, although they may differ at various isolated points. Practically, it is not significant. Hence, the inverse of a given transform is essentially unique.

$L(t^{-1/2}) = \int_0^\infty e^{-st} t^{-1/2} dt = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^{-1/2} \frac{du}{s}$ $\quad st = u, \; s\,dt = du$

$= \frac{1}{\sqrt{s}} \int_0^\infty e^{-u} u^{-1/2} du = \frac{1}{\sqrt{s}} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{s}}, \; s > 0$

Now note that the conditions in the existence theorem are sufficient rather than necessary. The conditions given in the theorem are sufficient rather than necessary because we have functions for which Laplace transform adjust even without their satisfying conditions of this theorem. For example Ft=1/ root t at t=0 it assumes the value infinity but Laplace transform exists. Laplace transforms of t to the power –1/2.

We can find as 0 to infinity e to the power – st and t to the power –1/2 dt. You take st= u. so sdt=du. Then if s is positive the limits of integration will remain t0 and infinity. And we will have e to the power – u. t will be u/s to the power – 1/2 and dt will be du/s. So, what we will get is 0 to infinity= 1/root s 0 to infinity/ e to the power – u and u to the power –1/2 du. This is 1/root s* gamma 1/2=root pi/root s where s is positive.

So Laplace transform of ft= 1/root t exist but it is not piece wise continuous in a finite interval when t is >= 0. When we take finite interval say 0 to some number a so from this interval it is not piece wise continuous. Now further if the Laplace transform exist, it is unique conversely if 2 functions have the same Laplace transforms then they cannot differ over an interval of positive length, although they may differ at various isolated points.

Practically it is not significant. Hence we shall assume that the inverse of a given transform is essentially unique.

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Properties:

1. **Linearity:** $L(af(t)+bg(t)) = aL(f(t)) + bL(g(t))$
2. **Laplace transform of derivative:**

$$L(f'(t)) = sL(f(t)) - f(0) = sF(s) - f(0) \quad (3)$$

where $F(s)$ is the Laplace transform of f and $f'(t)$ is piecewise continuous on every finite interval in the range $t \geq 0$.

The Laplace transforms of higher derivatives can be found by iterating (3).

$$L(f''(t)) = s^2F(s) - sf(0) - f'(0).$$

In general

$$L(f^{(n)}(t)) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - .....f^{(n-1)}(0).$$

The Laplace transform satisfy the linearity property if f and g are any 2 functions, where Laplace transform exists then Laplace transform of aft+bgt= aLft+bLgt where a and b are constants. Laplace transforms of derivatives suppose f is a function whose Laplace transform exist and f prime t is piece wise continuous and every finite interval in the range t>=0 then Laplace transform of ft is given s/ Laplace transform of the function ft – f 0.

If we denote Laplace transform as ft/fs then we get sfs – f0. And the result iterating we can get the Laplace transform of f double dash t n higher order derivatives. Suppose if we want to find out Laplace transform of f double dash t then we apply this formula of f dash in place of f. So, then L f double dash twill be equal to s times L f dash t-f dash 0 replacing f by f dash. Now f times Laplace transform of f dash t=S*Lfs*Lft-f0-f dash 0.

So, what we will have s square Lft-sf0-f dash 0. So, we will get Laplace transform of f double dash ts s square*fs-sf0-sf0. We can again use the iteration over this n can be Laplace transform f triple dash t and so on. So, Laplace transform of the n rh derivative of f I sn to the power fs-sn-1 f0-s n-2f dash 2 and so on fn-10.

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Laplace transform of the integral of a function:

If $L(f(t)) = F(s)$ then

$$L\left(\int_0^t f(u)du\right) = \frac{F(s)}{s}.$$

Now, we study two very important properties of the Laplace transform concerning the shifting on the s-axis and the shifting on the t-axis.

Now Laplace transform of the integral of a function, suppose Laplace transform of Ft is Fs, then Laplace transform 0 to t udu is Fs/s. So, when we take the integral of a function then it is Laplace transform gives the Laplace transform of the function F/s. Now, let us study 2 important properties of the Laplace transform concerning the shifting on s-axis and shifting on t-axis.

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First shifting theorem: Let $L(f(t)) = F(s),\ s > \alpha$, then

$$L\left(e^{at} f(t)\right) = F(s-a), \quad \text{where } s > a + \alpha.$$

Thus, replacing s by $s-a$ in the transform $F(s)$ corresponds to multiplying $f(t)$ by e^{at}.

Second shifting theorem: If $L(f(t)) = F(s)$, then

$$e^{-as}F(s) = L(\tilde{f}(t)),\ a > 0$$

where

$$\tilde{f}(t) = \begin{cases} 0, & \text{if } t < a \\ f(t-a), & \text{if } t > a. \end{cases}$$

In the first shifting theorem if the Laplace transform of Ft is Fs for s>alpha then, the Laplace transform of e to the power at*ft is Fs-a means there is a shifting on the x axis where s>a+alpha.

So, now placing s by s-a in the transform Fs corresponds to *Ft/e to power at, In the second shifting theorem if L of FT×Fs then e to the power-F(s)=Laplace Transform of F to T where a >0 after that function is defined as a 0.

If t <a then f(t-a) if t>a. So this can be alternately written like this

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or $$L\big(f(t-a)u_a(t)\big) = e^{-as}F(s),$$

where $$u_a(t) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } t > a. \end{cases}$$

is the unit step function.

You need a step function L of Ft-at=e to the power as FS if you use the unit step function uat is defined as 0 if t <a and 1 if t>a, so u at ×Ft-a will be 0 if t <a and ft-a then t>a then Ft-a×uat is F t.

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Differentiation of Laplace transform: If $L(f(t)) = F(s)$, then

$$L(tf(t)) = -sF(s),$$

i.e. the differentiation of the Laplace transform $F(s)$ of a function $f(t)$ corresponds to the multiplication of the function $f(t)$ by $-t$.

In general

$$L\big(t^n f(t)\big) = (-1)^n F^{(n)}(s), \quad \text{where } n \in \mathbb{N}.$$

Now, Differentiation of Laplace transform: If L of ft=Fs, then let us see what happen when we differentiate the function Fs which will be the Laplace transform of t function the differentiation of Fs with respect to S corresponds to multiplication of ft/-t. So, L of tft=-d over ds of fs and when you use mathematical induction you get L of t to the power ft=-1 to the power n Fns where Fns is the nth derivative with respect to s.

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Integration of Laplace transform: If $L(f(t)) = F(s)$, then

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(u)du,$$

i.e. the integration of the Laplace transform $F(s)$ of a function $f(t)$ corresponds to the division of the function $f(t)$ by t.

Convolution: Another important property of the Laplace transform is concerned with the product of transforms. Quite often, we wish to determine $L^{-1}(F(s)G(s))$, where $L^{-1}(F(s)) = f(t)$ and $L^{-1}(G(s)) = g(t)$ are known to us.

Integration of Laplace Transform if Lft=Fs let us see what happens in integrate the function Fs with respect to S. So when you integrate Fs the function ft/ t that is the integration of Laplace transform Fs of a function f corresponds to the division of the function ft/t. The more property which we have to consider in the Convolution. Another important property of the Laplace Transform is concerned with the product of transforms.

Quite often, we wish to determine L inverse of Fs*Gs where inverse Laplace transform of Fs is ft and inverse Laplace transform of Gs is Gt. So, we know the Laplace transform of Ft and Gt we want to determine the inverse of the product of Fs and Gs.

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Let $h(t) = L^{-1}(F(s)G(s))$ then $h(t) = (f * g)(t)$ is called the convolution of *f* and *g* and defined as

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Periodic functions: These functions occur in many physical problems and mostly they are more complicated than single sine or cosine function.

Let *f(t)* be a piecewise continuous function for all $t > 0$ and

$$f(t+p) = f(t), \forall t > 0$$

Then

$$L(f(t)) \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)dt, \; s > 0.$$

It turns out that if ht is the inverse of Laplace inverse of Fs*Gs and ht it will be the convolution of F and G. This convolution of F and G is defined as integral over 0 to t F tau Gt-tau d tau and inverse of Fs*Gs is the convolution of F and G that will be call it as Convolution theorem, now let us look at periodic function. These Functions occur in many physical problems and mostly they are more complicated than single sin or cosine function.

Let ft be a piecewise continuous function for all t>0 and ft+p=ft, for all t>0.Then Lft=0, 1 over 1 minus integral 0 to p e to the power-st*ftdt where s is >0.

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Example: Let us consider the mathematical model of forced oscillations of a body of mass '*m*' attached at the lower end of an elastic spring whose upper end is fixed. Let '*k*' be the spring modulus and $K_0 \sin pt$ be the driving force or input. Then, we have the initial value problem

$$my'' + ky = K_0 \sin pt, \; y(0) = 0, y'(0) = 0.$$

$$y'' + \frac{k}{m}y = \frac{K_0}{m}\sin pt$$

$$y'' + \omega_0^2 y = K \sin pt$$

$$L(y'') + \omega_0^2 L(y(t)) = K L(\sin pt)$$

$$[s^2Y(s) - sy(0) - y'(0)] + \omega_0^2 Y(s) = K\frac{p}{s^2+p^2}$$

$$(s^2+\omega_0^2)Y(s) = K\frac{p}{s^2+p^2}$$

$$\text{or } Y(s) = K\frac{p}{(s^2+p^2)(s^2+\omega_0^2)} = \frac{Kp}{p^2-\omega_0^2}\left[\frac{1}{s^2+\omega_0^2} - \frac{1}{s^2+p^2}\right]$$

$$L^{-1}(Y(s)) = \frac{Kp}{p^2-\omega_0^2}\left[L^{-1}\left(\frac{1}{s^2+\omega_0^2}\right) - L^{-1}\left(\frac{1}{s^2+p^2}\right)\right]$$

$$= \frac{Kp}{p^2-\omega_0^2}\left(\frac{\sin \omega_0 t}{\omega_0} - \frac{\sin pt}{p}\right), \; p^2 \neq \omega_0^2$$

We get $Y(s) = \dfrac{Kp}{(s^2+\omega_0^2)(s^2+p^2)}$, where $\omega_0 = \sqrt{\dfrac{k}{m}}$ and $K = \dfrac{K_0}{m}$. (4)

Let us consider the mathematical model of forced oscillations of a body of mass m attached at the lower end of an elastic spring whose upper end is fixed. Let k be the spring modulus and k_0 sin pt be the driving force or input. Then, we have the initial value problem and my double dash ky=k0 sin pt, y0=0 y dash=0**.**Let us solve this equation by using Laplace transform**.** So, we can write it as, y double dash+k/m*y=k0/m sin pt.

Now, let us de0e k/m=omega 0 square so then y double dash+omega 0 square*y=K sin pt. Let us take the Laplace transform L y double dash because Laplace is a linear operation. So, omega 0 square Laplace transform of yt and then k times Laplace transform of sin pt and Laplace transform of y double dash will be s square/s-s/0-y dash 0+omega 0 square/s ys is the Laplace transform of yt=K times.

Laplace transform of sin pt is p upon s square+p Square now we are given y=0 and y dash=0. So, using the initial conditions s square+Omega 0 Square*k times p upon s square+p square or we get ys=k times p upon s square+p square*s Square+omega 0 square where omega 0=root kym and capital K=k0ym We can then find the inverse Laplace transform. So, what we will do is we can break it into partial sections.

And if k times p then upon s square+omega 0 Square-then upon s square+p square/p square-omega 0 square. ys can be written as kp upon p square-omega 0 square so omega 0 square+b square. Now take inverse Laplace transform L inverse of ys=kp upon p square-omega 0 square L inverse of 1 upon s square+omega 0 square=L inverse of 1 upon s square+p square kp upon p square-omega 0 square+omega 0 square is the inverse Laplace transform.

The inverse Laplace transform of s square+omega 0 square is sin omega t/omega 0 T and omega 0 and this inverse Laplace transform 1 over s square+p square is sin pt/p. Here we assume that p s is not=omega 0 square.

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Hence, $y(t) = \frac{K}{p^2 - \omega_0^2}\left[\frac{p}{\omega_0}\sin \omega_0 t - \sin pt\right]$, if $p^2 \neq \omega_0^2$ (5)

i.e. case of no resonance.

The equation (5) represents a superposition of two harmonic oscillations whose frequencies are the natural frequency of the freely vibrating system and the frequency of the driving force.

In the case of resonance, we have $p = \omega_0$ and therefore

$$Y(s) = \frac{K\omega_0}{(s^2 + \omega_0^2)^2}$$

and consequently,

$$y(t) = \frac{K}{2\omega_0^2}\left[\sin \omega_0 t - \omega_0 t \cos \omega_0 t\right]$$

Here we get the function yt/t is the inverse Laplace transform of ys, this is yt. So, with the function yt=k over omega 0 square p over omega 0 square omega 0-sin omega 0 t into pt, p square is 0=omega 0 square. Which is the case of no resonance the equation 5 represents a superposition of 2 harmonic oscillations sin omega 0t sin pt whose frequencies are the natural frequency of the freely vibrating system.

And the frequency of the driving force given by sin pt. So, in the case of resonance, we have P=omega 0 and when p=omega 0 t by P=omega 0 but we have here p=omega 0 ys will become kp upon S Square+omega 0 S square whole square we get ys=k omega 0 s square s square+omega 0 square. And when we take the inverse Laplace transform of this we will get yt=K/omega 0 square sin omega 0 t-omega 0 t-cos omega 0 t.

Let us see how we get this so L inverse of ys=k omega 0 is a constant. We can write like this L inverse of then overs square+omega 0 square. Now here whole square Let us remember that let us call that L inverse of 1 over s square+omega 0 square=sin omega 0t. Is=sin omega 0 t/omega 0. Okay, so when we differentiate, suppose we know Laplace transform of ft=fs, then differentiation of fs with respect to s, gives the multiplication of ft/-t. t.

So, what we have Laplace transform of – t* sin omega 0 t/omega 0. When you differentiate, whenever s is square and omega 0 square, the sin omega 0 t omega 0 will be multiplied by-t, so d/ds upon of 1/s square+omega 0 square. And this will be-1 upon s square+omega 0 square the whole square*2s. So, what we get Laplace transform of t sin omega 0 t ÷ by omega 0=2s upon s square+omega 0 square the whole square.

Now we want the inverse Laplace transform of 1 upon s square+omega 0 square the whole square. So, we have to use the formula which divides the Laplace transform fs/s and that is the Laplace if ft=fs then Laplace transform of 0 to t is fudu is fs/s. So, this means that l inverse of 1/s square+omega 0 square the whole square. This means we are dividing this by s okay. So, this will be integral/0 to t u sin omega 0 u/2 omega 0 du.

We are bringing this 2 here, so this becomes t sin omega 0 t/2 omega 0 and then the right side becomes s/s square+omega 0 square the whole square. We divide this by s and that means that we are integrating this/0 to t, okay. So, this can be either integrated 1/2 omega 0, integral of this is u sin omega 0 u. so u times-cos omega 0 u/omega 0-0 to t, derivative of u is 1. So, -cos omega 0 u/omega 0 du and here we put 0 to t.

So, when we put t here, we get what 1/2 omega 0-t cos omega 0 t/omega 0. When we put 0 it becomes 0, then here we get+0 to t . So,0 to t cos omega 0 u y omega 0, so 1 omega 0 and cos omega 0 u. When you integrate you get sin omega 0 u/omega 0 0 to t. So, what we get 1/2 omega 0-t cos omega 0 t/omega 0 and this becomes 1 /omega 0 sin omega 0 square sin omega 0 t , okay 0 it becomes u becomes 0. So, we get 1/2 omega 0 s, so what do we get the upper omega 0.

You can take outside, so 1/2 omega 0 square you can take omega 0 square outside. So what you get there, you get 1/2 omega 0 q. Okay, 2 omega 0 q. what you get there, -omega 0 t cos omega 0 t+sin omega 0 t that is the inverse Laplace transform of 1/s square+omega 0 square and this we multiply by p omega 0. So, we get k upon 2 omega 0 square sin omega 0 t minus omega 0 t cos omega 0 t.

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Remark: When the input and the natural frequencies are same, we have resonance which is of basic importance in the study of vibrating systems.

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The inverse Laplace transform: The Laplace transform can be applied to a wide range of initial value problems. The main difficulty lies in taking the inverse Laplace transform. Usually we refer to the table of Laplace transforms but if this does not work then we use the inversion formula for the Laplace transform.

If $L(f(t)) = F(s)$, then $f(t)$ can be obtained from

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}\,ds$$

To obtain $f(t)$, the integral is performed along a line AB parallel to the imaginary axis in the complex plane such that all the singularities of F(s) lie to its left.

The inverse Laplace transform the Laplace transform can be applied to a wide range of initial values problems. The main difficulty lies in taking the inverse Laplace transform. Usually we refer to the table of Laplace transforms but if this does not work then we use inversion formula for the Laplace transform. So, if Lft=fs, then ft can be obtained from the formula, ft is=1 /2 pi I integral/a-sin infinity to a+sin infinity fs e to the power of st ds.

To obtain this function ft, the integral is performed along a line parallel to the imaginary axis in the complex plane such that all the singularities of fs lie to its left.

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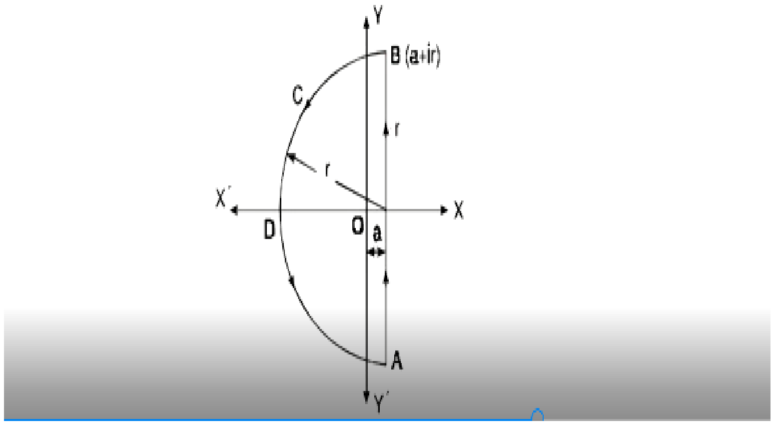
The contour C includes the line AB and the semicircle C′ (i.e. BDA).

$$\int_{AB} e^{st} F(s)ds = \int_{C} e^{st} F(s)ds - \int_{C'} e^{st} F(s)ds.$$



So, you can see this figure. This is the line AB parallel to the imaginary axis in the complex plane and this line is drawn in such a way that, all this similarities of fs lie to its left. Now let us consider the contour c, the contour c includes the line AB and the semicircle c dash, this BCDA the semicircle c dash. So, integral/AB=integral/C, e to the power st fsds-integral/ c dash e to the power st fsds.

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Theorem: If there exists positive constants k and K such that

$$|F(s)| < Kr^{-k}, \quad \text{for every point on } C'$$

then

$$\int_{C'} e^{st} F(s)ds \to 0, \quad as\ r \to \infty$$

hence

$$\lim_{r\to\infty} \int_{AB} e^{st} F(s)ds = \lim_{r\to\infty} \int_{a-ir}^{a+ir} e^{st} F(s)ds$$

$$= 2\pi i \left(\text{sum of residues of } e^{st} F(s)ds \text{ at the poles of } F(s)\right)$$

or,

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} F(s)ds = \text{sum of residues of } e^{st} F(s)ds \text{ at the poles of } F(s)$$

Now we have a theorem, this theorem says that if there exists positive constants k and K such that mod of fs is less than k times r to the power-k for every point on c dash. So this is r, the circle is drawn with center at this point with radius r so when this r goes to infinity. Mod of fs must be < constant times r to the power-k for every point on c dash then integral/c dash e to the power st fsds goes to 0 as r goes to infinity. So, this theorem we shall use in this result, okay.

So, what we have here, when we take the limit r tends to infinity here in this formula, we get limit r tends to infinity integral /AB, e to the power st fsds is=integral limit r tends to infinity integral /a-ir because this is A is a-ir B is a+ir. So, integral a-ir to a+ir e to the power st fsds and this=this integral 2 pi i by residue theorem 2 pi I into some of residues at the poses of fs, and this goes to 0 as r goes to infinity so we have this formula.

Now r we can say ft is=1/2 pi i. Now ft is 1/2 pi i is integral a-i infinity e power fsds becomes some of the residues of e to the power st*fsds. Sum of e to the power st*fsds the poles of fs.

1/2 pi i. 1/2 pi i integral/a-i infinity a+i infinity, e to the power st into fsds will be then=some of residues of e to the power st into fs, the poles of fs.

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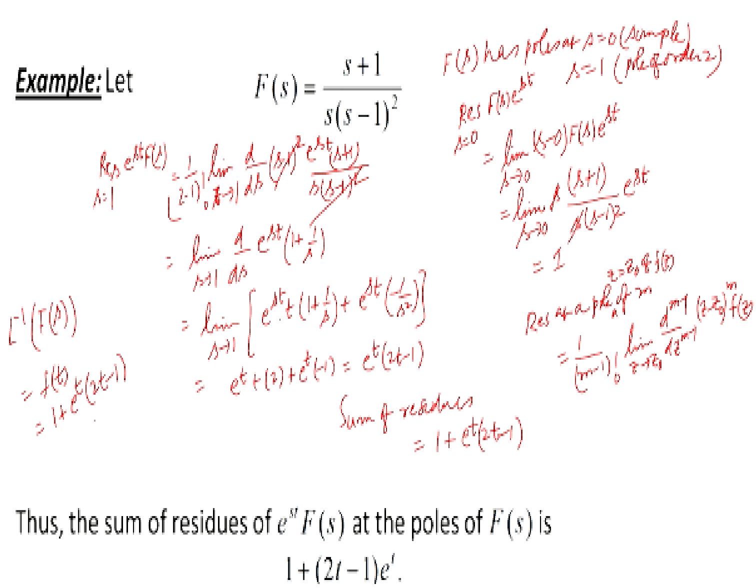

Now let us take for example fs=s+1/s*s-1 the whole square. Fs has poles that s=0 which is simple pole and s is=1 pole of order 2. So, let us find the residue at residue of fs at s is=residue of e to the power st*fsds. So residue of st*fs at s=0 so limit s tends to 0, because it is a simple pole s-0*fs*e to the power st which is=limit s tends to 0 s times fs is s+1/s*s-1 whole square* e to the power st.

So when s goes to 0 e to the power 0 is 1 and we get here 1 and here we get-1 whole square=1. Now residue at s=1 which is the pole of order 2 now residue at a pole of order 1 has the following formula.1/m-1 factorial residue at pole of order z=z 0 of order m. We have limit z tends to z 0 at dm-1/dm-1 z-z 0 to the power m*fz. If f z has a pole of order m at z=z 0 then we apply this formula.

Residue at a pole z=z 0 of fz of order m. okay. So by this formula here a pole of order 2-2 factorial limit s tends to 1 this is s here d/s. okay s-1 whole square* e to the power st*fs. So s+1/s*s-1 whole square. This cancels with this and/1 factor release 1 limit s tends to 1 d/ds of e to the power st* 1+1/s. So this is limit s tends to 1 derivative of e to the power of st *t 1+1/s and then e to the power st *-1/s square.

When s goes to 1 what we get here. So e to the power t and then we get t here 1+1 is 2 and here we get e to the power t and then-1. So we get e to the power t*2t-1. And therefore sum of residues is 1+e to the power t times 2t-1. So inverse Laplace transform of fs okay and inverse of fs=ft and ft=1+e to the power of t*2t-1. This is the inverse Laplace transform of s+1/s*s-1 whole square.

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Example: Determine

$$L^{-1}\left(\frac{e^{-b\sqrt{s}}}{s}\right), b > 0.$$

Let

$$F(s) = \frac{e^{-b\sqrt{s}}}{s},$$

then

$$L^{-1}\left(\frac{e^{-b\sqrt{s}}}{s}\right) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\frac{e^{-b\sqrt{s}}}{s}ds.$$

The integrand has a branch point at the origin. So we take a contour ABCDEF, which encloses no singularity. Then

Now let us determine L inverse e to the power –b root s/s where b >0. So, let us take Fs=e to the power –b root s/s then L inverse e to the power –b root s/s by inverse formula is 1/2 pi i integral/a-i infinity to a+i infinity e to the power st*e to the power –b root s/s* ds. Now the integrand has a branch point at the origin. So we take a contour ABCDEF, which encloses no singularity. Let us see the contour.

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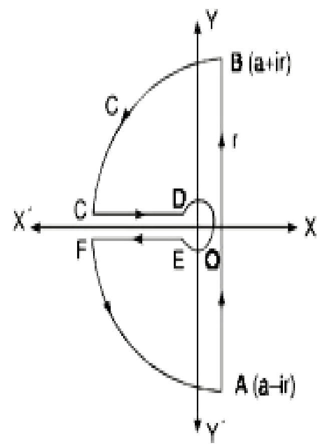

This contour we will take at z=0 we have trans point that includes the similarity and the contour is this. A B C D E F A. We take this contour and then we write the integral/this contour into parts. Integral/AB BC CD DE EF FA e to the power st e to the power –b root s/s*ds=0. Now what do we notice. Let us notice that this Fs okay is e to the power st*e to the power –b root s/s. Let us apply this theory which we have earlier stated.

If there exists positive constants k and k such that modem F s < K times r to the power –k. for every point on c dash then this integral goes to 0. So what to happens is that this integral okay the integral BC and FA goes to 0 by that theory. Because mod of Fs < some constant/S in some constant/R.

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$$\left\{\int_{AB} + \int_{BC} + \int_{CD} + \int_{DE} + \int_{EF} + \int_{FA}\right\} \frac{e^{st} e^{-b\sqrt{s}}}{s} ds = 0 \qquad (6)$$

Let $OC = \rho, OD = \varepsilon$; then (6) implies

$$\int_{a-ir}^{a+ir} e^{st} \frac{e^{-b\sqrt{s}}}{s} ds + \int_{\varepsilon}^{\rho} e^{-Rt} \frac{e^{ib\sqrt{R}} - e^{-ib\sqrt{R}}}{R} dR - 2\pi i = 0.$$

Taking $\varepsilon \to 0, \rho \to \infty$ and $u^2 = tR$, we get

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} \frac{e^{-b\sqrt{s}}}{s} ds = 1 - \frac{2}{\pi} \int_{0}^{\infty} e^{-u^2} \frac{\sin\left(\frac{bu}{\sqrt{t}}\right)}{u} du$$

And that goes to 0 and R goes to infinity integral/BC and FA goes to 0 and if you take OC=rho and OD=epsilon then 6 implies that integral/a-ir to a+ir e to the power st e to the power –b root s/ s* ds and integral/epsilon to rho this-2 pi i=0. When we take epsilon to 0 and rho to infinity and u square=tR, we come to this okay this is = this.

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In view of $\int_0^\infty e^{-u^2}\cos 2lu\,du = \frac{1}{2}\sqrt{\pi}e^{-l^2}$ and $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$

we obtain

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\frac{e^{-b\sqrt{s}}}{s}\,ds = 1 - erf\left(\frac{b}{2\sqrt{t}}\right).$$

And then what we use is we use this result. Integral/0 to infinity e power –u square cos 2 lu du=1/2 root pi e power –l square and the error function. Error function of x=2/root pi integral/0 to x e to the power of –u square *du and then we obtain value of this integral is=1-error function of b/2 root t. So we get the value of this integral okay and inverse of e to the power of-b root s/s=1-error function of b/2 root t.

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Integral equations: The Laplace transform can be used to solve certain types of integral equations also.

An integral equation is an equation in which the unknown function, say $y(t)$, apears under the integral sign.

For example, let

$$y(t) = t + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$$

Then by taking Laplace transform on both sides, we get

$$Y(s) = \frac{s^2+1}{s^4}$$

$L(y(t)) = \frac{1}{s^2} + L\left(\int_0^t y(\tau)\sin(t-\tau)\,d\tau\right)$

$Y(s) = \frac{1}{s^2} + Y(s)\frac{1}{s^2+1}$

$Y(s)\left(1 - \frac{1}{s^2+1}\right) = \frac{1}{s^2}$

$Y(s) = \frac{s^2+1}{s^4}$

$Y(s) = \frac{1}{s^2} + \frac{1}{s^4}$

$L(t^n) = \frac{n!}{s^{n+1}}$, n is a positive integer

$L^{-1}(Y(s)) = L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{1}{s^4}\right) = t + \frac{t^3}{6}$

Now the Laplace transform can be used to solve certain types of integral equation also. An integral equation is an equation in which the unknown function, let us call it as y t, appears under the integral sign. For example, let us say y t=t+integral/0 to t y tau sin t – tau*d tau. Then by taking Laplace transform of y t we can write as=Laplace transform of t is 1/s square+Laplace transform of integral is 0 to t/tau sin t-tau d tau.

And Laplace transform of ut let us take ys so ys=1s square. Here we use convulsion theorem. So, Laplace transform of 0 to t y times sin t-tau this is=Laplace transform of yt which is ys*Laplace transform of sin t which is 1/s square+1. So, we get Y s times 1-1/square+1=1/s square and this gives you ys=s square+1/s to the power 4 or we can say ys=1/s square+1/s to the power 4.

Inverse Laplace transform of ys let us take=inverse Laplace transform of 1/s square+Inverse Laplace transform of 1/s to the power 4. Inverse Laplace Transform is 1/s square t+inverse Laplace transform of 1/s power 4 is 3 factorial/t cube. So we get t+this is t upon 3 factorial so t+t cube/6. Because Laplace transform of t to the power n is n factorial upon s to the power of n+1 if n is a positive integer.

So, inverse Laplace transform of 1/s to the power of 4 will be t cube /3 factorial.

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Again by taking inverse Laplace transform, we get

$$y(t) = t + \frac{t^3}{6}, \quad \text{because} \quad L(t^n) = \frac{n!}{s^{n+1}}.$$

Again by taking inverse Laplace transform of yt function=t+t cube/6. With this I would like to end my lecture. Thank you very for your attention.