Ordinary and Partial Differential Equations and Applications

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Lecture - 45
Classification and Characteristic Curves of Second Order PDEs

Welcome to my lecture on Classification and Characteristic Curves of Second Order PDEs. So we will classify the second order PDE according to the sign of discriminant S square-4RT. So, let us see how we define how we classify the second order PDE.

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The classification of second order PDE depends on the form of the leading part of the equation consisting of the second order terms.
The type of the second order PDE (1) at a point (x_0, y_0) depends on the sign of the discriminant defined as

$$\Delta(x_0, y_0) = \begin{vmatrix} S & 2R \\ 2T & S \end{vmatrix} = S^2(x_0, y_0) - 4R(x_0, y_0)T(x_0, y_0).$$

If $\Delta(x_0, y_0) > 0$, the equation is called hyperbolic; $\Delta(x_0, y_0) = 0$, the equation is parabolic and $\Delta(x_0, y_0) < 0$, the equation is elliptic.
Note that the given PDE may be of one type at a specific point and of another type at some other point.

The classification of second order PDE depends on the form of the leading part of the equation consisting of the second order terms. And this leading part is Rr+Ss+Tt. Because this is the only part which consist of second order terms. Remaining is fxyzpq=0. No second order term is involved here. So this part is called the leading part so the classification of second order PDE depends on the form of this Rr+Ss+Tt which is known as the leading part.

Or the principal part of the second order PDE. Now the type of second order PDE at a point x0 y0 depends on the sign of the discriminant. It is defined as delta at x0 y0=S 2R 2T S that is S square-4RT at the point. So, delta we calculate the values of SR and T at the point x0 and y0 and then determine S square-4RT at the point x0 and y0. If this s square-4 rt at x0 y0 is>0 then the equation will be called hyperbolic.

And if this is=0delta delta0 it is called parabolic and if this is delta<0 we called it elliptic. The given PDE may be one type at a specific point and of another type at some other point. So it may happen that at x0 y0 it is of hyperbolic type and at another point x1 y1 it is parabolic type or it is elliptic type. So, the given may be of one type at a specific point and of another point at some other point.

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For example, the Tricomi equation

$$u_{xx} + xu_{yy} = 0$$

is hyperbolic in the left half plane $x < 0$; parabolic for $x = 0$ and elliptic in right half plane $x > 0$, since $\Delta = -4x$.

A PDE is called hyperbolic (or parabolic or elliptic) in a region Ω, if the PDE is hyperbolic (or parabolic or elliptic) at each point of Ω.

The terminology hyperbolic, parabolic and elliptic chosen to classify PDEs reflects the analogy between the form of the discriminant $S^2 - 4RT$, for PDEs and the form of the discriminant $S^2 - 4RT$, which classifies conic sections given by

$$Rx^2 + Sxy + Ty^2 + ux + vy + w = 0$$

For example, let us consider this Tricomi equation uxx+xuyy=0. So, if you compare this with this standard form Rr+ Ss+ Tt+ f x y z p q= 0. Here uxx the co efficient of uxx is 1 so R= 1 and S= 0 and T= x. So, delta which is s square 4RT the discriminant will be= S is 0 R is 1 and T is x so we get-4x. So, delta is-4x so now you can see at the point at for any value of x in the in the left half plane let us say this is your 0x axis y axis so this is your left half plane.

So, x is<0 and in the right half plane okay x>0 So, in the left half plane that is here x<0 and when x<0 delta is positive so delta is positive in this side okay parabolic when x=0 that means on the y axis, that is on the okay this Tricomi equation is parabolic on y axis hyperbolic in the left side of the plane left half plane and elliptic in the right half plane. Because in the right half plane x>0 and when axis is > 0 delta is<0 okay.

So, at different points in the xy plane the nature of the Partial differential Equation is different. A PDE is called hyperbolic or parabolic or elliptic in a region omega of the xy plane if the PDE is hyperbolic or parabolic or elliptic at each point of omega. The terminology hyperbolic, parabolic and elliptic chosen to classify PDEs reflects the analogy. Between the form of the discriminant S square-4RT for the PDEs and the form of the discriminant S square- 4RT.

Which classifies the conic sections. You know that conic sections are given by the second-degree equation Rx2+ Sxy+Ty2+ ux+ vy+ w=0. And we say that this conic section represent hyperbola if S square- 4RT>0 and it represents parabola if S square- 4RT= 0 and it represent ellipse if S square -4RT<0. So, by analogy okay of the given PDE of second order the given PDE as hyperbolic, parabolic or elliptic according as S square- 4RT>0 0 or<0.

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The above conic section represents a hyperbola, parabola or ellipse respectively according as $S^2 - 4RT > 0, = 0$ or < 0. There is no other significance to the terminology. The terms hyperbolic, parabolic and elliptic are simply three convenient names to classify PDEs.

The above conic section represents a hyperbola, parabola or ellipse respectively according as S square- 4RT>0 0 or<0. So, this is the region why we have called the given PDE of second order as hyperbolic parabolic and elliptic. The terms hyperbolic parabolic and elliptic are simply the 3 convenient names to classify PDEs. There is no other significance to this terminology.

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Consider the second order PDE

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad (1)$$

where R, S and T are functions of x and y only. The Cauchy's problem consists of the problem of determining the solution of (1) such that on a given space curve Γ it takes on prescribed values of z and $\frac{\partial z}{\partial n}$, where n is the distance measured along the normal to the curve.

Corresponding to (1), consider the λ-quadratic

$$R\lambda^2 + S\lambda + T = 0 \qquad (2)$$

Consider the second order PDE Rr+ Ss+ Tt+ f x y z p q=0 where RS and T are functions of x and y only. The Cauchys problem consists of the problem of determining the solution of second order PDE such that on a given space curve gamma it takes on prescribed values of z and the partial derivative of z where n is the distance measured along the normal to the curve corresponding to this equation 1 we know that lambda quadratic is given by R lambda square+S lambda+T=0.

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If $S^2 - 4RT > 0$ (i.e. if (1) is hyperbolic), then (2) has two distinct real roots λ_1 and λ_2, say, so that we have two characteristic equations

$$\frac{dy}{dx} + \lambda_1(x, y) = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2(x, y) = 0.$$

Solving these equations, we get two families of characteristic curves or simply characteristics of the second order PDE (1).

If S square- 4RT>0 then just now we have seen the equation 1 is hyperbolic and in this case what happens is that the lambda quadratic R lambda square+S lambda+T=0 has 2 distinct real roots say lambda 1 and lambda 2 and so that we have 2 characteristic equations dy/dx+lambda1 xy=0 and dy/dx+lambda2 xy=0. Now Solving these characteristic equations, we get 2 families of characteristic curves or we call them as simply characteristics of the second order PDE given by equation 1.

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If $S^2 - 4RT = 0$ (i.e. if (1) is parabolic), then (2) has two equal roots λ and λ so that we get only one characteristic equation

$$\frac{dy}{dx} + \lambda(x, y) = 0.$$

On solving this equation, we obtain only one family of characteristics.

Now if $S^2 - 4RT < 0$ (i.e. if (1) is elliptic), then (2) has complex roots. Hence, there are no real characteristics.

$\frac{dy}{dx} + \lambda_1(x,y) = 0, \frac{dy}{dx} + \lambda_2(x,y) = 0$

$\Rightarrow$ complex characteristics

If it so happens that S square-4RT=0 then the equation 1 we defined as parabolic differential equation in that case what happens the lambda quadratic R lambda square+S lambda+T=0 has 2 equal roots okay let us call them as lambda and lambda so that we get only one characteristic equation dy/dx lambda xy=0. When we solve this equation we get only one family of characteristics.

Now in the case S square- 4RT<0 the lambda quadratic R lambda square+S lambda+T=0 has 2 complex roots which are conjugate of each other then we notice that the characteristic equations are dy/dx+lambda1 xy=0 dy/dx+lambda2 xy=0. So, these are the characteristic equations in this case S square-4Rt<0 lambda 1xy and lambda 2 xy are complex roots so these 2 will give complex family of characteristics.

When we solve them we get complex characteristics. So, there are no real characteristics in this case.

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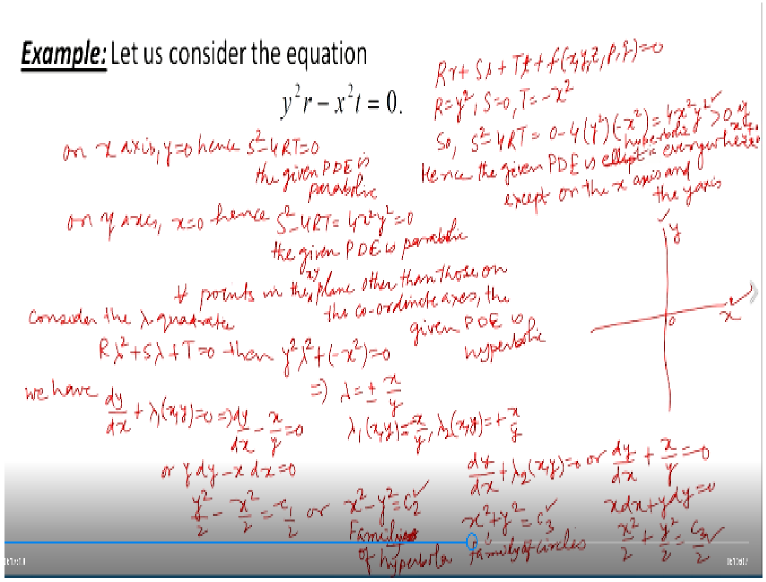

Now let us consider the co efficient r y square r-x square t=0 so comparing with the equation Rr+ Ss+ Tt+ fx, y,z, p,q= 0 you get R=y square, S=0, T=-x square. So, s square=4RT=S is 0 so we get 0-4 R=y square and t is-x square. So, you get 4x square y square which is > 0. Okay. Hence the given PDE is elliptic everywhere okay except on the x axis and the y axis. So, it is>0 if x is not 0 and also y is not 0.

Okay, this means that, this is our x axis and this is y axis so on x axis y= 0 so, S square– 4RT=0. Okay this will come here so the given equation is PDE is, the given equation is parabolic. Okay same thing happens on y axis on y axis, x= 0 hence S square–4RT which is 4 times x square y square so this is 0 so again the given PDE is parabolic. So on x axis okay y axis the given PDE is parabolic elsewhere.

Okay it is hyperbolic for all points in the xy plane xy plane for all points in xy plane other than those on the co-ordinate axis. The given PDE is hyperbolic. So, now let us consider the lambda quadratic which is R lambda square+ S lambda+ T= 0. Okay So, then since R= y square, s= 0 so we have y square lambda square+ T T is-x square. So this gives you lambda=+ or – x/y. So, we have 2 characteristic equations.

We have dy/dx+ lambda1 x,y= 0 which gives you taking lambda1 x,y= x/y, lambda2 x,y=-x/y we get here dy/dx – x/y= 0 or we can say ydy – xdx= 0. So, we get y square/2-x square/2= a constant we can take as c1/ 2 or you can say – c1/2 you take so Rx square-y square=some constant c2> okay now if you take other characteristic equation dy/dx+ lambda2 x,y= 0. So, we have dy/ dx here we have taken –x/y now we take lambda1+ x/y we have taken as-x/y.

So, this is x/y and lambda 2 xy we can take as+x/y. So, this is+x/y=0 okay so xdx+ ydy= 0 gives you x square / 2+ y square by 2= some constant c3/2. So, this gives you x square+ y square= some constant c3. So, now we get 2 characteristic 2 family characteristic characteristic curves one is given by x square – y square= c2 and one is given by x square+ y square= c3 okay. So, this represents a family of hyperbolas and this represents family of circles.

So, the given PDE the characteristic curves are 2 families of, one is the family of hyperbolas and the other is the family of circles.

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Hence, the characteristics are given by

$$x^2 + y^2 = c_1, \quad \text{and} \quad x^2 - y^2 = c_2$$

which are the required families of characteristics. Here these are the families of circles and hyperbolas respectively.

The characteristic of curves or characteristic are given are given by x square+ y square=some constant and x square–y square=some constant which are the required families of characteristics. So these are the families of circles and hyperbolas.

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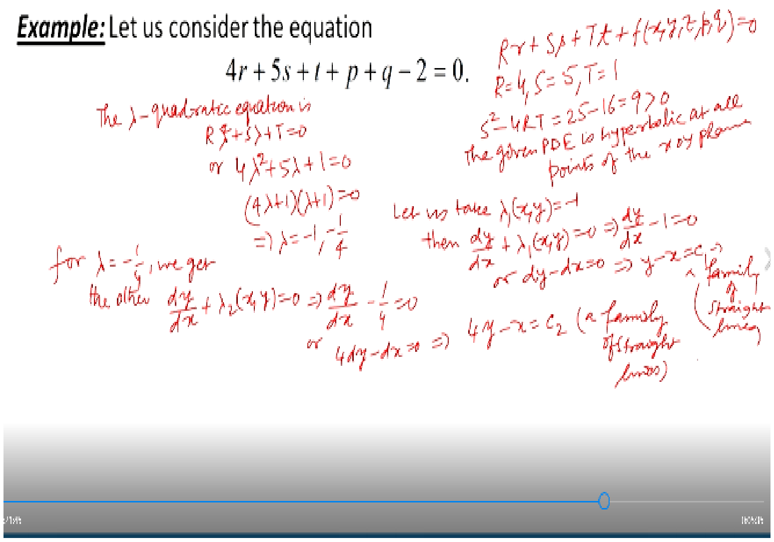

Now, let us consider the equation 4r+ 5s+ t+ p+ q-2= 0. So again let us compare, then what we have R= 4, S= 5, T= 1. So s square–4RT here will be 25–16 that is 9 okay but it is strictly positive so the given equation is the given PDE is hyperbolic for all xy at all points of the xy plane. So, now let us write the lambda characteristic lambda quadratic equation R lambda square+ s lambda+ T= 0.

So in this case R is 4 so 4 lambda squares=5 so 4 lambda square+ 5 lambda+ 1= 0 and its factors are 4lambda+ 1* lambda+ 1 4lambda square+ lambda+ 1, yeah this gives you lambda=-1, and-1/4. Let us take lambda 1 to be-1 okay the characteristic equations are dy/dx+ lambda1 x,y= 0 so this gives you dy/dx lambda 1 is-1 or dy-dx= 0.

So, we can find y – x= some constant c1 okay so this is one familiar of straight lines Now for lambda=-1 / 4 we get the other characteristic equation as dy /dx+ lambda 2xy= 0 which gives you dy /dx+ – 1 / 4= 0 or 4dy – dx= 0. So, this implies 4 /– x= some constant c2 which is family of straight lines. So, this is a family of straight lines y- x/ c1 is a family of straight lines 4y – x= c2 is as also a family of straight lines So in this case the characteristics are 2 families of straight lines

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Hence, the characteristics are given by

$$4y - x = c_1, \quad \text{and} \quad y - x = c_2.$$

Thus, we get two families of straight lines.

So, this is what we have.

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Example: Find the D' Alembert's solution of the Cauchy's problem:

$$z_{xx} = \frac{1}{c^2} z_{tt}, \quad (c > 0)$$

satisfying $z(x,0) = f(x)$ and $z_t(x,0) = g(x)$

where $f(x)$ and $g(x)$ are given functions representing the initial displacement and the initial velocity, respectively.

$t = \frac{\partial^2 z}{\partial y^2}$ here $(y = t)$

$Rr + Ss + Tt + f(x,y,z,p,q)$

The given PDE is hyperbolic $\forall (x,y)$. Here $z_{xx} - \frac{1}{c^2} z_{tt} = 0$, therefore $R = 1, S = 0, T = -\frac{1}{c^2}$

$S^2 - 4RT = 0 - 4 \cdot 1 \cdot (-\frac{1}{c^2}) = \frac{4}{c^2} > 0$

The λ-quadratic is given by $R\lambda^2 + S\lambda + T = 0$

or $\lambda^2 - \frac{1}{c^2} = 0$ we have $\lambda = \pm \frac{1}{c}$

Let $\lambda_1(x,y) = \frac{1}{c}$

& $\lambda_2(x,y) = -\frac{1}{c}$

then $\frac{dy}{dx} + \lambda(x,y) = 0$ we have $\frac{dt}{dx} + \frac{1}{c} = 0$ or $dx + c\,dt = 0 \Rightarrow x + ct = c_1$

$\frac{dt}{dx} - \frac{1}{c} = 0$ or $dx - c\,dt = 0 \Rightarrow x - ct = c_2$

So, now let us go to the differential equation z x x= 1/c square ztt So here zxx–1/c square ztt= 0 We have the standard form of differential equation as Rr+ Ss+ Tt+ F xyzpq= 0. So, therefore R= 1 S= 0 T=- 1/c square this is small c S square– 4 RT is what S is 0 – 4 R is 1 okay and T is-1 / C square= 4/c square so this is > 0. So, S square–4RT is > 0. Hence the given PDE okay the given PDE is hyperbolic for all xy for all the points xy.

Now let us find its characteristics we have lambda to be quadratic is given /R lambda square+ S lambda+ T= 0 okay so r=1 so we have Lambda square plus S= 0 and t–1/C square. So, we have lambda=+-1/c okay let lambda 1 xy be 1/c and lambda 2 xy=-1/c then from dy/dx+ lambda 1 xy= 0. We have dy /dx lambda 1 xy is1/c so 1/c=0 or dx. We have to take x and t so that other variable y is to be replaced with T.

So, that we can do this I am taking this as T okay this T is delta square z/ delta y square in this standard form. Here it is given by small t here y=small t t is the time his is actually one dimensional wave equation where t represents the time. So, y is t here so this is actually dx+Cdt=0 So, this gives you x+ ct as a constant so c1 other equation is dy is again dt dt / dx– 1 / c= 0 or we can say dx–cdt= 0.

So this gives you x–ct= some other constant c2 So x+ ct= 1 and x – ct= some other constant c2 They are the families of characteristics in the case of one dimensional wave equation zxx= 1/C square ztt. That is all in this lecture, thank you very much for your attention.