Ordinary and Partial Differential Equations and Applications
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Lecture – 41
Charpit's Method- II

Hello friend, welcome to this lecture, in the lecture, we will start of our decision of Charpit method and if you recall in previous lecture, we have discussed the compatible system and based on compatibility, we discuss the method; Charpit method to solve first order non-linear PDE and in this regard, I just want to say one more thing that in previous lecture, we have defined compatible system in way that in many of the books, you will find the compatible system means every solution of the 2 equation are common.

It means that once every solution of first PDE is a solution of second PDE and vice versa and but in some book, you will find that compatibility system means they have at least one solution in common, so here I am not giving, so you can take an definition of compatible system that every solutions of first equation is a solution of second equation.

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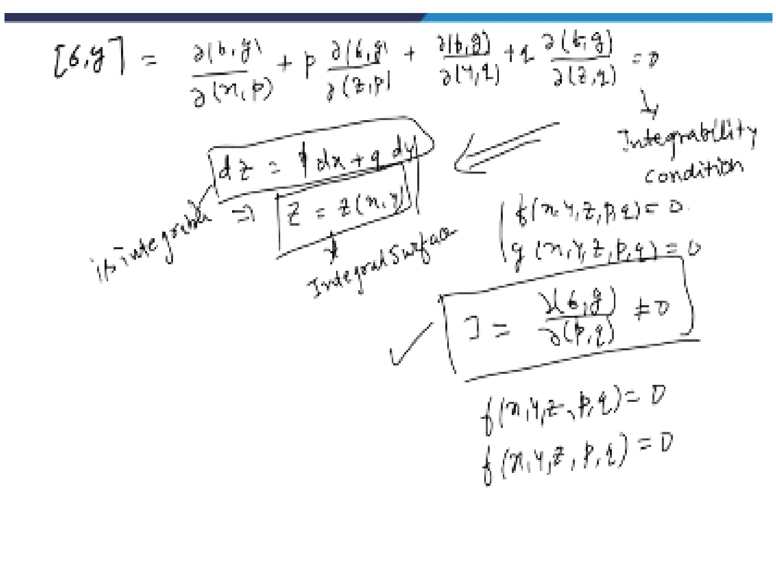

But the condition that box of fg is = 0 that is this that box of fg = 0 which is a Jacobian of fg with respect to xp + p times Jacobian of fg with respect to zp + Jacobian of fq with respect to yq + q times Jacobian of fq with respect to zq is = 0, this condition is basically integrability condition which ensure that your dz = pdx + qdy is integrable and hence we can find out by integrating this dz = pdx + qdy, we can find out a surface z = zxy, which we call as an integral surface which is a common solution of both the equation f x y z p q = to 0 and g x y z p q = 0.

And to obtain this condition, we assume that this Jacobian which is Jacobian of fg with respect to pq is nonzero, so we say that whatever definition of compatibility you choose but this box fg is = 0 is an integrability condition which help us to find out a one parameter family of common solution and here we assume that this Jacobian is nonzero because if you can consider this example, I will take x y z p q = 0.

And g also the same equation that is x y z p q = 0, so here we can say that Jacobian is 0 , box is 0 but I cannot find out means, if the formula of form of f is complex I cannot find out p and q in terms as a function of x, y and z, so here this Jacobian is nonzero is quite important which help us to find out p and q in terms of x, y and z, and this box fg is = 0 is an integrability condition which make sure that this for find differential equation is an exact derivative of some function; z = zxy.

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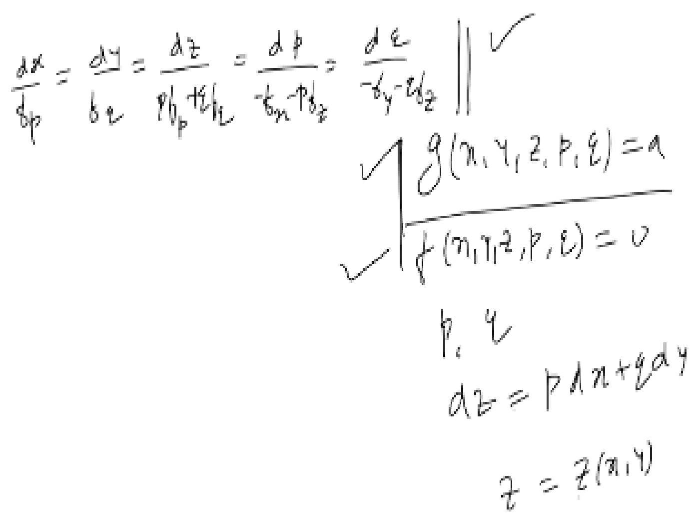

So, and with the help of this compatibility, we have obtained the Charpit auxiliary equation, which is dx upon fp = dy/ fq = dz/pfp + qfq = dp/ - fx – pfz = dq divided by –fy - qz and with the help of these auxiliary equation, we try to find out g x y z p q = a, which is compatible with your original f x y z p q = 0, so this is a given first order PDE which we want to solve, now with the help of Charpit auxiliary equation, we try to find out another first order PDE involving one parameter.

And we say that f and g are compatible to each other and then we try to find out the p and q from these 2 equation and then dz = pdx = qdy will solve and we will have the solutions z ss z xy that is the idea of Charpit method. Now, let us focus on the Charpit method and consider some example so that we can understand whatever we have discussed just now.

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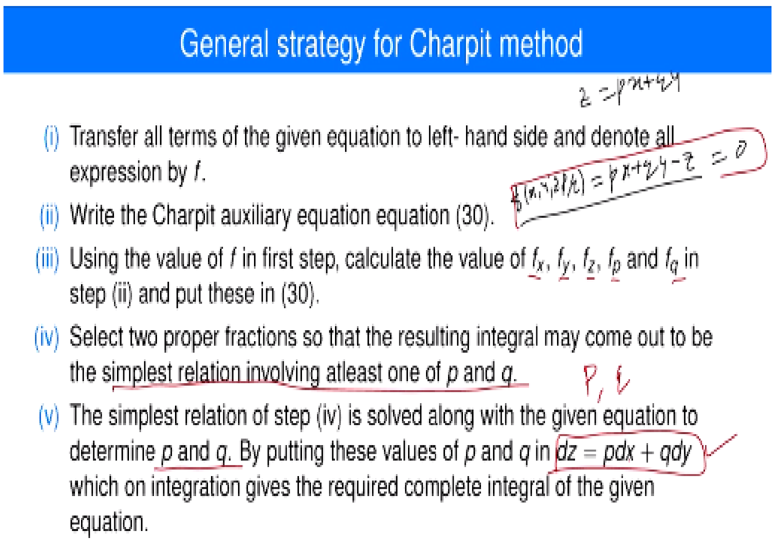

So, here, so let us again discuss the general strategy Charpit method, this I am repeating it, so first thing is if your PDE is given, you transfer all the terms on the given equation to one side, let us say left hand side and denote all expression by f, so means that whatever equations suppose, you equation is given as z = px + qy, so this you can write it px + qy - z and denote this as f of x y z p q, so you denote this as px + qy – z = 0.

So, this is f x y z p q is = 0, so once we do this, then write the Charpit auxiliary equation which we have just given and using the value of f in first step to calculate the value of fx, fy, fz, fp and fq and put these in Charpit auxiliary equation and then this is a very important step is this that here you have to select 2 proper fraction, so that the resulting integral may come out to be the simplest relation involving at least one of the p and q.

So, what we try to do; choose among these equation, you choose 2 in a way such that you can get a simplest expression for p or q or between p and q, so it cannot; we should not choose any equation which will give a relation between x and y, we must have involved our p or q or both. Now, the simplest relation of a step 4 is solved along with the given equation to determine p and q.

So, here in step 4, you may get a relation between pq, px, py, qx, qy, any one of them and once we have this relation between p and q, p in terms of x y z, use the original equation that is f of x y z p q= 0 to find out the value of q, so the other one, so it means that once we do this procedure, then we are able to find out p and q as a function of x, y and z and then put the value of p and q in this equation dz = pdx + qdy.

And we already know that these f and the new relation is compatible to each other using compatible we say that this has to be integrable because we have already utilise the box of fg is = 0, so this has to be integrable and then we do some work out and we solve this equation dz = pdx + qdy to find out our solution and with that solution; obtained solution contains 2 arbitrary parameter and we say that the solution is 2 parameter family of solution and known as complete solution.

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Example 5

Find a complete integral of $2(z + px + qy) = yp^2$.

Solution. Given equation is $f(x, y, z, p, q) = 2(z + px + qy) - yp^2 = 0$. Charpit's auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-f_x - pf_z} = \frac{dq}{-f_y - qf_z}$$

which reduce to

$$\frac{dx}{2x - 2yp} = \frac{dy}{2y} = \frac{dz}{p(2x - 2yp) + 2qy} = \frac{dp}{-4p} = \frac{dq}{p^2 - 4q}$$

Now $\frac{dy}{2y} = \frac{dp}{-4p}$ or $2\frac{dy}{y} + \frac{dp}{p} = 0$.

So, now let us are move to example, so find a complete integral of 2z + px + qy = yp sqaure, so here we have this equation now, as we are pointed out that let us shift everything in one side and denoted by f x y z p q, so let us say that 2z + px + qy - yp square = 0, so that is f x y z p q. Now, in a second step, you write down the Charpit auxiliary equation that is dx/ fp = dy/fq = dz/pfp + qfq = dp upon –fx –pfz = dq on – fy – qz.

F is already given to us, we can easily find out fp, so fp, you can simply say that it is 2x – 2yp, so 2x – 2yp dy fq; fq is what? Simply 2y, so that is written here, dz/2fp, so you can multiply and you can simply write it here and dp upon –fx, if you look at fx is what? Fx is simply 2p and fy is what? Fy is 2q and fz is 2 here, so we can write it here –fx – pfz as 2p – p and fz is 2, so here, so it is -2p, this is – 4p, so dp upon -4p.

Similarly, you can find out the –fy –qz, the –fy means – 2q - q times 2 here, so it is -4q, so fy is what? Fy is 2q, fy is 2q - p square, so 2q - p square because this term is also there, so 2q - p square, so here – p square will also be there, so that is p square. So p square – 4q will get it here, now, so we have; once we have auxiliary equation now the very important step will come into picture.

Now, choose among these relations in a way, choose 2 equations in a way such that we can simplify our p or q in terms of x and y, so here if you look at the dy upon 2y and dp upon -4p, we can choose this and we can write dy upon 2y = dp upon -4p and we can simply write down; simplify this and we can write 2dy/y + dp upon p = 0. Now, here one thing you may note that here I am choosing this and it may happen that somebody else choose some other 2, right.

The only thing is that you should have a relation between say p as xyz or q in terms of xyz or p in terms of q, so it may happen that a given partial differential equation may have more than 2 complete integral, right. So, here I am choosing one, so let us say that 2dy/y + dp upon p =0.

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On integrating, we get $\ln p + 2\ln y = \ln a$ or $py^2 = a$ i.e. $p = \frac{a}{y^2}$. Now, by using the value of p in $f(x, y, z, p, q) = 2(z + px + qy) - yp^2 = 0$, we get $q = -\frac{z}{y} - \frac{ax}{y^3} + \frac{a^2}{2y^4}$.
Therefore,

$$dz = pdx + qdy = \frac{a}{y^2}dx + \left[-\frac{z}{y} - \frac{ax}{y^3} + \frac{a^2}{2y^4}\right]dy.$$

Multiplying both sides by y and re- arranging, we get
$(ydz + zdy) - a\left(\frac{ydx - xdy}{y^2}\right) - \frac{a^2}{2y^3}dy = 0$ i.e.

And when you simplify, you will get p = a/ y square, by simply, once we have p = a/ y square, now use your equation f x y z p q = 0 that is this put p = a upon y square and solve for q, let us say that q is given as –z/y –ax upon yq + a square upon 2y4, so p and q is already given to us, now let us solve this dz = a/ y sqaure dx + now within bracket –z/y -ax upon yq + a square upon 2y4 and dy.

Now, here you can simply say that you can take out y here and we can write y dz, so y dz is = a/y dx – z dy – ax upon y square dy + a square upon dy + a square upon 2yq dy, now here one thing we have noted down that here you have to arrange in a way, such that it can be written as derivative of some known form. So, here since we say that it is ydz, so it means that you collect some terms of y and z.

Here, we have y dz and here if you look at it is – z dy, so this can be collected at a same place and here, a/ y dx is there here –ax/y square dy, so here you collect these two and these two and since it is only y and it is written with dy, so no problem, we can keep it there here, so we can write it here, y dz + z dy – a y dx – x dy divided by y square, so we are collecting these 2, - a square upon 2yq dy = 0.

So, here you have to use your experience in a way such that you write your dz = pdx + qdy in a way that it will come out to be derivative of some exact form, right so it is so much better if you remember some of form. So, here if you look at ydz + z dy, then I can write this as d of zy and here – aydx – xdy upon y, I can write it as –a d of say, x/y, right. Similarly, you can integrate this because it is quite easy, so we can simply integrate.

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$$d(yz) - ad\left(\frac{x}{y}\right) - \frac{a^2}{2}y^{-3}dy = 0.$$

On integrating, we get

$$yz - a(x/y) + (a^2/4y^2) = b$$

where a and b are arbitrary constants.

And we can write it that I can write this as d of yz – ad of xy; x/y – a square upon 2 y to the power -3 dy = 0, now you integrate, so when we integrate this will be yz and here it is –a x/y and here we have a square upon 4y square and then one more integration constant will appear here, so here we have a relation between xyz involving to the parameter a and b, this a you have obtained while solving your Charpit auxiliary equation.

And when you integrate this, you will have 1 arbitrary parameter coming to picture that is a here and one more parameter will come, when you solve dz = pdx + qdy and hence, we have 2 parameter; arbitrary parameter present in your solution and we say that this is the 2 parameter family of solution of f x y z p q = 0 and hence we call this as a complete integral of your partial differential equation, so this is one problem let us consider one more problem.

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Example 6

Find a complete integral of $(p^2 + q^2)y = qz$.

Solution. Given equation is

$$f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0. \qquad (31)$$

Charpit's auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-f_x - pf_z} = \frac{dq}{-f_y - qf_z}$$

which reduce to

$$\frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{qz} = \frac{dp}{pq} = \frac{dq}{-p^2}$$

Now $\frac{dp}{q} = \frac{dq}{-p}$ or $pdp + qdq = 0$.

So, here find a complete integral of p square + q square y - qz, so first step, write your equation as f x y z p q = 0, so you write down p square + q square y – qz = 0, so first thing is you transfer everything in one side and write as f x y z p q = 0, so once we have this and then you calculate you write on your Charpit auxiliary equation that is this, so you need to remember the form of Charpit auxiliary equation.

And then, once your f is given, you can easily find out f of x that is given a 0, f of y that is p square + q square, f of z that is –q, f of p is = 2py that is all, f of q = 2qy – z here and that is all and using these derivatives, now write dx/fp; dx/fp is 2py, so dx/2py dy upon fq, so fq is 2q – z, we have written here, dz/ pfp.

Now, pfp is what? P * 2py + q * 2, so 2q square y – qz, now if you look at this, this is what 2p square y +2q square y and if you use a equation this is what; 2qz, so 2qz – qz is qz here, so we have written qz, now dp upon - fx is 0 – p, fz is –q, so dp = pq and dq upon –fy; fy is p square + q square with – sign – qfz is –q, so we can simplify – p square - q square + q square, so cancel out and we have –p square.

Now, we have these relation, now if you look at here, we have quite easy pattern, so I can use many in fact, but how many we can use these in this, these 2, so let us use these 2 and let us say dp upon pq = dq upon –p square, when you simplify, it is what? dp upon q = dq upon – p, we simply cancel out this p and when you again simplify, it is pdp + qdq = 0, when you integrate it is a simply what; d of p square + q square/2.

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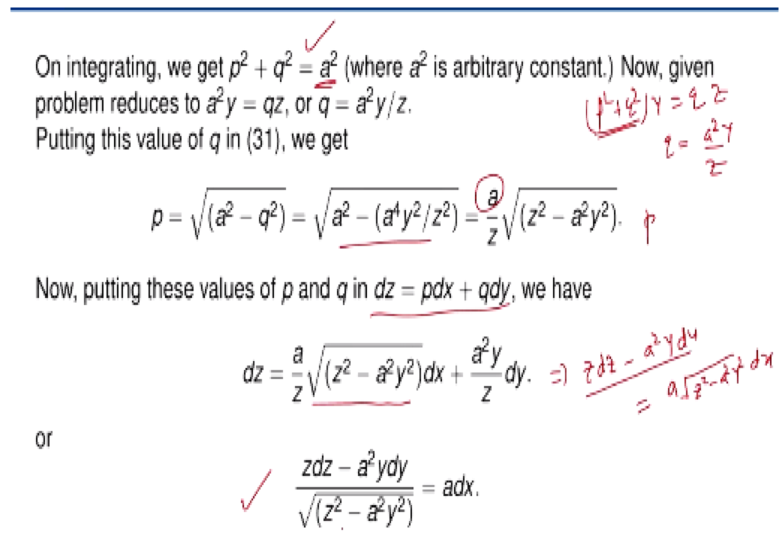

So, here what you will have? Here, we have been p square + q square = a square, where a square is some arbitrary constant, so here you can use any constant; c1, c2, d1, d2 but let us say that for simplicity, i am using a square, we will see that why this a square will have this, okay. Now, given problem, we have p square + q square = a square, but this is one relation between p and q and now you utilise your equation that is p square + q square y = qz.

So, p square + q square y = qz, now this value is coming out to be a square, so q is = a square y upon z and once we have q, you again use equation to find out p, so p is under root a square – q square. Now, a square – q square is a 4 y square upon z square, so we have this, so we can simplify a/z under root z square – a square/ y square. Now, here you can utilise that why we have started with a square.

So, otherwise it is simply under root some constant, so here we have just assumed that constant a square, it is just a simplicity, just for simplicity we have taken, so p is written as a/z under root z square – a square y square, I am taking only the positive value, you can take both positive and negative value. So, now once we have; so I am just giving you a method to find out in one case, when I am assuming q as a square upon y; a square y upon z.

And p as a + sign, you may take the negative sign, no problem, now putting these value of p and q in dz = pdx + qdy, we have this equation dz = a/z under root z square - a square y square dx + a square y upon z dy here. Now, if you look at here, here your z is involved, y is involved and that is all, so what we do here looking at this term, you need to arrange your thing in a way such that it is a derivative of this kind of thing.

So, let us write it here, you multiply both side by z, so what you will get? Z dz, right - a square y dy = a under root z square - a square y square dx and now you get the idea. The idea is that that you divide by this under root z square - a square y square and you write z dz - a square y dy divided by under root z square - a square y square = adx. So, you always look at the coefficient of the dx and dy.

And according to the coefficient of dx and dy, you arrange your equation in a way, such that it will become a derivative of some exact form, so here it is if you look at, it is derivative of under root z square - a square y square.

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On integrating, we get $\sqrt{(z^2 - a^2y^2)} = ax + b$ or $z^2 - a^2y^2 = (ax + b)^2$, which is a required integral, where a and b are arbitrary constants.

$z^2 = a^2y^2 + (ax+b)^2$

So, using this, let us say that you integrate and you will get under root z square - a square y square and here we have adz, so ax + b, so we have ax + b, you square it out and you have z square – a square y square = ax + b whole square, so we can write down equation, z square = a square y square + ax + b whole square and this involves 2 arbitrary parameters, so we call this as 2 parameter family of solution of the PDE, which we have discussed.

And this is a complete integral of, you can say that this is a complete integral of our first order non-linear PDE, so I hope that now things are little bit clear, so let us consider one very special type of first order equations. Here, special type of first order equation means, here f xyz pq is taking some kind of a simpler form and when it is simpler form, then your solution is also coming to be very easy.

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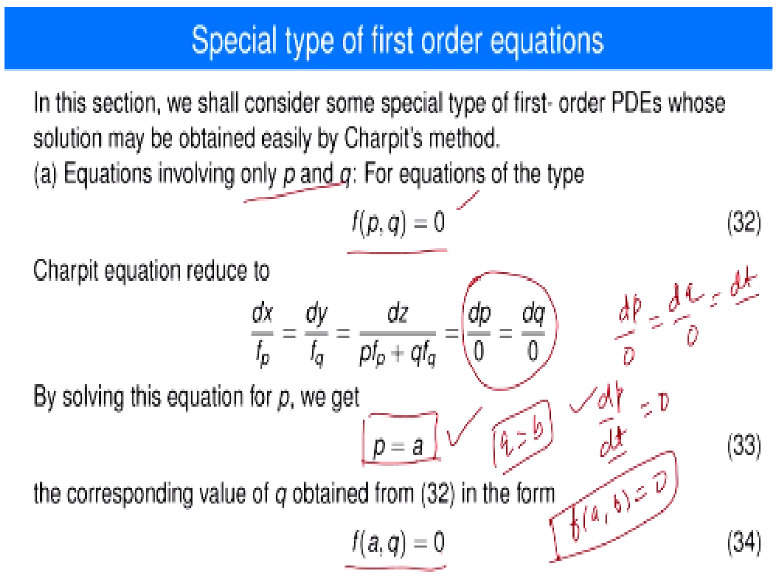

You may remember these things or you can understand the procedure by which we can find out the solution in these simpler cases. So, here they are 4 cases we are considering, so first case is that when f x y z p q involved only p and q, so it means that our equation is looking like fp, q = 0. So, if we have fp, q, then here we say that in this kind of equation you have a simpler way to solve this but let us use the Charpit method to find out that simpler way.

So, here your Charpit equation reduced to dxy upon fp = dy upon fq = dz upon p fp + q fq = dp upon, here fx is 0, fz is 0, so dp upon 0 and dq upon 0, so here you can say that this simply means what that your p has to be a constant, in fact if you look at dp upon 0 means what; if you look at dp upon 0 means that if we take this as some parameter say dt, then this simply say that dp/dt = 0.

So, it means that p is constant with respect to any parameter, so we say that p is = a, that is what it means that dp/0, it has this; you should not consider this as division of 0, it is a ratio of dp with 0, so this can be understood in a way that if you take these common ratio as some dt, where t is some parameter, then dp/dt is = 0 and we can say that p is constant with respect to any parameter, so we can say that p is = constant, so we are writing here.

Similarly, you can write q = some constant b, right or then of course, these constant cannot be taken as arbitrary, this a and b must satisfy the equation that is f of p, q = 0, so it means that either you use p = a = b and you put the condition that f of a, b is = 0 or you can say that you use only one that is p = a and find out the value of q from this equation that f of a q = 0, so it is up to you.

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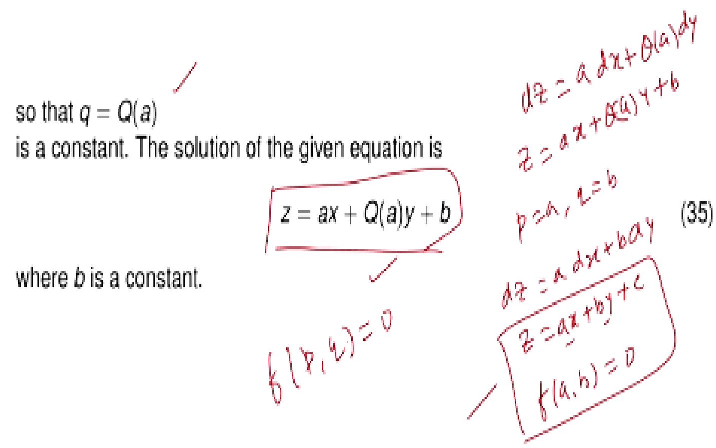

So that you can find q as some function of q of a, which you can obtained from here also that if f of ab = 0, then from here also you can find out b as a function of a, so here we can simply say that in this case, your solution is what? Z = ax + here because it is dz = a, p is = a dx + q; q is simply a dy and when you integrate, you have z = ax + q of ay + some v. If you use p = a and q = b, then what you will have? Dz = adx + b dy.

And when you solve, it is z = ax + by + c, now here we are getting 3 parameters but it is not the case because this a and b are not independent in fact, they are satisfying this relation that f of ab = 0, so when you write this kind of relation z = ax + by + c, then f of ab are related by this or you simply write z = ax + q ay + p, is that okay. So, you can use any of this relation, either this or this.

So, it means that when f of pq is = 0, it means that your PDE is independent of xyz, then you simply assume p as a and q as b and you can write down your solution as z = ax + qay + p, is that okay.

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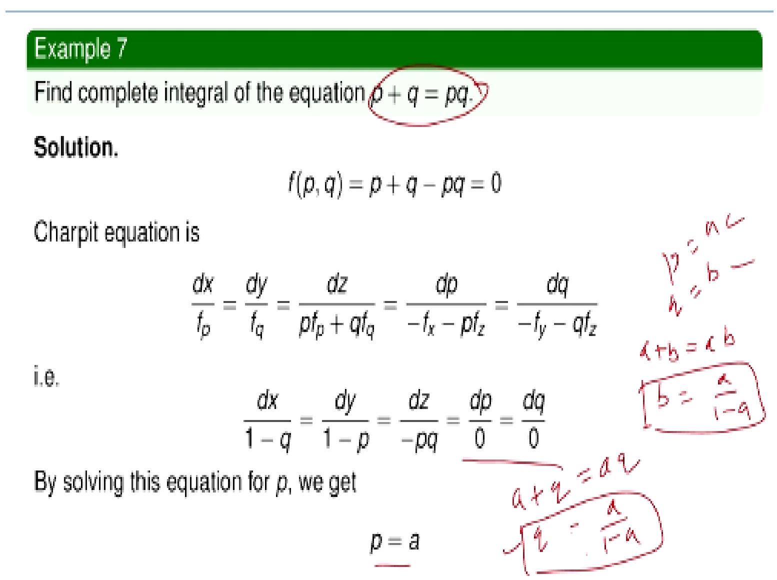

Now, move on to next example based on this that p + q = pq, here if you look at your equation is complete in terms of p and q, no xyz is appears, so here when you simply solve, then you have p = a and q = b, so here if p = a, you assume, then you can find out the value of q, so by a + q = aq, so q you can find out as a upon 1 – a, right.

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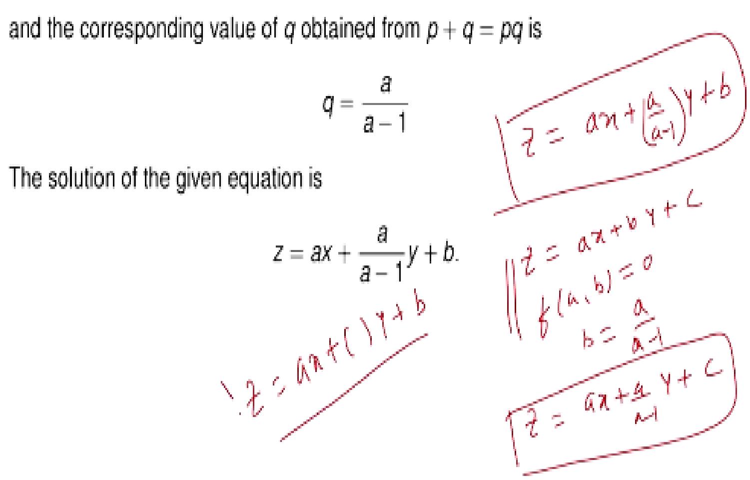

So, that is here but if I again as I said you can assume p = a and q = b and if you simplify this then a + b = ab, so here also you can find out the value of b as a upon 1 – a, so either you use this kind of thing to find out b in terms of a or you use q to find out; use p = a to find out q in terms of this, so whatever you want to use, you will have this. So, here as I pointed out, you can write z = ax + a upon a – 1y + b, so this is one relation.

Or if I look at; I have already given ax + by + c here, now we have already noted down that f of ab = 0 and by following this, we have b upon a upon a – 1, so here when you use this, then also we have ax + a upon a - 1 y + c, so you can see that in both the thing your solution is matching, the only thing is that your arbitrary parameter b is now presented as c that is all.

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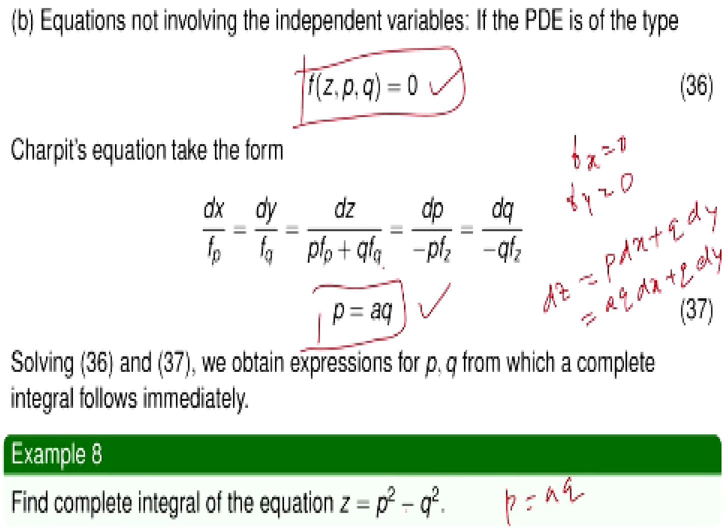

So, it means that here your solution is always lenient terms of x and y, you can write z = ax + some constant times y + some b, so when your PDE is independent of pq, then you have this kind of relation. Now, moving on second type that is f of z pq, here your equation is independent of x and y and when you write down the Charpit equation, then here we can write dx upon fp dy upon fq = dz upon pfp + qfq.

And now here, in last 2 equations, dp upon - pfz because fx is 0 and fy is 0, so we have dp upon - pfz = dq upon – qfz, so here since f is depending on z, so fz is non-zero, you can cut it out and when you solve this, we have p = aq, so one relation p = aq comes and using this and the problem means the original PDE, you can solve p and q and you can find out dz = pdx + qdy, so here we can write aq dx + q dy.

So, you need to find only q that we can find out using the first order PDE and we can find out our relation. Now, let us take one example based on this; z = p square – q square and we want to find out our complete integral. So, here it is independent of x and y, so here our relation is p = aq, you can simply put it here.

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Solution.

$$f(p, q) = p^2 - q^2 - z = 0$$

Charpit equation is

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-f_x - pf_z} = \frac{dq}{-f_y - qf_z}$$

i.e.

$$\frac{dx}{2p} = \frac{dy}{-2q} = \frac{dz}{2(p^2 - q^2)} = \frac{dp}{p} = \frac{dq}{q}$$

By solving this equation for p, we get

$$p = aq$$

By putting this value in $p^2 - q^2 - z = 0$, we get

$$q = \pm\sqrt{\frac{z}{a^2 - 1}}$$

And we can simply say that p = aq and you can find out the value of q here, so p square – q square – z = 0, when you put p = aq, so you can get your q like a square q square - q square = z, so q square is basically z upon a square -1, so you can write it under root of this plus minus, so q is basically + - under root z upon a square -1 and then p is also written as a + - a under root z upon a square -1.

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and

$$p = \pm a\sqrt{\frac{z}{a^2 - 1}}$$

The equation $dz = pdx + qdy$ reduces to

$$dz = \pm a\sqrt{\frac{z}{a^2 - 1}}dx + \pm\sqrt{\frac{z}{a^2 - 1}}dy$$

$$= \pm\sqrt{\frac{z}{a^2 - 1}}(adx + dy)$$

so

$$z^{-1/2}dz = \pm\sqrt{\frac{1}{a^2 - 1}}(adx + dy)$$

On integrating, we get

$$z = \left(b \pm \frac{1}{2\sqrt{(a^2 - 1)}}(ax + y)\right)^2.$$

which is the required solution

Then, you solve your dz = pdx + qdy, so now you simply see that in numerator, we have under root z that you can put it that side, so we can write + - under root z upon a square – 1 adx + dy and by putting root z here, we have z to the power -1/ 2 dz = this quantity, so when you simplify, you will simply get z = p + -1 upon under root 2a square – 1 ax + y whole square. So, here this integrating this is quite easy because it is simply integrating, no problem.

So, this is the required solutions so, in this case when f z p q = 0, you simply write p = aq and using this you can solve to find out both p and q and then you integrate it, is that okay d and we have complete evaluation like this.

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(c) Separable Equations: A first- order PDE is separable if it can be written in the form

$$f(x,p) = g(y,q) \quad (38)$$

and corresponding Charpit equation is

$$\frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{pf_p - qg_q} = \frac{dp}{-f_x} = \frac{dq}{g_y}$$

so that we have an ODE

$$\frac{dp}{dx} + \frac{f_x}{f_p} = 0$$

in x and p. Writing this equation in the form $f_p dp + f_x dx = 0$, its solution is $f(x,p) = a$ and p and q be determined by the relations

$$f(x,p) = a, g(y,q) = a \quad (39)$$

and then proceed in the general theory.

Now, in third, which is the case of separable equation, here your equation is not involving any z and it can be separated out into form that f of xp = g of yq, so it can be written like this, so it means that here, f of x y p q = 0 can be written as f of xp = g of yq, so you can separate like this. Now, in this case when you write down the Charpit equation then, here dxy fp =; so here fx p – gyq = 0 that is your fx value.

So, fx dx/ fp is nothing but small fp = dy upon -gq = this thing, you write down your Charpit equation and now if you simplify here, now if you simplify dx upon fp and dp upon –fx and then what you will get dp upon dx + fx upon fp = 0 and when you solve this, this is what fp dp + fx dx, so we have fp dp + fx dx = 0 means what? It is d of fx, p =0, so it means that your f of xp is a constant value.

If fxp is constant, so gyq is also constant; constant; the same constant, now with this to relation you find out p and q that is idea, so it means that when your equation is separable, then your; these value is just a coming out to be a constant and with this you can find out p and q and once we have p and q, you can solve dz = pdx = qdy.

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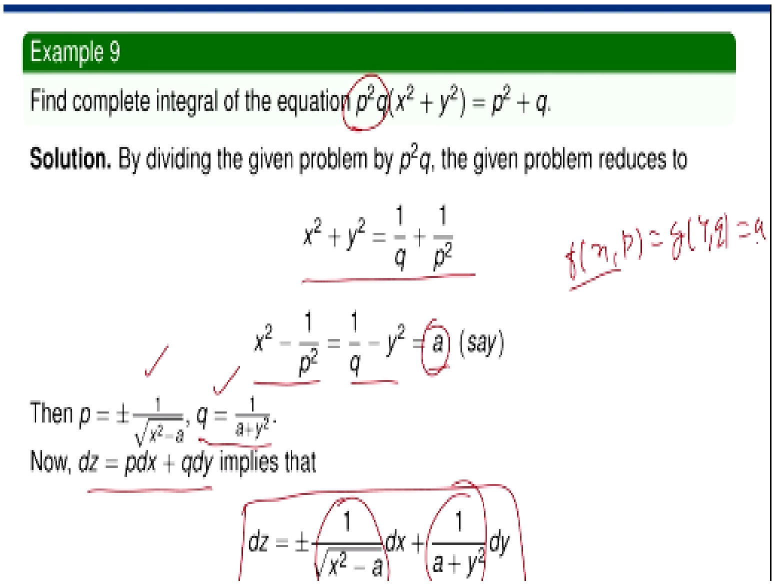

So, let us take one example, so p square q x square y = p square + q, here it may not be so obvious that it is a separable equation, so you have to work it out, you simply say that it is p square q divided by p square q and when you divide, you will be written as x square + y square = 1/ q + 1/ p square. Now, you can simplify and you can write x square – 1/ p square = 1/ q – y square.

And as we have pointed out these 2 are coming out to be constant, so x square - 1 upon p square = constant 1 upon q - y square is = same constant, so we can solve for p and q and p is coming to be + -1 upon under root x square - a and q is coming out to be this, you write down dz = pdx + qdy and when you write down, we have this equation and this is simple, quite easy dz = some integral, this can; you can integrate and here also you can integrate.

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On integrating, we get

$$z = \ln\left|x + \sqrt{x^2 - a}\right| + \frac{1}{\sqrt{a}}\tan^{-1}\frac{y}{\sqrt{a}} + b$$

which is the required solution.

And you can write down your solution, z = ln modulus of x + under root x square – a + 1 upon under root a tan inverse y / root a + some integration constant, so this is a solution. So, in case of separable equation, you separate it out and equate the constant value, so here you write f of xp = g of yq and that a constant value that is, right and then integrate, from this you find out p and q and you can integrate this.

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(d) Clairaut Equations: A first- order PDE is said to be of Clairaut type if it can be written in the form

$$z = px + qy + f(p, q) \tag{40}$$

The corresponding Charpit equation is

$$\frac{dx}{x + f_p} = \frac{dy}{y + f_q} = \frac{dz}{px + qy + pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

so that we can take $p = a,\ q = b$. If we substitute these values in (40), we get the complete integral

$$z = ax + by + f(a, b) \tag{41}$$

which is complete integral of equation (40).

Now, the very important part or we can say it is quite common equation that is Clairaut equation, so here first order PDE is said to be Clairaut type, if it can be written in the form z = px + qy + f of p q. So, here if you look at your x y z p q everything is here but it is a kind of a particular form z = px + qy and the component which involves only p and q, so this is linear in xyz and non-linear maybe in p and q. So, if you write down the corresponding Clairaut equation, then here we can say that here we have dp upon 0 = dq upon 0.

So, it means that here if you look at fx is = what? P, fy = q and fz = -1, so we are taking this as px + qy + f of pq – z, so here if you look at – fx – p of z, so you will get what? –fx is –p –p and this is -1, so you will get 0. So, dp upon 0 dq upon 0, so here you can say that your p = a and q = b you can get but this a an b again it will satisfy this equation, so we simply say that pq satisfy this equation.

So, it means that your equation will be what? Z = ax + by + f of ab and if you look at this is a relation between xyz involve in 2 parameters and hence we can call this as a complete integral, is that okay, so here what we did? we have obtained p = constant, q = constant and when you put it back because here I cannot choose both as constant because a and b cannot be an arbitrary constant, so a and b must satisfy the PDE.

When you put it back into PDE, then it is a relation between xyz and involving 2 parameters that is a and b, so it looks; it will be your complete integral, here you need not to solve dz = pdx + qdy that is the important part here because here a and b are not independent say, they must satisfy the PDE, when you put it back to PDE, then you see that it is not involving any of p and q, it is simply written a relation between xyz and 2 arbitrary parameter.

So, it will come out to be your complete integral, you need not to solve dz = pdx + qdy has solved, so that is the idea here in Clairaut equation, okay, so let us take one example based on this.

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Example 10

Find complete integral of the equation $pqz = p^2(xq + p^2) + q^2(yp + q^2)$.

Solution.

$$z = px + qy + \frac{p^2}{q} + \frac{q^3}{p}$$

Charpit equation is

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-f_x - pf_z} = \frac{dq}{-f_y - qf_z}$$

i.e.

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

By solving this equation for p and q, we get

$$p = a$$

Find complete integral of this equation because pqz = this, again it may not look like that it is a Clairaut equation but you have to simplify, so divide by pq, when you divide by pq, you can see that it is z = px + qy + p square upon q + q cube upon p square, so it is linear in xyz and maybe nonlinear in terms of p and q, so when you; since it is Clairaut form, you simply write p = a and q = b and once we have p =a and q = b, you simply put it here.

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and

$$q = b$$

Hence, the solution of the given equation is

$$z = ax + by + \frac{a^2}{b} + \frac{b^3}{a}$$

And you can write our solution as z = ax + by + a square upon + b cube upon a, so with this, I will and my our decision in fact, in this lecture what we have discussed, we have discussed some example based on Charpit method and how we can utilise Charpit method to find out the complete integral of a given first order non-linear PDE and also we have seen some special cases where your Charpit equation will give you very easy answer.

And we can get our solution in a very quick manner and these are very common in say, problem, so we utilise Charpit method and these for a special type of PDE's, so with this I end here and thank you for listening us, thank you.