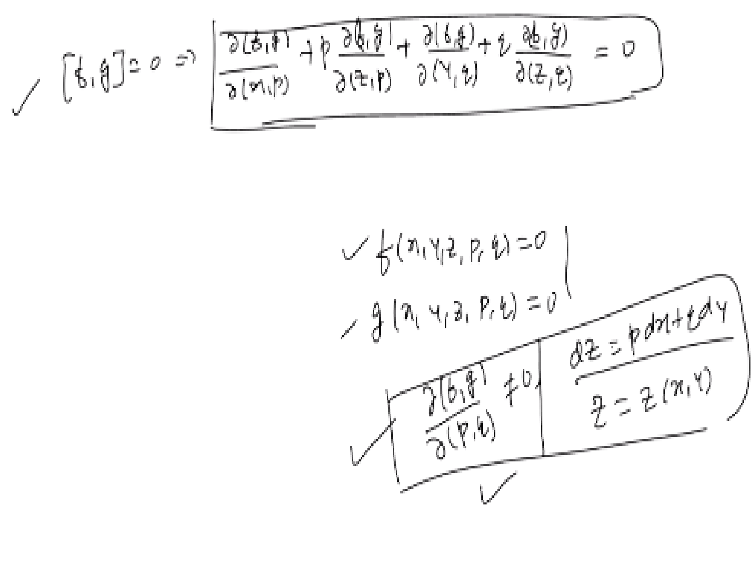
Ordinary and Partial Differential Equations and Applications

Dr. D. N. Pandey
Department of Mathematics
Indian Institute of Technology, Roorkee

Lecture – 40
Charpit's Method- I

Hello friends, welcome to this lecture and in this lecture, we will continue our study of compatible system and if you recall in a previous lecture, we have discussed the definition of compatibility as follows.

(Refer Slide Time: 00:39)



Here, we say that f of x y z p q = 0 and g x y z p q = 0 are said to be compatible if every solution of first equation is a solution of second solution and vice a versa and then we have considered a method of finding at least one solution in common that is you; let us assumed that Jacobian of fg with respect to pq is nonzero and corresponding dz = pdx + qdy is integrable to obtain; give a solution like z = zxy.

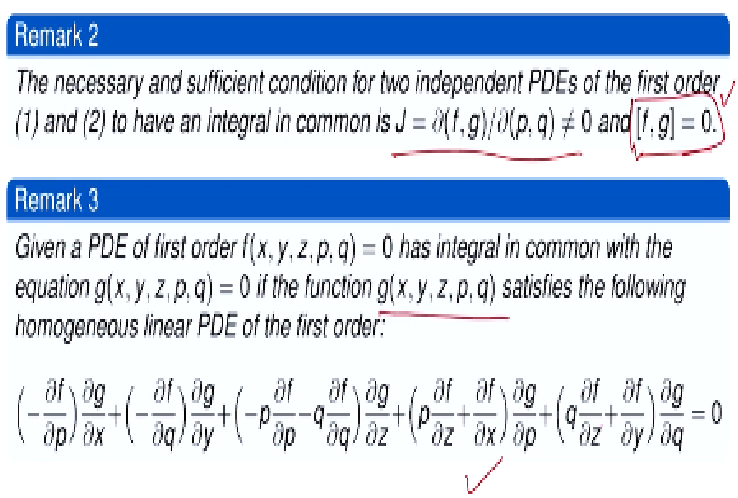
And so, in fact this procedure is to find out the common solution they have, so this method is equally applicable in any 2 equations where they have one solution in common, so based on this in some book or in some literature, they define compatibility in this manner that f and g are compatible to each other, if they have at least one solution in common and the procedure listed here that Jacobian is nonzero.

And the dz = pdx + qdy is integrable is the method to find out that common solution, so then we also discussed the procedure that in place of writing these 2 condition separately, can we simply look at the equation number ; equation f and g to say that whether they are that by f and g can be find out once common solution and that we have obtained in the following thing that if bracket fg is = 0 that is Jacobian of fg with respect to xp + p times Jacobian of fg with respect to zp+ Jacobian of fg with respect to yq + q Jacobian of fg with respect to zq.

If this quantity is = 0, then we are able to find out one common solution of f and g here, so here I am not saying that compatibility means what but I am giving you both the condition that in many of books, you say compatibility means every solution common and in some book, you will find out that they have at least one solution is common but this definition may be say, given in different, different but they give this procedure to find out that one common solution.

So, I am assuming that we are using this condition that bracket of fg is = 0 to say that they have at least one solution in common and the procedure that dz = pdx + qdy is integrable to find out that common solution is that okay. So, now proceed further and see what we can further achieve.

(Refer Slide Time: 03:52)

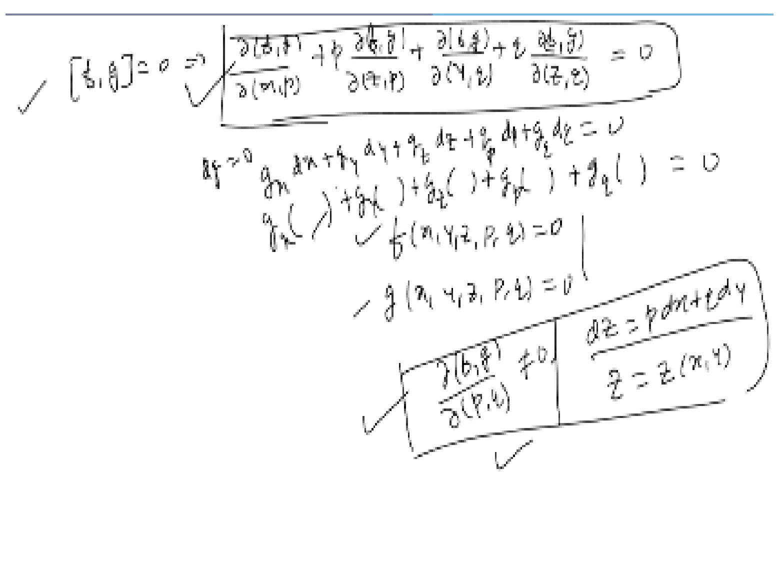


So, here the necessary and sufficient condition for 2 independent PDE of the first order 1 and 2 to how an integral in common is this that Jacobian is nonzero and the bracket is 0 and if you look at the bracket fg is = 0 is already involving this condition that Jacobian is nonzero and basically this is an integrability condition that dz = pdx + qdy is integrable to this, so basically I will assume that this is nothing but an integrability condition.

But in many books you will find this condition as a compatibility condition, so whatever you; if you look at compatibility means they have one solution in common, I agree that bracket fg = 0 is a compatibility condition but I will assume this bracket fg = 0 is a method to find out the common solution that f and g are having. So, next remark is that given a PDE of first order f x y z p q = 0 has integral in common with the equation g x y z p q = 0.

If the function g x y z p q satisfy the following homogeneous linear PDE of the first order that is -fp gx + - fq gy – p fp – q fq * dz + p of z + fx gp + q fz + fy * gq is = 0 and in fact, this is nothing but if you have; we say that here f and g have one solution common means, the bracket fg is = 0 and when you expand this, then we can arrange like this.

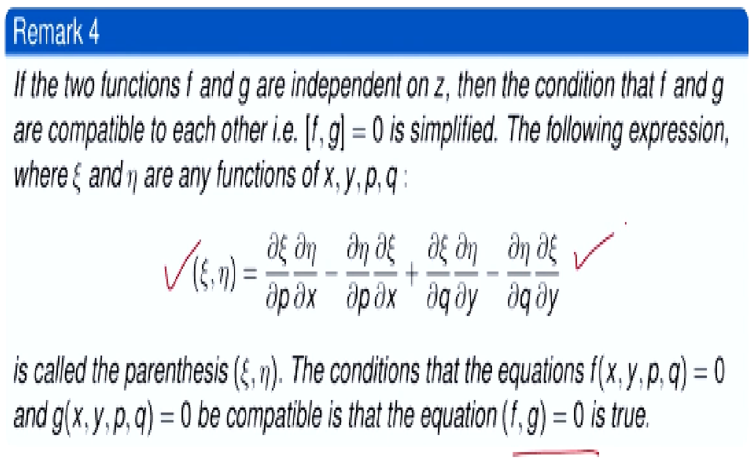
(Refer Slide Time: 05:40)



Here, we can write it that dg is = 0 or we can say that gx dx + gy dy + gz dz + gp dp + gq dq = 0, so when you expand this, then you can write gx into this + gy * this + gz * and gp and this is fresh and you can get it from this in fact, if you look at the coefficient of gx will be what? Coefficient of gx will be fp; -fp, so that is what we have written, so this condition that g satisfy this homogeneous linear PDE of the first order, we can obtain by expanding this bracket fg is =0.

And we can get that g will satisfy the following homogeneous linear PDE of the first order and this is very, very important remark and we will use this remark in Charpit method to find out the complete integral of a first order nonlinear first order PDE, so let us move ahead and look at the remark 4.

(Refer Slide Time: 06:50)



And this remark 4 says that if the 2 function f and g are independent of z, it means that they are writing only f of x y x p q and g of x y z p q, then the expression that bracket fg = 0 is simplified and we can say that that can be written as only this and that parenthesis fg is = 0, where parenthesis fg is defined like this, parenthesis of Xi eta is dou Xi/ dou p dou eta/ dou x – dou eta/dou p dou Xi/ dou x + dou Xi/dou q dou eta/dou y – dou eta/ dou q dou Xi/ dou y.

So, it means that here f and g both are independent of z, then we simply look at the parenthesis fg is = 0, where parenthesis xi, eta is given by this expression, so it is just a case of the bracket fg = 0, when f and g are not involving any z. Now, let us consider one example based on this compatibility procedure.

(Refer Slide Time: 08:04)

Example 4

Show that the equation $z = px + qy$ is compatible with any equation $f(x, y, z, p, q) = 0$ that is homogeneous in x, y, and z. Solve the simultaneous equations $z = px + qy$ and $2xy(p^2 + q^2) = z(yp + xq)$.

solution. Consider the pde $f(x, y, z, p, q)$

$$f(\alpha x, \alpha y, \alpha z, p, q) = \alpha^n f(x, y, z, p, q)$$

$$x f_x + y f_y + z f_z = n f$$

and denote

$$f(x, y, z, p, q) = 0$$

$$g(x, y, z, p, q) := px + qy - z$$

Now to show that f and g are compatible we have to show that

$$[f, g] = 0.$$

Here

$$[f, g] = \frac{\partial(f, g)}{\partial(x, p)} + p\frac{\partial(f, g)}{\partial(z, p)} + \frac{\partial(f, g)}{\partial(y, q)} + q\frac{\partial(f, g)}{\partial(z, q)}$$

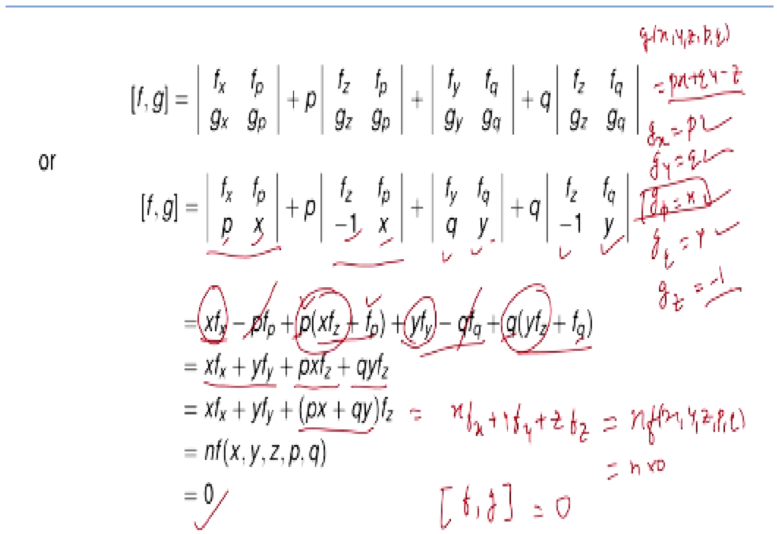
So, here we say that show that the equations z = px + qy is compatible with any equation f of x y z p q = 0 that is homogeneous in x, y and z and once we show that they are compatible to each other, we try to find out the 2 equations, z = px + qy and 2xy p square + q square = z y p + x q and we want to find out the common solution they have. So, here I just want to note that the homogeneous equation in x y z is we say that any equations say, f of x y z p q, we are talking about.

We say that f of x y z p q is homogeneous in terms of x y and z, if we replace x y z by alpha x, alpha y, alpha z, then we will should get alpha 2 power n f of x y z p q and in this case we say that your equation f is homogeneous in terms of x y z of the degree n, where n is the power of alpha here. So, here by this we can find out the degree and they also have very important relations that is if f is homogeneous in x y and z, then we have this relation that f of x fx + y f of y + z f of z is = n times f.

Here, n is the degree of the; n is the power of this alpha, so we call this as degree of the homogeneity. So, here with this information, now let us see that if f satisfy this condition, then it is compatible with z = px + qy, so now let us assume this equation f of x y z p q = 0 and another equation, g x y z p q as px + qy - n and we want to show that they are compatible to each other, so let us look at the bracket fg and we try to show that this bracket fg is = 0.

Now, look at the bracket fg here, it is what? Dou of fg upon dou fg dou xp + p times Jacobian of fg with respect to zp + Jacobian of fg with respect to yq + q times Jacobian of fg with respect toz q.

(Refer Slide Time: 10:39)



Now, let us simplify this further, so I can write, Jacobian of fg with respect to xp as this fx fp gx gp + p times fz fp gz gp fy fp gy gq + q times fz fq gz gq, now, here we already know that g x y z p q is = px + qy – z, let me see whether it is correct, it is what? Px + qy – z, okay, so this is the bracket fg. Now, let us simplify, since we do not have any expression for f, we simply know the properties of f that it is homogeneous in terms of x y and z, that is all, nothing is given neither the degree is given nor the exact formulation is given.

So, here let us use this expression for g only, so let us calculate g of x; g of x is = p, g of y is = q, q of p = x, g of q = y and g of z = -1, so using this let us again calculate bracket of fg, so fx fp now in place of gx let us; gx is p here, so let me write it here, gp is x here, let me write it x here + p times fz fp gz is -1 that is given here and gp is x, which is already calculated here + fy fy, now gy is what?

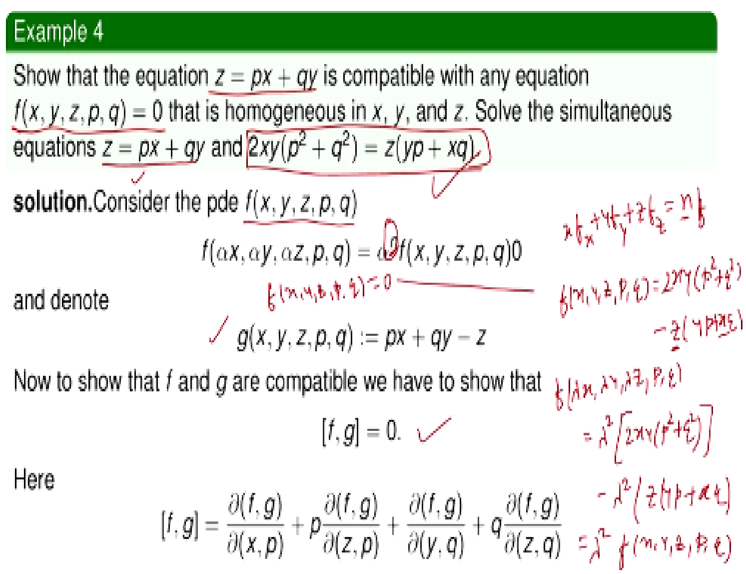
Gy is q here and gq is y here, let me write it here + q fz fq and gz; gz is -1 and gq is y, so now let us simplify this, it is what; x fx, so this is x fx – p fp, we have simplified this + p times, it is what; x fz + fp, so this is the expression plus, here we write y fy - q fq + q times y fz + fq, so we have simply written down. Now, we can simply say that x fx + y fy, let us write it like this; x fx + y fy + px of z.

Now, here you can see that this p fp, here this will be cancelled out from this and similarly, q fq will be cancel out like this, so what we will have is px fz + qy fz, so this term is left and this term is left, so we are writing here, x fx + y fy + px fz + qy fz, now if you look at, if we take out this fz common, then we have px + qy and we already know the value of px + qy in terms of z and we can write this as x fx + y fy + z fz.

And we know that if f is homogeneous in terms of x y and z, then this expression is nothing but n times f, now we already have that the expression f of x y z p q, this value is coming out to be 0, so n is 0, so we have this value is 0. So, what we have shown here, we have calculated the bracket fg and we have say using the property of homogeneity, this value is coming out to be 0. So, it means that any equation which is homogeneous in terms of x y z.

And px + qy - z are compatible to each other, now once they are compatible to each other then we can find out the common solution and to find out the common solution, we need to know what are the value of p and q, so here for that we have simply taking expression for f of x y z that is 2 xy * p square + q square = z yp + xq. Now, here if you look at this is homogeneous equation in terms of x y and z, how we can check?

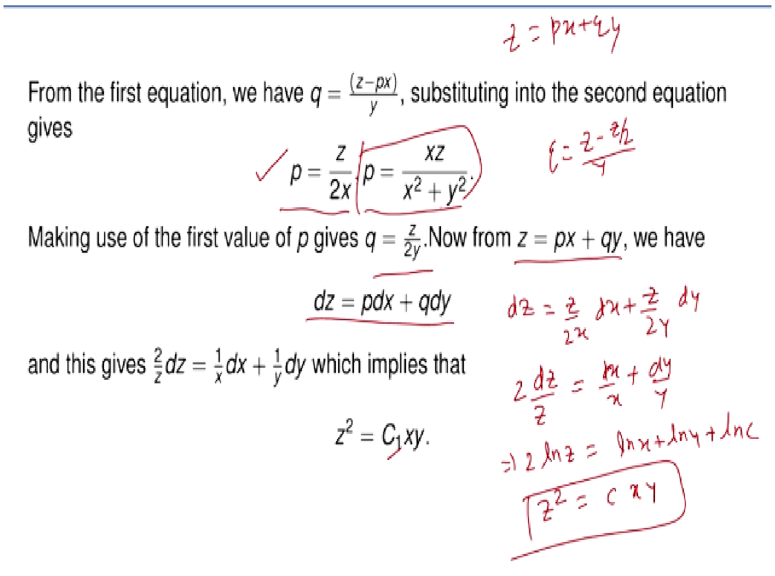
(Refer Slide Time: 14:33)



You simply write this as f of x y z p q = let me write it here 2xy p square + q square – z yp + xq, now what you simply write in place of x, let us write lambda x lambda y lambda z pq and if you simplify what you will get? Here, you will get a lambda square in common from 1st term 2xy p square + q square and -; here if you write lambda z Lambda y, so again you can take out lambda square and what you will get?

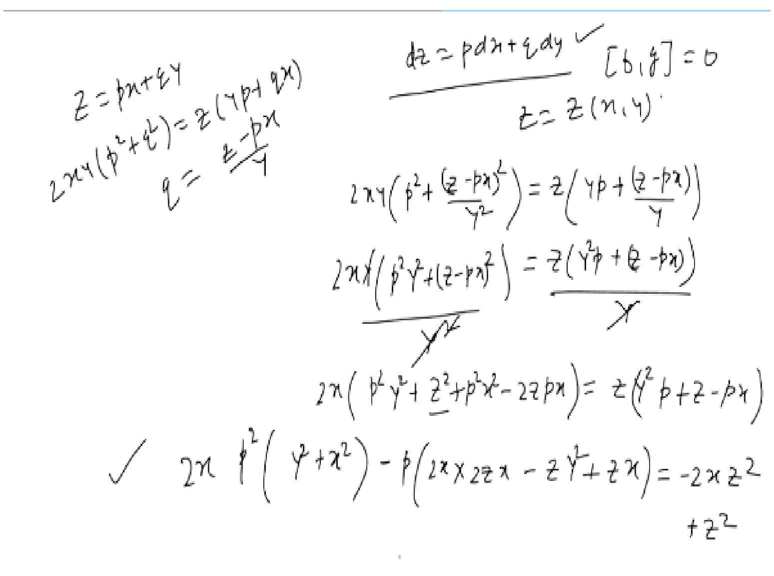
Zy p + here also from this one lambda and from this one lambda, so lambda square you can take it out, so xq, right, so when you simplify this, then it is lambda square times f of x y z p q, so we can say that this is a homogeneous equation in terms of x y z with degree 2, right. Now, you want to show that since we have already proved that if f is homogeneous with respect to x y z and z = px + qy, then they are compatible to each other.

(Refer Slide Time: 16:02)



Now, we want to show that let us take a particular form of that is this and z = px + qy compatible means we can find out one common solution, so let us try to find out one common solution here. So, here we have z = px + qy.

(Refer Slide Time: 16:10)



And let me do it in a separate page, so here we have z = px + qy and second equation is what? 2xy, let me use here, it is 2 xy p square + q square is = z yp + qx, I hope I have written correctly; yp + xq, is that okay. Now, we already proved that it is a compatible to each other and we need to find out the common solution, it means that dz = pdx + qdy is integrable to each other. Now, we already know that since bracket fg is 0.

So, it means that this has to be integrable because that is a condition we already proved, so it means that what we need to do here; we need to find out p and q from these 2 equation and put it here and then use our experience to solve this equation and to get this equation as z of x y and maybe one integration constant will also appear, so we can get that it is a one parameter family of solution.

So, now let us solve this, so for this, you simplify first equation as z - px upon y, so we simply write down the value of q in terms of z p and x and y and put it in a second equation, so let us say 2xy p square + q square is z - px whole square y square = z yp + q is z - px upon y here, so let me simple this as z y square p + z - px and here, we have y here. Now, here again we can write 2xy, it is p square y square + z - px whole square and we have y square.

Now, we can see y cancel and y, here y cancel, so here we can simplify further, we have 2x, now inside it is p square y square + z square + p square x square – 2z px = zy square p + z – px, it is given by this. Now, let us calculate the coefficient of p square here, so p square coefficient will be what? Here, it is what and to get 2x also common, so here, what you will get; here we will get y square and here we have x square, is there any other term?

No, we do not have any term, now let us calculate the coefficient of p here, so let me write it here, -p; what you will get here; 2x is here and here 2 zx here, so let me write it, 2x * 2zx, so that is what we have obtained here and here, if we look at the coefficient here; coefficient here is what? – zy square and + zx.

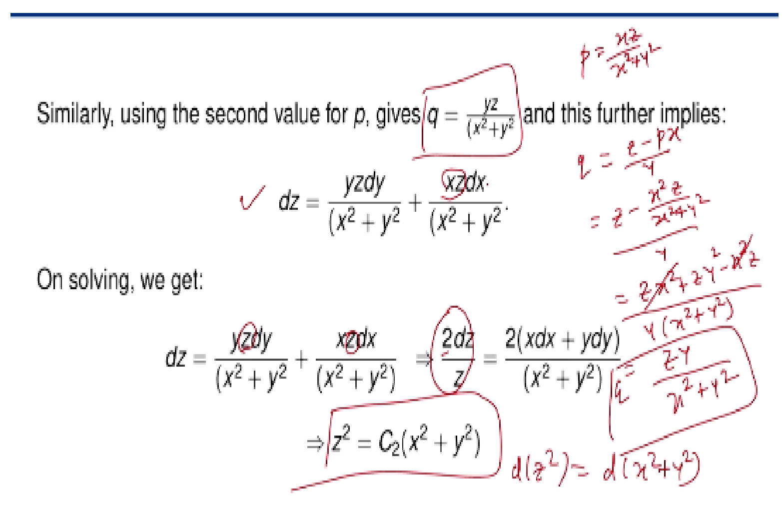
And the remaining is just a constant value, so which we can write it here as; what you will get? 2 z square is here, so I can write – 2x z square and here I will get + z square, so here we can get a expression for quadratic equation terms of p square and we can simplify for p using the theory of quadratic equation and you can get the value of p by solving this quadratic equation and we have equation as p = z/ 2x and p = xz/ x square + y square.

I am leaving it to you, I in fact we have obtained a quadratic equation and then we can solve for p and you will have 2 values of p out of this, so I am saying that let us do this work and we have obtained and we say that p = z/ 2x is one value of p and another value is p = xz upon x square + y square. Now, once we have 2 values of p, then of course you will have 2 values of q as well, so let us solve for q.

Let us take the first value as xz/2x then q = z – px; px is going to be z/2 divided by y, so it means q is going to be z/ 2y, so it means that we have now values of p and q, then look at your equation, dz = pdx = qdy, so let us say that dz = pdx + qdy is here, then you simply write it dz and p is what? P is z/2x dx and q is what? Q is your z/2y, so z/2y dy and if you simplify this, it is what; dz/z in fact, you can take 2 here, then it is dx/x + dy/y here.

And here, if you simplify this is what? This is; you can simplify you can write it 2ln z = ln x + ln y + ln c, so we can write z square = c times xy, I am using c1 here, so you can write the one solution is z square = c xy, so once we have one value of p, you have another value for q, which we can obtain using the equation as well as this value and we have a common solution z square = c xy.

(Refer Slide Time: 22:11)



Now, we if we use the other value that is xz upon x square + y square as p, then you can write p as xz upon x square + y square and q is what; q is z – px upon y; z – px upon y here, so let me write it here, z – px, so I can write it x square z upon x square + y square divided by y and if you simplify zx square + z y square – x square z divided by y times x square + y square, right. So, zx square is cancel out here, so we have zy, y is also cancel out.

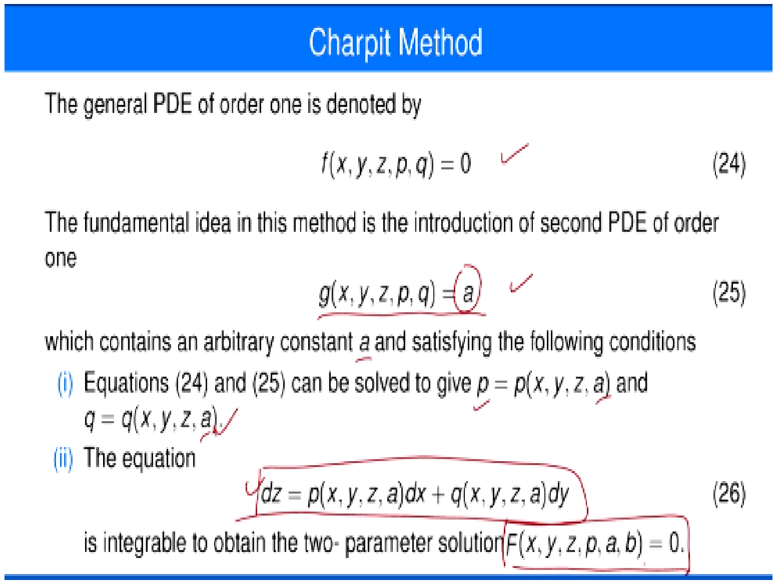
So, we have zy divided; sorry sorry, so here we will get zy upon x square + y square, so that is a value of q, so q is zy upon x square + y square, which we have written here, the q is zy upon x square + y square. So, we have q, we have p, now let us simplify dz = p means, p here xz upon x square + y dx + qdy, so yz dy upon x square + y square. Now, let us; since we already know that it is integrable, the only thing is we have to make it in a form such that exact integration is possible.

So, here you can take out that z here, so we can right this as, you can take out z here, we can write dz = yz dy upon x square + y square + xz dx upon x square + y square, then we can simply write, you can take that common and you can write dz upon z and then you multiply by 2, so it is 2x dx + y dy divided by x square + y, so this is what it is the derivative of z square and if you look at this is what derivative of x square + y square.

So, we can simply say that you can write a z square = some constant time c x square + y square, so we can say that this is an another common solution they have, if we use a value p as xz upon x square + y square, so that is what we have achieved here. So, what we have shown here; we have shown here that any equation, which is homogeneous in x, y, and z is compatible with z = px + qy and then we have taken a particular form of f x y z p q that is 2xy p square + q square = z yp + q yp + xq.

And we have found out the common solution they have and in this case, they have 2 common solutions which we have obtained by z square = c1 xy and z square = c 2 times x square + y square here.

(Refer Slide Time: 25:08)

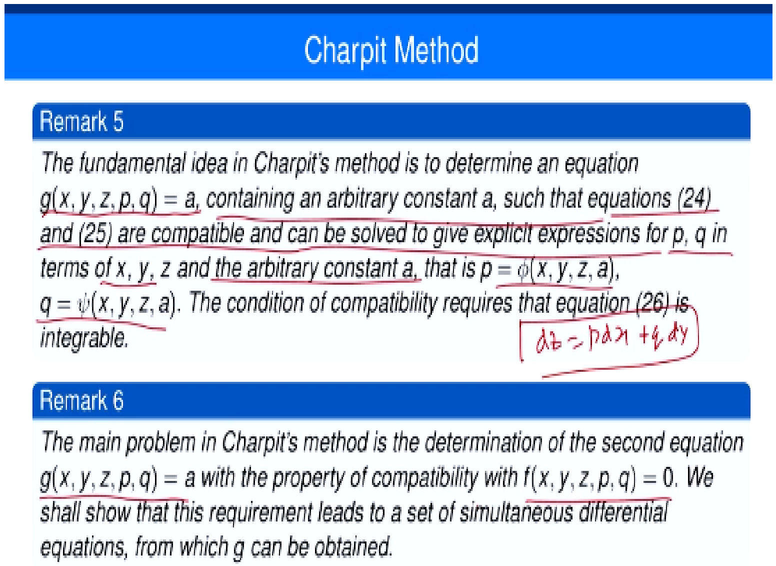


So, now based on this, now let us define a important method that is Charpit method, so what we try to do here, our aim is to find out the complete solution of a first order non-linear PDE that is f of x y z p q = 0, so here we want to find out a complete solution of this equation and how we obtain xz following that using the concept of compatibility, so it means that if we are able to find out another say, PDE that is say, g x y z p q = a.

Here, a is some kind of parameter, so we say that if the fundamental idea in this method is the introduction of second PDE of order one that is g x y z p q = a, which contains an arbitrary constant a and satisfying the following condition, following condition is that, we can solve these 2 equations for p and q and of course since the equation of g which involve 1 parameter a, so p and q will also involve that parameter a.

And then once we have the value of p and q, then dz = p x y z a dx + q x y z a dy, a is integrable exactly, so when we find out the solution, then we have one parameter and we say that our solution is given as f of x y z p, x y z a b = 0 here and this is given as 2 parameter family of solution and we say that this solution as complete solution.

(Refer Slide Time: 26:58)



So, here let me here the fundamental idea in charpit method is to determine an equation g x y z p q = a, so here, what we say that our main problem is how to find out differential equation; 2nd differential equation which has the following property that by which we can find out the expression for p and q and that the dz = pdx + qdy is integrable and here it means that the another equation we need to find out that is g x y z p q = a containing an arbitrary constant a, such that equation 24 and 25 are compatible to each other.

And can be solved to give explicit expression for pq in terms of xyz and the arbitrary constant a that is p = phi x y z a and q = psi x y z a and the condition of compatible require that equation 26 is integrable that is dz = pdx + qdy is integrable. So, idea is; are the problem is how to find out this one parameter equation of PDE, so that is the problem. So, the main problem in Charpit method is the determination of the second equation g x y z p q = a having the property that it is compatible with f of xyz p , q = 0.

And we will; so that the g and f are compatible to each other will give you an idea how to find out this equation g and this is quite obvious if you remembered one of the remarks that g is compatible with f, if g satisfy certain homogeneous partial differential equation.

(Refer Slide Time: 28:58)

When such a given function g satisfying the above two conditions has been found, the solution of equation (26) containing two arbitrary constants will be a solution of equation (24). The condition (i) will be satisfied if $J = \frac{\partial(f,g)}{\partial(p,q)} \neq 0$.
Assume that equation $[f, g] = 0$ is satisfied and equation (26) exists.

$$[f,g] = \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix} + p \begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix} + \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix} + q \begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix} = 0; \text{ or} \qquad (27)$$

$$(-f_p)g_x + (-f_q)g_y + (-pf_p - qf_q)g_z + (pf_z + f_x)g_p + (qf_z + f_y)g_q = 0 \qquad (28)$$

Now compare this with $dg = 0$, *i.e.*

$$g_x dx + g_y dy + g_z dz + g_p dp + g_q dq = 0$$

So, when such a given function f satisfying the above 2 condition has been found, it means that if we find out g such that g and f are compatible to each other and g is already having one parameter, then we can say that our solution of equation 26 is contained 2 arbitrary constants will be a solution of equation f x y z = a and the condition 1 will be what? Condition 1 means that f and g can be solvable in terms of p and q is equivalent that Jacobian fg with respect to pq is nonzero.

And that f and g are having one solution in common means dz = pdx + qdy is integrable is simply say that we can say that our bracket f g has to be 0, the value of fg has to be 0, now look at the bracket fg, it is fx fp gx gp + p times determinant of fz fp gz gp and so on, so this expression has to be = 0 and if we simplify I can write this expression that minuses –fp gx + - fq gy – pf p - q fq gz + pf z + fx gp + qfz+ fy gq = 0.

So here, we simplify this bracket fg = 0 in a way such that we write in in terms of coefficient of gx, gy gz, gp and gq and now let us compare this equation number 28 with the following thing that dz = 0, here you have g x y z p q = a, so if you find out the total derivative, then total derivative dg is coming to be 0 and when we expand it, we have the following thing that gx dx + gy dy + gz dz + gp dp + gq dq = 0.

(Refer Slide Time: 31:12)

$$\frac{dx}{\frac{-\partial f}{\partial p}} = \frac{dy}{\frac{-\partial f}{\partial q}} = \frac{dz}{\frac{-p\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p(\frac{\partial f}{\partial z})} = \frac{dq}{\frac{\partial f}{\partial y} + q(\frac{\partial f}{\partial z})} \quad (29)$$

Equation (29) is known as Charpit's subsidiary equations which can also be written in the form

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-f_x - pf_z} = \frac{dq}{-f_y - qf_z} = dt \quad (30)$$

So, it means that if g is given like this then it must satisfy the equation number 28 and this equation and if we compare these 2 equations, then we have the following thing that dx upon –fp, if you look at compare here, dx upon –fp, dy upon –fq, dz upon – pfp – qfq, dp upon pfz + fx and dq upon this quantity and we have a set of equations which is known as equation number 29 and we call this as Charpit subsidiary equation.

So, here managing this – sign out and we have dx upon fp + dy upon fq =; sorry, dx upon fp = dy upon fq = dz upon pfp + qfq = dp upon - fx – pfz = dq upon –fy - qfz and these equations are known as Charpit subsidiary equation. Now, if you recall the Cauchy method of characteristics then if you equate this value as some dt, where t is some parametric; parameter, then it is; the equation which you have a obtained in Cauchy method characteristic, an equation what we have obtain right now as same.

The only thing is that they been have some initial condition here, we do not have any initial condition, so that is only difference we have now, so it means that when we solve this equation, so then we will get an expression for g x y z p q, here what we have seen that here, dx dy dz dp basically, they will characterise the surface g x y z p q; g x y z = a; g x y z p q = a, so that is our idea how to solve this equation number 30 to find out the expression g x y z p q.

(Refer Slide Time: 33:19)

General strategy for Charpit method

(i) Transfer all terms of the given equation to left- hand side and denote all expression by f.

(ii) Write the Charpit auxiliary equation equation (30).

(iii) Using the value of f in first step, calculate the value of f_x, f_y, f_z, f_p and f_q in step (ii) and put these in (30).

(iv) Select two proper fractions so that the resulting integral may come out to be the simplest relation involving atleast one of p and q.

(v) The simplest relation of step (iv) is solved along with the given equation to determine p and q. By putting these values of p and q in $dz = pdx + qdy$ which on integration gives the required complete integral of the given equation.

When we solve this, we try to do some kind of integration and by which we have 1 parameter involved, so what we try to do here, we discuss the general strategy for Charpit method, so here transfer all terms of the given equation to left hand side and denote all expression for f, so here whatever equation we have, you just transfer everything into one side and denote that expression by f and write the Charpit auxiliary equation 30.

And using the value of f in first step calculated the value of fx, fy, fz, fp, fq and put in equation number 30 and then this is very important step that select 2 proper fraction, so that the resulting integral may come out to be the simplest relation involving at least one of p and q, so here out of these set, they are 1, 2, 3, 4, 5 you choose at least 2, so that we can get expression; a simpler expression for p and q, right, p or q.

So, once we have an expression for p and q, use the equation, the simplest relation of step 4 is solved along the given equation to determine p and q, given equation means, f of x y z p q = 0, so once we have p, then of course since they have one solution in common then that must satisfy this given equation that is f of x y z p q = 0, so by this you find out the expression for q, so in this way we are able to find out both p and q.

And once we have the values of p and q put in dz = pdx + qdy and solved for your integral and in this way we can find out the integral and that is the required complete integral of the equation, f of x y z p q = 0 and this is the method, which we follow and we find the complete integral and here we will stop and in next lecture will apply this strategy of Charpit method to find out complete integral, so thank you for listening us, thank you.