Ordinary and Partial Differential Equations and Applications

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Lecture - 04
Existence and Uniqueness of Solutions of Differential Equations - II

Hello friends. Welcome to this lecture. In this lecture, we will continue our discussion of existence and uniqueness theorem for nonlinear problems. So here we have to discuss this. Before we do anything with this successive approximation we must show that they are defined properly. This means that in order to define yj on some interval I, we must first know that the point s, yj s remains in the rectangle R for every s in I.

So we have already defined the successive approximation and we also discussed the Lipschitz condition of the nonlinear function f and with the help of this lemma we want to show that whatever approximation we have constructed they belongs to the rectangle R.

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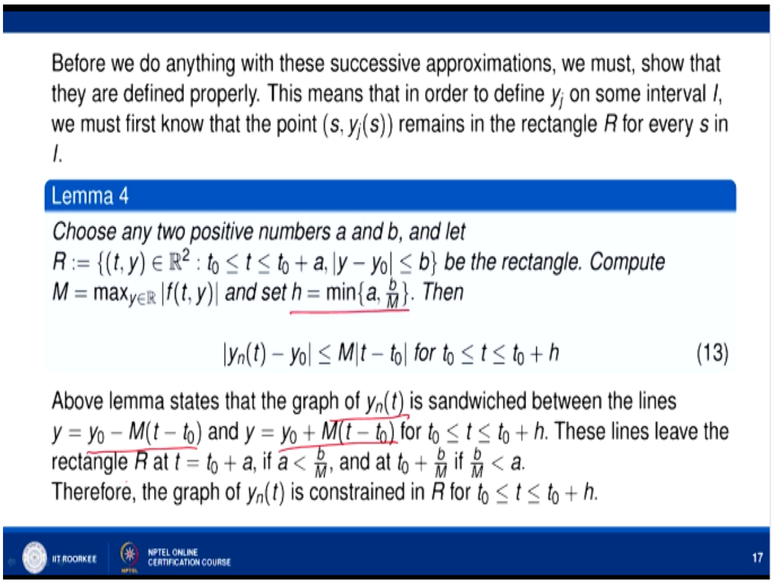

So before we do anything with this successive approximation we have already constructed the successive approximation and we have also considered the definition of Lipschitz condition. We must show that they are defined properly, so it means that we have to first show that the iteration which we have constructed they are defined properly and this means that in order to define yj on some interval I we must first know that the point s, yj s remains in the rectangle R for every s in I.

So first we have to show that all the iteration belong to the same rectangle for all the time t in that particular interval I. So that we want to see, so for that we have a lemma 4. So choose any 2 positive numbers a and b and let R is defined this, R is the set of all t, y belonging to R2 such that t is lying between t0 to t0+a and modulus of y-y0 is<=b be the rectangle and we can compute the maximum of modulus of f t, y in this rectangle R.

And we can set h as minimum of a and b/M then modulus of yn t-y0 is<=M t-t0 for all t belonging to t0 to t0+h where h is minimum of a and b/M. So what we mean by this lemma 4, it means that this lemma state that the graph of yn t is sandwiched between the lines y=y0-M t-t0. You can say open this if you look at this it is what yn t-y0 is<=M t-t0. If you open this, you can say that yn t is lying between y0-M t-t0 and y0+M t-t0 that we can easily see like this.

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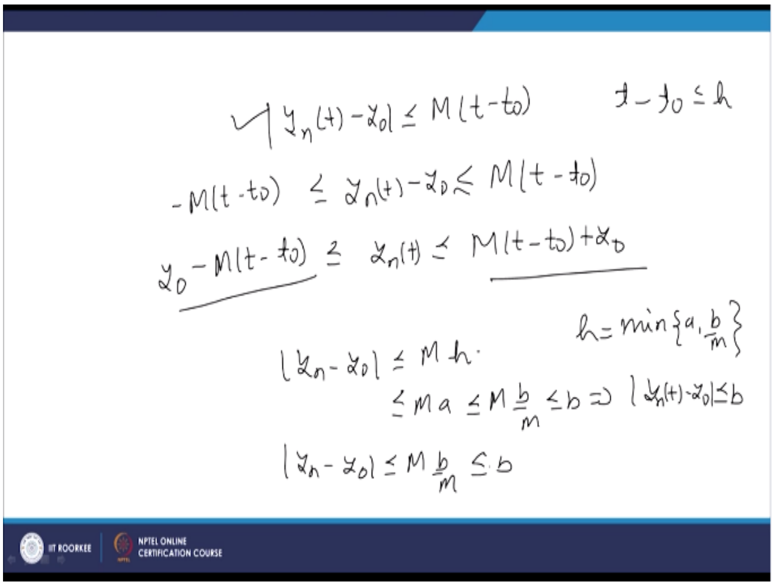

Here modulus of yn t-y0 is<=M t-t0 okay. So we can write it yn t-y0<=M t-t0<=-M t-t0 and you can put y0 here so yn t is lying between M t-t0+y0 and here it is y0-M t-t0. So we can say that this yn t is lying between this line and this line. That is what we wanted to show it here that the graph of yn t is sandwiched between the lines y0-M t-t0 and the line y=y0+M t-t0.

And here this is true for all t between t0 to t0+h and these lines leave the rectangle R at t=t0+a if a is<b/M and t0+b/M if b/M is<a. By showing this if you look at here we already know that t-t0 is<h right. So it means that we can say that yn t-y0 is<=Mh right and h is what we have defined is minimum of a and b/M right. So it means that we can say that if a is minimum we can say that it is Ma and this is nothing but M b/M that is<=b.

So it means that modulus of yn t-y0 is<=b it means that your yn t remains in the rectangle but if minimum of a and b/M is b/M we can simply say that yn t-y0 is<=M, you can say that h is b/M so it is<=b. So it means that for this h your yn will always remain in the rectangle R that is what we wanted to show here. So it means that therefore the graph of yn t is constrained in R for t between t0 to t0+h, here h is minimum of a and b/M.

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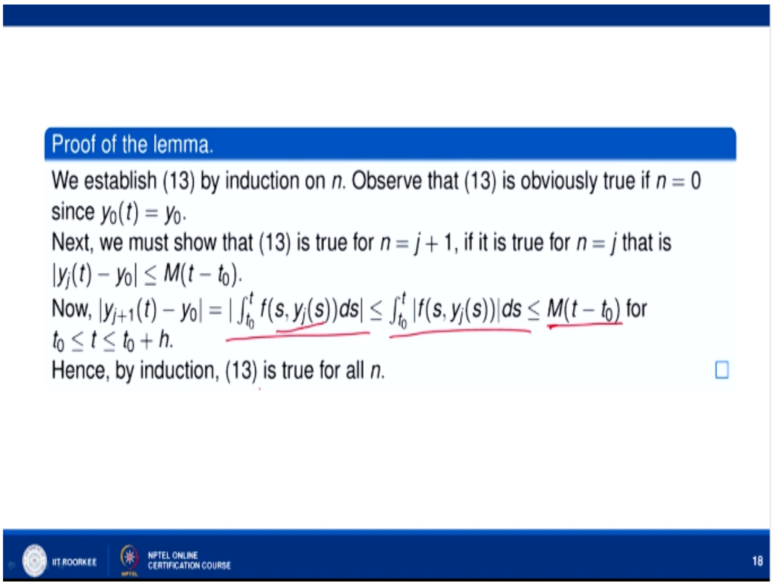

So this proof we can do by induction on n. So observe that this is true for n=0, in fact for n=0 y0 t is nothing but y0 so this is trivially true. So this is obviously true for n=0 because y0 t=y0. Next, we must show that this condition is true for j+1 if it is true for j. So that we want to show that if we assume yj t-y0 is<=M t-t0 then we want to show that modulus of yj+1t-y0 is also<=M t-t0.

So for that let us calculate this quantity modulus of yj+1t-y0=t0 to t f of s yj s ds. Now here we already know that this yj s belongs to the rectangle. So it means that this s yj s belongs to the rectangle and we already know that in this rectangle your f is bounded by M, so we can say that this thing is bounded by this and here we already know that this is bounded by M so we can say that M integration from t0 to t ds so which is given as M t-t0.

So it means that modulus of yj+1t-y0 is<=M t-t0 for all t lying between t0 to t0+h and hence we can say that by induction this is true for all M. So it means that this successive approximation will remain in the rectangle which we have defined like this. So it means that all the constructed sequence will be in this rectangle R for all time t between t0 to t0+h and in this rectangle your modulus of f t, y is bounded by this M.

So we have shown that with the previous lemma that all the successive approximation satisfy this inequality 13. So it means that all yn t belongs to this rectangle R.

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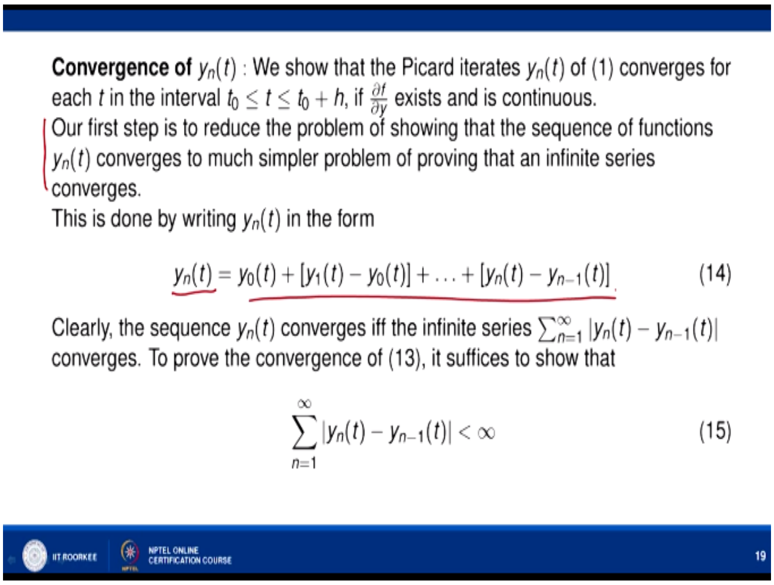

Now we consider the question which is very important question that convergence of yn t. So we show that the Picard iteration yn t of 1 converges for each time t in the interval t0 to t0+h if dou f/dou y exist and is continuous. We may replace this condition dou f/dou y exist and is continuous by the Lipschitz condition also. So our first step is to reduce the problem of showing the sequence of function yn t converges to much simpler problem approving that an infinite series converges.

So this we can do by showing the equivalence between yn t and an infinite series. So here if you look at we can write yn t as y0 t+y1 t-y0 t+1 yn t-yn-2. So we can write yn t as the following sum. So here we can say that clearly the sequence yn t converges if and only if the infinite series n from 1 to infinity modulus of yn t-yn-1 t converges. In fact, if you look at this means what that limit n tending to infinity yn t is going to be what limit.

So y0 t+summation n=1 to infinity yn t-yn-1t. So it means that limit exist provided this series is convergent series. So to prove the convergence of this yn t it suffices to show that summation n=1 to infinity modulus of yn t-yn-1 t is<infinity. So we need to show the convergence of this infinite series.

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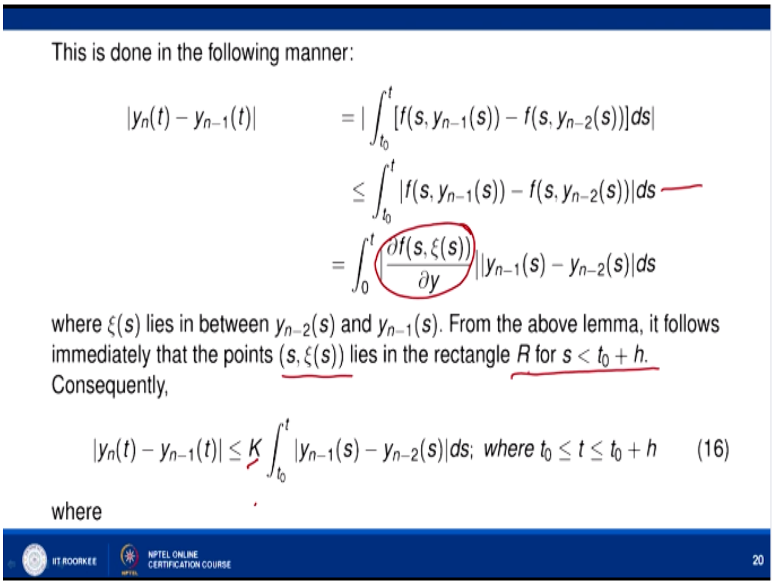

So this is done in the following manner. So modulus of yn t-yn-1 t, let us calculate this quantity and it is coming out to be t0 to t f of s yn-1 s-fs, yn-2 s ds and this is further<t0 to t modulus of fs yn-1 s-fs yn-2 ds ds. Now here we can apply either the Lipschitz condition or the mean value theorem such that this is going to be <= modulus of dou f/dou y at some point s is Is*modulus of yn-1 s-yn-2 s.

So here at this point either you use mean value theorem or you use the Lipschitz condition and we can show that here if we are using this mean value theorem we can say that this xi s lies between yn-2 s and yn-1 s and from the above lemma which we have just proved it follows immediately that this point lies in the rectangle R for this. So it means that if this point lies in this rectangle R it means that this is bounded by the constant K here right.

If you use the Lipschitz condition, then this is bounded by the Lipschitz constant K here. So if you are using the Lipschitz condition from here we can directly move to this. If we are using the bounded-ness of partial derivative here then we use this condition that s, xi s lies in this rectangle and in this rectangle this is bounded by K. So here from this we can say that modulus of yn t-yn-t is<=K times t0 to t modulus of yn-1 s-yn-2 s ds.

So what we do we are able to get a recursive relation in terms of yn t-yn-t and here t is lying between t0 to t0+h. Now here K right now we are assuming the maximum of this quantity over the rectangle R.

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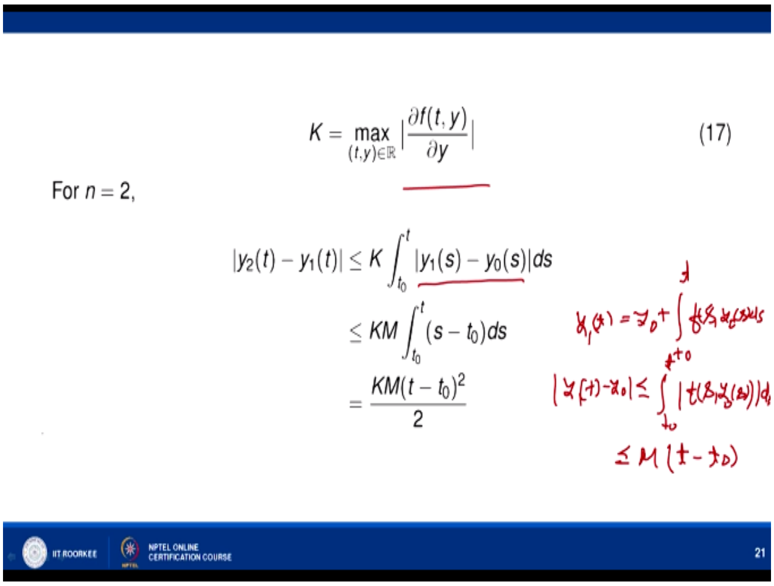

So here we have defined K. If we are using the Lipschitz condition, then we can say that K is a Lipschitz constant. So here for if you look at the recursive relation this yn t-yn-1 t<=K t0 to t yn-1 s-yn-2 s for say n=2. So let us define it for n=2 so it is y2 t-y1 t is<=K t0 to t y1 s-y0 s ds and if we already know that when we consider approximation y1 t=y0 t+t0 to t f of s y0 s ds and we already know that y1 t-y0 is bounded by t0 to t modulus of f of s y0 of s ds.

And we already know that this is bounded by M and this is what t-t0. So we already know that this y1 s-y0 s is bounded by this quantity. So we can say that this is bounded by KM t0 to t s-t0 here y1 t-y0 we are saying that it is bounded by this quantity, so y1 s-y0 s is bounded by s-t0 and if you simplify this integration then it is coming out to be KM t-t0 square/2.

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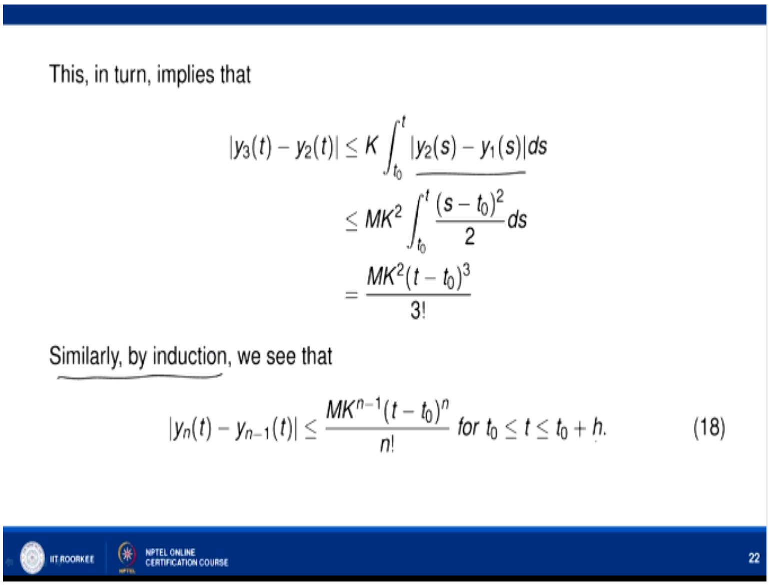

This we have shown for n=2, now we can also say that y3 t-y2 t is<=K times t0 to t y2 s-y1 s ds and just now we have calculated the bound for this y2 s-y1 s that is this we have just calculated that it is bounded by KM t-t0 square/2. So using this bound we can write it MK square t0 to t is –t0 square/2 ds and when you calculate this it is coming out to be MK square t-t0 to power 3/factorial 3.

And if we proceed like this and by induction we can show that modulus of yn t-yn-1 t is<=MK to power n-1 t-t0 to power n/factorial n. Here t is lying between t0 to t0+h.

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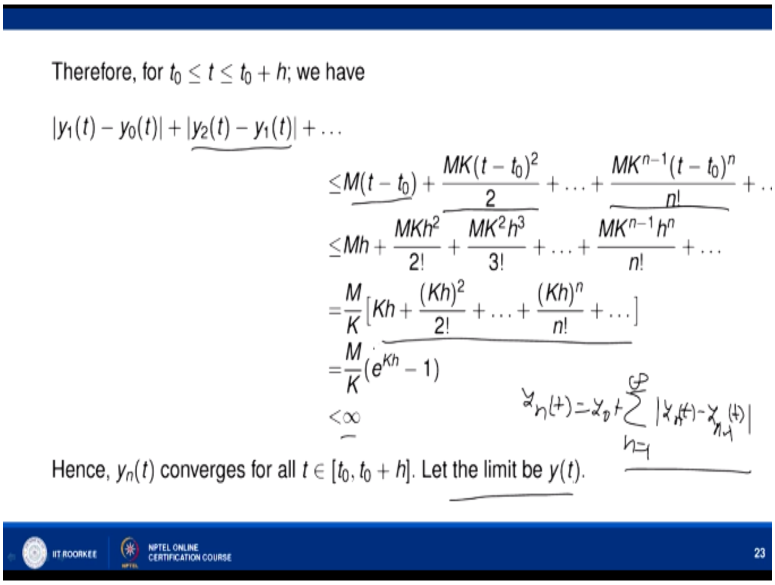

So we can show that this series which we are considering that modulus of y1 t-y0 t+y2 t-y1 t and so on is<=y1 t-y0 t is bounded by M t-t0 and y2 t-y1 t is bounded by this quantity and so on we can say that yn t-y1-1 t is bounded by this and if you take the summation here since t is lying between t0 to t0+h so t-t0 is bounded by h so we can say that this is further bounded by Mh+MK h square/factorial 2+MK square h cube/factorial 3 and so on.

And here if you look at this is kind of known series and we can make that form by multiplying by K and divided by K. So we can say that M/K then it is written as Kh+Kh whole square/factorial 2 and so on and if you look at this is an exponential series and this is written as M/K e to power Kh-1. So we can say that this series is finite series, so we can say that this series is bounded by some finite quantity and we can say that it is < infinity.

So what we can see that yn t converges for all t belonging to t0 to t0+h. So we have shown that this series is finite. So it means that yn t is basically what yn t is y0+summation n=1 to infinity yn t-yn-1 t and we have shown that this part is finite so adding this y0 keep this quantity finite so we can say that yn t converges for all t belonging to t0 to t0+h and since it is convergent we say that limit is y t. So y t is given as the limit of this infinite series.

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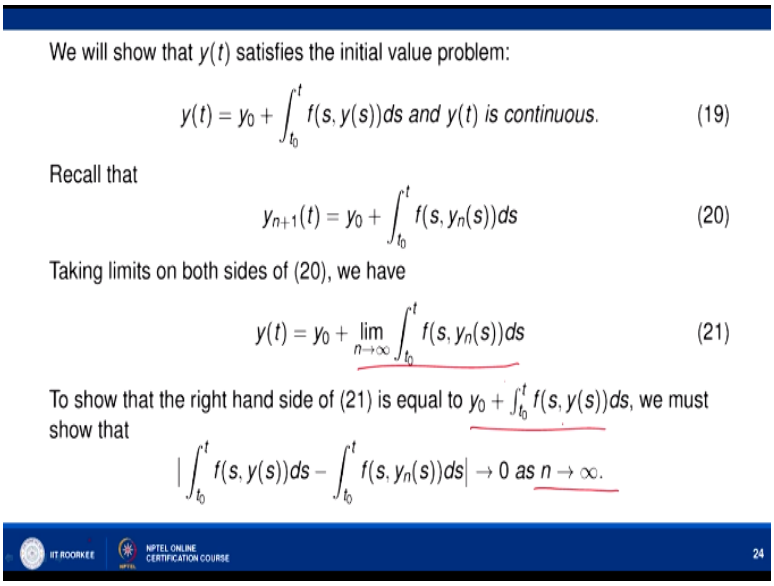

So now we want to show that y t satisfies the initial value problem. So what we have done, we have done first 2 steps. First step is the definition of successive approximation we have done in previous lecture and then we have to show that this approximation converges to some limit. So that we have just shown that the sequence yn t converges to some limit call it y t and now we want to show that this limit satisfy the initial value problem.

So it means that it satisfies the equation y t=y0+t0 to t f s, y s ds and also want to show that this y t is continuous right. So by showing that y t is a solution of integral equation and we can show that it means that y t satisfy the initial value problem as well. So to show this let us recall the equation yn+1 t=y0+t0 to t f s yn s ds. Now if you apply the limit here then limit will be what since limit of yn+1 t is yt so we can say y0 is independent of n.

So we can say that if we take the limit it is given as y t=y0+limit n tending to infinity t0 to t f s, yn s ds and to show that this right hand limit is going to be equal to this quantity we must show the following thing that modulus of t0 to t f s, ys ds-t0 to t f s, yn s ds is tending to 0 as n tending to infinity. So it means that we have to show that this is nothing but this quantity right. So this we want to show here.

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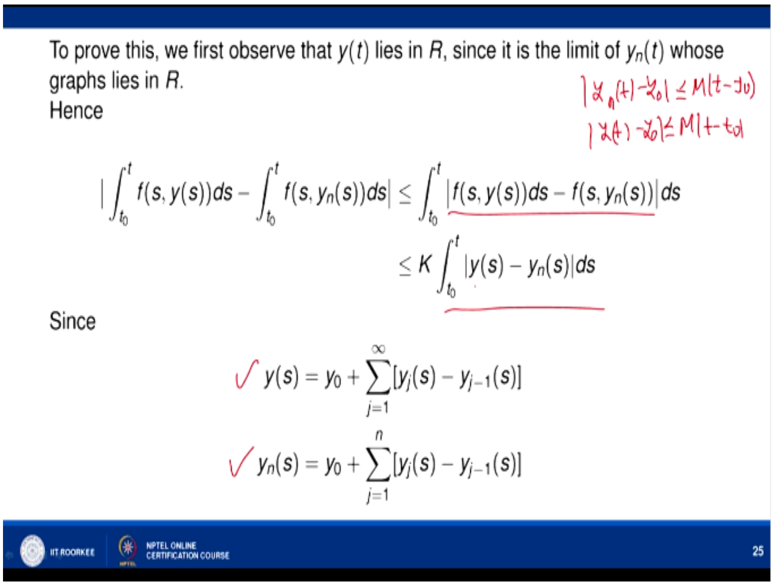

So how we can show, again we can prove this by observing that first of all that this y t lies in the rectangle R since every yn t is lying in this rectangle R it means that here we know that yn t-y0 is<=M t-t0 and if you take the limit n tending to infinity we can say that this implies that y t-y0 will also satisfy the condition t-t0. So y t will also belong to the rectangle R. So now look at this quantity modulus of t0 to t f s, ys ds-t0 to t f s, yn s ds.

So it is further < t0 to t modulus of this quantity and here we can apply either the Lipschitz condition or say boundedness of partial derivative of f. We can say that this is<=K times t0 to t ys-yn s ds. Now since here we have assumed that y lies in rectangle R so it means that to find out bound of this we have to look at the quantity of this. So here what is ys, ys is y0+j=1 to infinity yj s-yj-1s and yn s is defined and y0+j from 1 to n yj s-yj-s. So using these 2 expressions, we can find out ys-yn s as follows.

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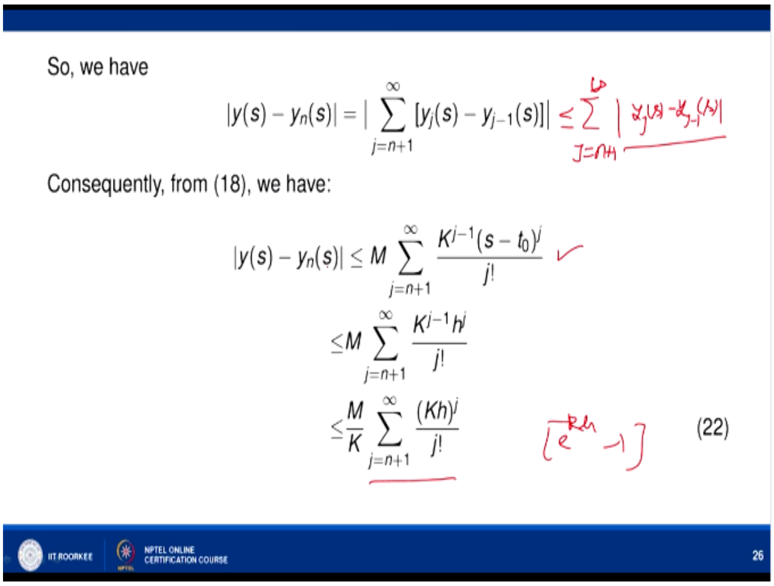

So modulus of ys-yn s is given as j from n+1 to infinity yj s-yj-1 s. Now we want to show that this quantity can be made arbitrary small as n tending to infinity. So we can see that modulus of ys–yn s<=M j from n+1 to infinity Kj-1 s-t0 to power j/factorial j. Here for this particular step we are using the bond of yj s-yj-1 s so here we are showing that this is further<=j from n+1 to infinity modulus of yj s-yj-1 s this thing.

And we already know the bound of this. The bound of this is K to power j-1 s-t0 to power j/factorial j here. Now again we can simplify this. This s-t0 is bounded by h so we can write this as M j from n+1 to infinity K to power j-1 s to power j/factorial j. Again we can multiply by K and divide by K and we can write M/K j from n+1 to infinity Kh to power j/factorial j. If you look at this is the tail of the series e to power Kh-1.

So this is the tail of the series of e to power Kh-1 and we know that since this series is convergent it means that as n tending to infinity this tail is tending to 0. So it means that as n tending to infinity modulus of ys-yn s is also tending to 0.

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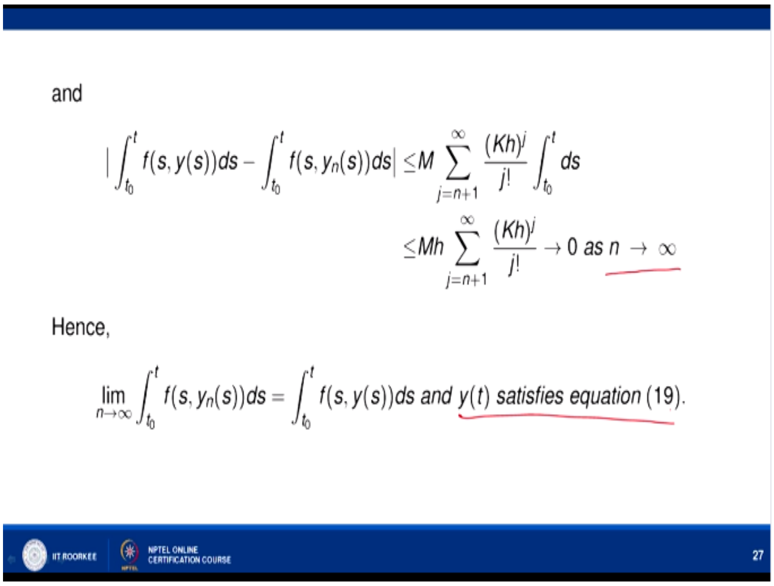

And this implies that this is also tending to 0. In fact, we can write this as modulus of t0 to t f s, ys ds-t0 to t f s, yn s ds which is bounded by this quantity M j from n+1 to infinity Kh to power j/factorial j t0 to t ds and we can write it like this and we know that this is tending to 0 as n tending to infinity and hence we can say that limit n tending to infinity t0 to t f s, yn s ds=t0 to t f s, ys ds.

And we can say that this implies that y t satisfy the integral equation 19. So we have shown that y t satisfy the integral equation.

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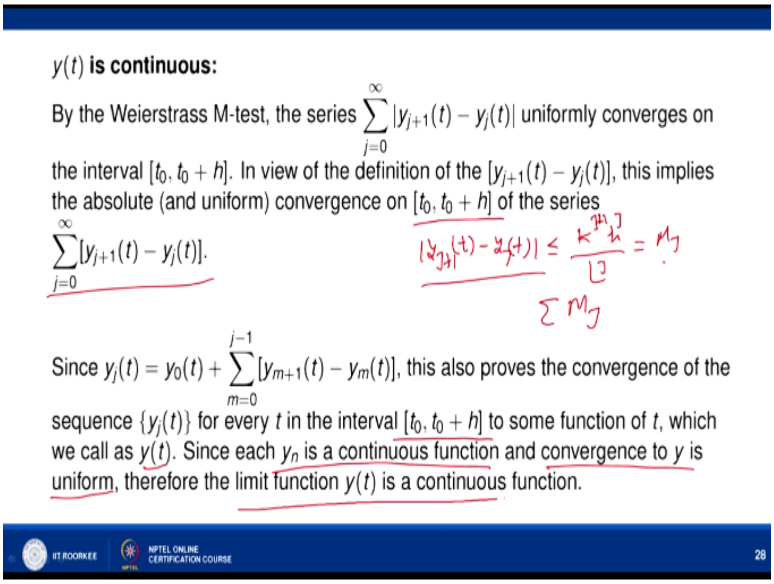

Now we want to show that y t is continuous function. So this we can achieve by Weierstrass M-test, in fact while proving the convergence part we have observed that the series j from 0 to infinity yj+1 t-yj t is uniformly convergent series in the interval t0 to t0+h. So how we can say that if you look at the convergence part in view of the definition of this yj+1 t-yj t this implies the absolute convergence on t0 to t0+h of the following series.

We can remember this is what modulus of yj+1 t-yj t is bounded by if you look at this is bounded by the following thing K j-1 s-t0 to power j/factorial j here. So this is bounded by this K to power j, h to power j-1/factorial j-1. Is it okay? I think it is here. So here it is j here and here it is j here so we can say that this is bounded by K j+1 s to power j/factorial j. Now we can apply Weierstrass M-test and we can say that we can call this as M j and we can say that summation of M j is converging.

So this series is convergence so we can say that by Weierstrass M-test this is convergent and this convergence is uniform convergence. So we can say since yj t=y0 t+m from 0 to j-1 ym+1 t-ym t, this also prove the convergence of the sequence yj t for every t in the interval t0 to t0+h to some function of t which we call as y t. Now we know that each y n is a continuous function and convergence to y is uniform.

So we can say that limit function is also a continuous function right. So we need to observe only this thing that the convergence which we have discussed is a uniform convergence and each yn t is basically if you look at yj t is basically what it is sum of this thing and each component is a continuous function so we can say that it will converge to a continuous function, so we can say that y t is a continuous function.

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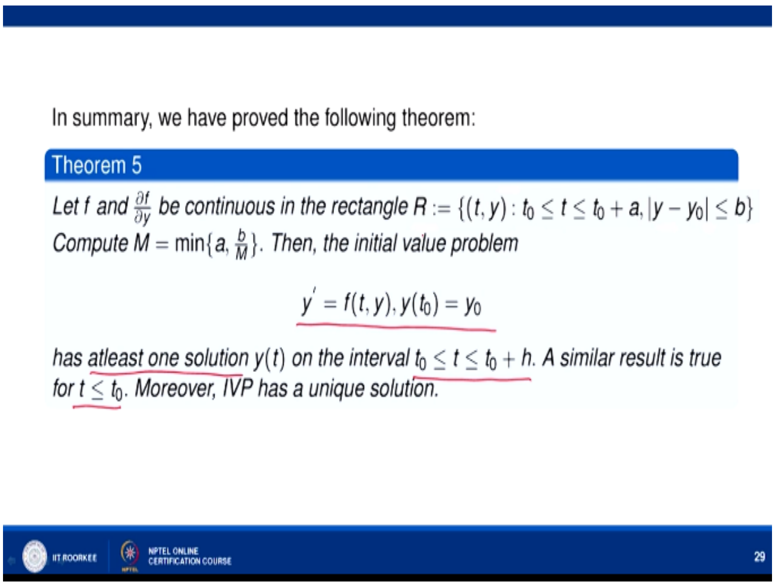

So if we summarize the discussion which we have done today we can say that we have proved the following theorem that let f and dou f/dou y be continuous in the rectangle R where R is defined as the set t and y defined as that t is lying between t0 to t0+h modulus of y-y0 is<=b and we can compute the bound M which is minimum of a and b/M. Then the initial value problem y dash-f t, y, y t0=y0 has at least one solution y t on the interval this t0 to t0+h.

So this theorem guarantees that there exist at least one solution and we can also use the similar argument to show that the solution exist for some interval t<=t0 in a same way and now we want to show that not only it has a one solution but it has a unique solution. So we have assumed that dou f/dou y be continuous in this rectangle that condition is sufficient to show that the solution is unique as well.

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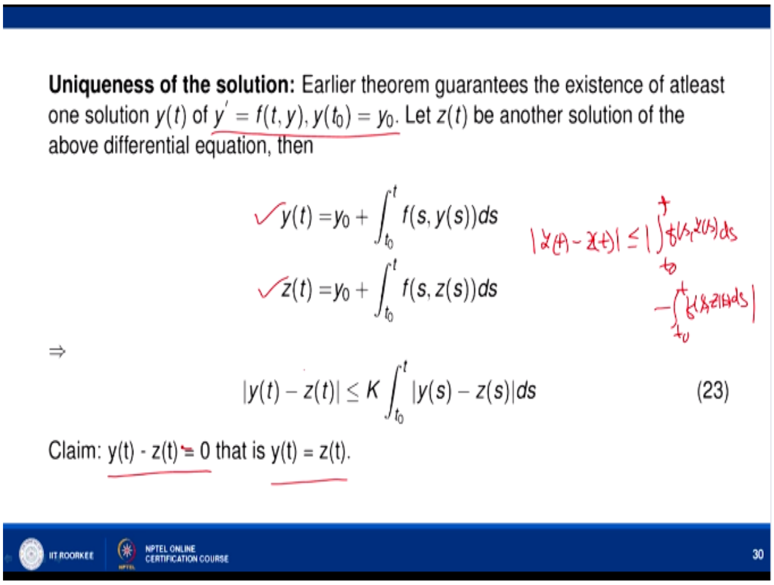

To show the uniqueness part let us consider the following thing. So in this we want to prove that solution is unique. So earlier theorem guarantees that existence of at least one solution, so it means that we have one solution of this. Now to show that it has a unique solution let us assume that we have one more solution call that as z t. So let z t be another solution of the above differential equation.

So it means that y t is a solution means this equation is true, z t is the solution means this equation is also true. Now with the help of this we can consider modulus of y t-z t and it is coming out to be if you look at modulus of y t-z t is<=modulus of t0 to t f of s y of s-t0 to t f of s z of s ds. Now here either we can use the existence of dou f/dou y or say Lipschitz condition on f we can show that modulus of y t-z t is<=K times t0 to t modulus of y s-z s ds. So here we can say that in fact this is inequality.

So we can say that modulus of y t-z t is<=modulus of t0 to t f s y s-f s z s ds. Now here we can apply the Lipschitz condition or the bound of the partial derivative of f. So here we can say that modulus of y t-z t<=K times t0 to t y s-z s ds. Now we want to show from this inequality that y t is identically = z t. So this we can show by saying that y t-z t is identically = 0 that is y t=z t.

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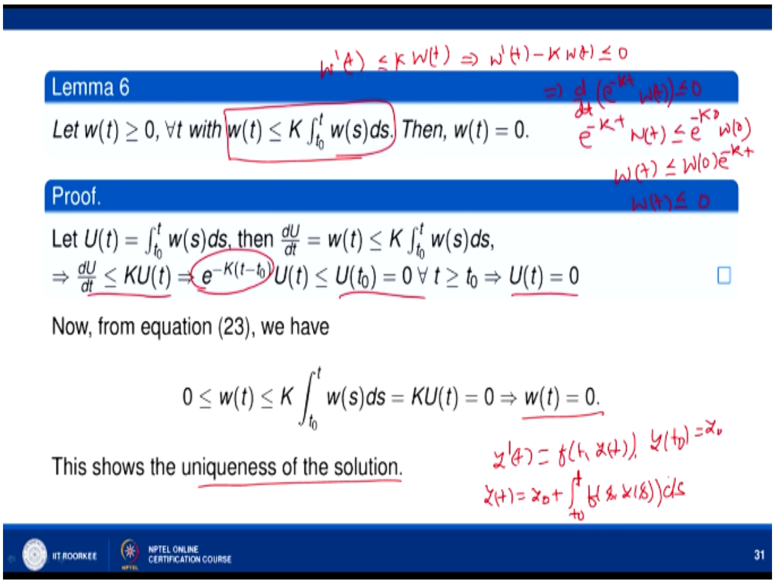

So we want to show that this y t-z t is basically 0. So we want to prove that let w t is nonnegative and for every t with w t satisfy the following inequality that w t is<=K times t0 to t w s ds. Then we want to show that w t is=0. So here if you simply want to prove like this then we can simply differentiate this like w dash t is<K w t and this we can write it w dash t-K w t<=0 and this I can write it d/dt of e to power-K t w t<=0.

And if you integrate from t0 to t we can write it e to power –K t-t0 w t<=w t0 and we already know that this w t0 is 0 because here we have just calculated here w t0 is 0 so we can write it this as w t is<=0 or you can if you do not want to do it like this then we can simply say that since du/dt of e to power –K t w t is<=0 and this if we integrate we can simply write from 0 to t we can simply write this as the following.

We can simply say that e to power –Kt w t<=e to power –K0 w of 0 and we can simply say that this implies that w t is<=this is w0 e to power –Kt and we can say that w0 is 0, this we can show it here w0 is<0 so we can say that w t is<=0 and we are done right but the problem is that I cannot use this okay why because I cannot differentiate this directly right. So to avoid this direct differentiation what we do let us assume this right hand side a new function U t.

So let U t is from t0 to t w s ds then we can differentiate this and we can find out du/dt as w t right and w t we already know that it is bounded by K times t0 to t w s ds so it is<=K times t0 to t w s ds and by the definition of U t I can say that this is nothing but KU t. So du/dt is<=K times U t and this implies that e to power –Kt-t0 U t is<=U t0 and we already know that U t0 is what U t0 is given as 0 because if we put t=t0 then it is nothing but 0.

So it means that e to power –Kt-t0 U t is<=0 now this is nearby 0 value so it means that U t has to be bounded above by 0 and we already know that U t is positive so it means that U t is=0 so it means that what we have proved here that U t is 0 and now this is possible only when w t is 0. So it means that what we have proved that if w t is nonnegative function along with this inequality then we can show that w t is=0 only.

So how we are going to utilize this lemma in our result, here we have shown this that modulus of y t-z t is<=K times t0 to t modulus of y s-z s ds. So here you can assume w t=modulus of y t-z t. So if we assume this then it is written as w t is<=K times t0 to t w s ds and just now we have proved this lemma that this implies that your w s has to be 0 so it means that 0<=w t.

Now w t is<=K times t0 to t w s ds which is nothing but K times U t and U t is coming out to be 0 so it means that 0<=w t<=0 so this implies that w t=0. Now what is w t here, w t is defined as modulus of y t-z t. So it means that y t=z t so what we have proved we have proved the uniqueness of the solution.

So in this lecture what we have shown here that first we have shown that this y dash t=f t, y t with condition y t0=y0. This is equivalent to the integral equation y t=y0+t0 to t f of s y of s ds and then we have constructed the approximate solution of this integral equation and we have shown that the sequence of approximation converges to a limit y t. We have shown that y t is continuous and we have shown that such a y t it means that y t is a solution and we have shown that a solution which satisfies the integral equation is going to be unique solution.

So what we have shown that if f is continuous and satisfy the Lipschitz condition in a rectangle R then initial value problem has a solution. So that is what we have stated as theorem 5 that let f and dou f/dou y be continuous in the rectangle R where R is defined as t from t0 to t0+h modulus of y-y0 is<b and we can compute M as minimum of a and b/M.

Then the initial value problem y dash=f t, y y t0=y0 has at least one solution y t on the interval t0 to t0+h and we have shown that it has a unique solution. So that is the content of this lecture. We will continue in next lecture this theory okay. Thank you for listening us. Thank you.