Ordinary and Partial Differential Equations and Applications
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Lecture – 38
Cauchy Method of Characteristics -II

Hello Friends! Welcome to this lecture. In this lecture we will continue our Cauchy method of Characteristics to solve first order non PDE and if you recall in previous lecture, we have discussed the theoretical idea of Cauchy method of characteristics and how this method will help us to find out the integer surface as totality of characteristic curves. So let us consider some examples so that we can understand the established theory. So for that let us consider 1 example.

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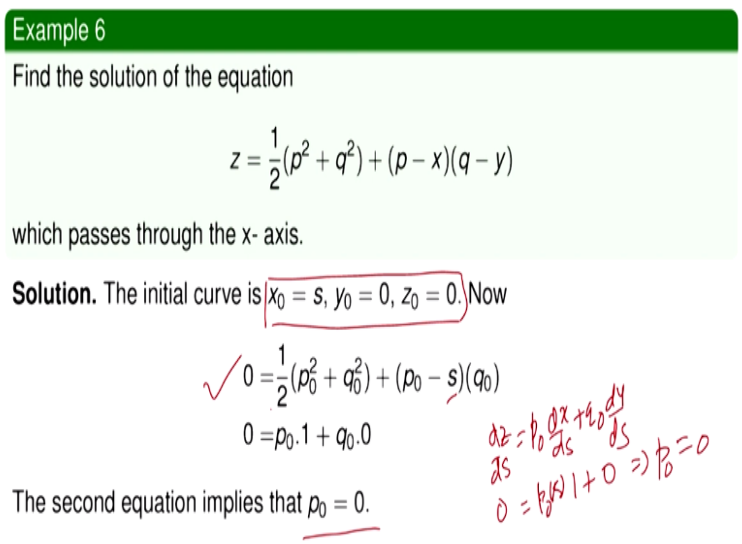

So here we need to find out the solution of the equation z=1/2(p square + q square) + (p - x)(q - y) and which passes through the x-axis. Now here, so first with the given data we try to find out the initial curve. So since data of the initial curve is that it passes through the x-axis so we can characterize the initial curve as x0 = some S, let us say S is a parameter by which we are characterizing the initial curve. So x0 = s and y0 = 0 and z0 = 0.

That is the parameterization of the x-axis. So it means that data is at the integral surface which should pass through the x-axis. So we parameterize the x-axis by this so x0 = s and y0 = 0 and z0 = 0. So that will give you the initial value of x0, y0, z0. Now look at the initial value of p0 and q0. So for that we use the relation that your initial data or initial curve will lie on surface that must satisfy the equation of the partial differential equation that is here put the value of x0, y0 and z0, so z0 is 0.

So this is 0 = 1/2 (p0 square + q0 square) + (p0 - x) x is s here and q0 and y0 is simply 0. So we have one relation 0 = 1/2 (p0 square + q0 square) + (p0 - s) (q0). And one more relation that your dz/ds = p0(dx/ds) + q0(dy/ds). So dz/ds is 0, p0 you need to find out. P0(s) you need to find out. dx/ds is 1 + q0(dy/ds) is 0 so again it is 0 so this will give you that p0 = 0. So p0(s) we are getting as 0. Now using this p0 = 0 you can find out q0 from this equation.

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On substituting in the first equation, we get

$$\frac{1}{2}q_0^2 + (-s)(q_0) = 0,$$

$$q_0 = 0 \text{ or } q_0 = 2s.$$

Therefore there are two initial strips

$$\text{Case (i)} \quad x_0 = s, y_0 = 0, z_0 = 0, p_0 = 0, q_0 = 2s, \qquad (23)$$

$$\text{Case (ii)} \quad x_0 = s, y_0 = 0, z_0 = 0, p_0 = 0, q_0 = 0. \qquad (24)$$

And the value of q0 you can say that 1/2 (q0 square) (- s)(q0) = 0. Then if you solve this you can see that you are getting 2 values of q0 that is q0 = 0 and q0 = 2s. Now we will consider this as 2 different cases. So let us see case1, case2. So initial strips are 2. Let us write it in a case1 and case2. So let us say in case1 x0 = s, y0 = 0, z0 = 0. This is common in both the cases. The only thing is that p0 = 0 and q0 = 2s in this case. In another case your p0 = 0 and q0 = 0.

So first in case1, we try to find out the integral surface and then in case2 also we will find out the solution of integral surface.

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The characteristic equation of this PDE is given by

$$\frac{dx}{dt} = p + q - y$$

$$\frac{dy}{dt} = p + q - x$$

$$\frac{dz}{dt} = p(p + q - y) + q(p + q - x)$$

$$\frac{dp}{dt} = p + q - y$$

$$\frac{dq}{dt} = p + q - x.$$

Therefore

$$\frac{d}{dt}(x - p) = 0 \text{ and } \frac{d}{dt}(y - q) = 0. \quad (25)$$

So here your characteristic equation of this is PDE is given by this following thing. So how you can obtain? So here if you look at your z = to what 1/2 (p square + q square) + (p - x)(q - y) right so we can write your F(x, y, z, p, q) equal to you take this z this side and you can write 1/2 (p square + q square)) + (p - x)(q - y) – z right. And then you look at what is your Fp. So Fp is basically, you differentiate this, so we will get p and here you will get q - y and that is all.

So this is Fp and similarly Fq is what. If you look at here Fq is again q + here you will get p – x and PFp you can easily calculate. Now what is Fx? Fx is basically - q. So it is y – q basically and Fy is basically x - p and Fz is basically - 1. So using this you can find out dx/dt as Fp that is p + q - y. So let me write it here. dy/dt = Fq. So Fq + p - x we have written here. dz/dt is PFp +QFq. So PFp is this and Fq is this. So we have written dz/dt. Dp/dt is - (Fx+PFz).

So Fx is x - p I am sorry Fx is y – q and this PFz is that is this so we can say it is p + q – y is that okay. So that we have written because - sign is inside so you can make it inside. So dq/dt is - (Fy+qFz) here. Let me solve this. It is - (Fy is basic (()) (06:17) x - p - q) and z is again q so this you can write it p + q - z. So dq/dt is also given. Now we need to solve these 5 set of ordinary differential equation and initial returns are already obtained like case(i) x0 = s, y0 = 0, z0 = 0, p0 = 0, q0 = 2s.

So theoretically, we can solve this ordinary differential equation and we should have a solution, but here now theoretically it is possible now your experience will come into picture and we solve this depending on the 2 equations. So here you can easily see that if you look at this equation, this dx/dt = p + q - y and dp/dt = p + q - y. So we can write d/dt of x - p = 0. Similarly, you can look at this and this. So it means that d/d(y - q) = 0 right. So here you can simply say that x - p is constant with respect to t.

Similarly, y - q is constant with respect to t. So we can write it that x - p = c0 and y - q = c1. Now we need to find out what is the value of c0 and c1. For that you can use the initial value. Now initial value is what x0 = s and when looking at the first case y0 = 0, z0 = 0, p0 = 0 and q0 = 2s. So here you can use c0. This implies that c0 = what. c0 = s here and c1 = what? c1 = - 2s is that okay. c1 = - 2s.

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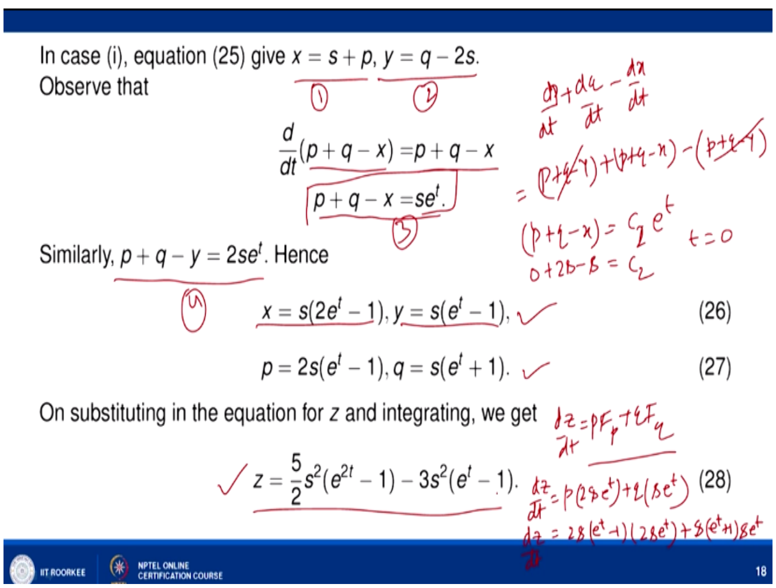

So we can write our equation as x - p = s and y - q = - 2s. That is what we have written here. This x - p = s and y - q = - 2s that is what we have written here. So x = s + p, y = q - 2s. But, we cannot utilize this also because the value of x, y is given in terms of p and q. So we have to find out the value of p and q. We can like say that it is a solution here. So for that look at your equations.

So here you look at the value of p + q - x and we can observe if we look at d/dt of (p + q - x) then it is given as p + q - x. For that you look at dp/dt + dq/dt - dx/dt. So if you look at dp/dt is what? dp/dt is this thing p + q = - 1. So it is (p + q - 1) + what is dq/dt? dq/dt is p + q - x - what is dx/dt, it is same as p + q - y. So it is p + q - y. So this we will cancel out you will have p + q - x. So here we simply say that d/dt of p + q - x = p + q - x and we can say that p + q - x is e to the power t.

So here we will get that p + q - x = some constant. Let us say c2 e to the power t. Now I need to find out the c2 and again we use the initial condition that is 0 + 2s - s = c2. So at t = 0 your initial conditions are given like this. So here you can say that c2 is s. So we can write p + q - x = se to the power t. Similarly, we can find out the value of p + q - y and it is coming out to be 2se to the power of t.

So that also you can find out. Now once we have this 1 relation, 2 relation, 3, and 4. Now using 1, 2, 3, 4 we try to solve x, y, p, and q. Here z is not involved. So we can have 4 relation and 4 unknown to find out. So we can easily solve our x, y, p and q. So using this you can solve as how will you solve. You just look at the value of p + q here in both the things and you can find out the value of p + q here and just equates and you will get x and y and p and q here.

So this I am leaving it to you that verify that your x = s (2 e to the power t - 1). y = s (e to the power t - 1), p as 2s(e to the power t - 1) and q as s(e to the power t + 1). So, so far what we have obtained is we obtained the say 2 parameter family in terms of x and y, but we have not get any expression for z also. So we also want 2 parametric representation of z. So for that we have to look at the equation dz/dt. So here we have look at dz/dt = PFp + qFq right.

So using this you go back to this equation. Then p + q - y you already know. p + q - x you know. We know p and q all these are known to us. Now using this you can write dz/dt. So let me write it what is dz/dt. So dz/dt is what? dz/dt = pFp. Fp is p + q - x right p + q - y. So p + q - y is 2s(e to the power t + q (s e to the power t). Now you write p and q also in terms of this. So 2 s(e to the power t - 1) (2se to the power t + q is what se to the power t + 1) s e to the power t.

So this is dz/dt. So now it is totally in terms of s and t. S is constant with respect to t. So you can solve this dz/dt. Now this again I am leaving it to you to show that your z is coming by this 5/2 (s square e to the power 2t - 1) - ds square e to the power t - 1. So now we also have the 2 parameter representation of z. So now x, y, and z will define n integral surface and now integral surface is given in terms of 2 parameter representation of surfaces.

But if you want the relation which is not given in terms of 2 parametric representations then we have to remove the parameter s and t and when you remove the parameter s and t so from this 2 equations you can find out the value of s and t in terms of x and y and you can see that you can solve the first 2 equation for x and y and you can get the value of et and s like this.

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On solving for t and s from (26), we get

$$e^t = \frac{y - x}{2y - x}, s = x - 2y.$$

On substituting for t and s in terms of x and y in (28), we get

$$z = \frac{1}{2}y(4x - 3y).$$

In Case (ii), equation (25) give $x = s + p$, $y = q$.
Observe that

$$\frac{d}{dt}(p + q - y) = p + q - y$$

$$p + q - x = -se^t.$$

Similarly, $p + q - y = 0$.

So et is y - x /2y - x, s as x - 2y. So using the value of e to the power of t and s you can now solve for z and you can use this expression for this s you know, e to the power t you know and you can put it here and you can simplify and you can get your solution as z = 1/2 y(4x - 3y) so that I am leaving it you that simplify and you can get this. Now this is the case when we have assumed the value of q0 as 2 of s, but if we have another case also when q0 = 0.

Then in this case, we again solve this set of ordinary differential equation and we have x as s + p and y = q. Now once we have this then again we can apply the same thing that d/dt of p + q - y = p + q - y and we can get this value p + q - y = - s * e to the power t and p + q - y = 0. So these 2 equation we can obtain and from 1, 2 dash, 3 dash, 4 dash you can solve x, y, p, and q and we can get our solution like this.

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Therefore $p = 0$, $q = y$, $x = s$, $q = y = s - se^t$. Hence

$$s = x,$$
$$e^t = \frac{x - y}{x},$$
$$\frac{dz}{dt} = y(y - x),$$
$$= s^2(e^{2t} - e^t).$$

On integrating and using the initial conditions, we get

$$z = \frac{s^2}{2}(e^{2t} - 2e^t + 1),$$
$$= \frac{s^2}{2}(e^t - 1)^2.$$

That p = 0, q = y, x = s, q = y = s - se to the power t. So we can write here s = x by solving this and e to the power t as x - y/x. when you simplify then dz/dt = p(p + q - y) and if you look at p + q - y is coming out to be 0. So this is 0 + q * p + q - x that is written as - se to the power t and you can use the value of q and then you can simplify z/dt as y(y - x) and we can simplify this as dz/dt = s square(e to the power 2t - e to the power t).

And we can again integrate and using the value hat z0 = 0 you can simply that z = s square/2 e to the power 2t - 2e to the power t + 1 which is nothing by the whole square of e to the power of t - 1. So z = s square upon 2 e to the power t - 1 whole square. Now we already know the value of s that is s = x and you can similarly you can find out the value of y as from the last equation that is y = s - se to the power t. So we can get the value of e to the power t from this.

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On eliminating s and t, we get

$$z = \frac{x^2}{2}\left(\frac{x-y}{x} - 1\right)^2 = \frac{y^2}{2}.$$

$$\Rightarrow z = \frac{y^2}{2}$$

Example 7

Using the method of characteristics, find the integral surface of $pq = xy$ which passes through the line $z = x$, $y = 0$.

So we can see that on eliminating s and t we get the following relation that is z = x square/2(x - y/x - 1) whole square and when you simplify it is nothing but y square/2. So you can say that z = y square/2 is also an integral surface. So now what you have obtained in this example we have obtained 2 set of integral surfaces right and these 2 set of integral surfaces are obtained because your initial data is basically 2 initial data that your integral curve passes through the x axis right so, here corresponding to 2 set of integral data.

We have 2 integral surfaces okay. So that is what we have obtained in this particular example. Now let us consider 1 more example. In fact, we will consider 2 more example to get the correct thorough understanding of Cauchy method of characteristic. So this example is using the method of characteristic find the integral surface of pq = xy and which passes through the line z = x, y = 0. So, first thing is the get the parametric representation of initial curve. So here how you can find out the parametric representation of z = x, y = 0.

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Solution. The initial data curve is

$$x_0(s) = s, y_0(s) = 0, z_0(s) = s.$$

Now, using this information, we get

$$p_0(s)q_0(s) = 0,$$

and

$$1 = p_0.1 + q_0.0$$

Therefore

$$p_0 = 1, q_0 = 0.$$

So you use the parametric s to represent this so you simply say that x0 = s. If x0 is s then z0 is also s, y0 is 0, z0 is s. So x0, y0, z0 is there how to find out p0, q0 for that you use partial differential equation that is pq = xy and the relation dz = Pdx + qdy. So this is true for any parametric so dz/ds is basically 1 = p0 you need to find out dx/ds is 1 + q0 dy/ds is 0. So p0 is coming out to be 1. So p0 is 1.

Now using this since it is p0 is 1, q0 we need to find out, x0 is s, y0 is 0. So q0 is coming out to be 0. So here we have only 1 initial data that is s, 0, s, 1, 0. So these are x0, y0, z0, p0, q0. Now initial data is given with us. Now we must have initial data. Now we look at the set of ordinary differential equation.

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The characteristic equations are

$$\frac{dx}{dt}=q$$
$$\frac{dy}{dt}=p$$
$$\frac{dz}{dt}=2pq$$
$$\frac{dp}{dt}=y$$
$$\frac{dq}{dt}=x.$$

Hence

$$x = Ae^{t} + Be^{-t}, q = Ae^{t} - Be^{-t}.$$

So we have for that we simply say that pq - xy we call as F(x, y, z, p, q) equal to this. Then F of p = q, Fq = p, Fx = - y. Fy = - x. and Fz = 0. So once we have then dx/dt = Fp so q. so dy/dt = q, dy/dt = Fq that is p here. dz/dt = PFp + QFq so that is pq + pq. So it is 2 pq. dp/dt is - (Fx + PFz), right. So Fz is 0 so it is Fx. Fx is - y. So it is coming out to be y here. dq/dt is x. So here if you look at our equation is what dx/dt = q, dy/dt = p, dz/dt = 2pq.

So to solve, our idea is to find out x, y, z, in terms of s and t. So but here to solve x, y, z, in s and t we need to solve first for p and q because your dx/dt is in terms of q and dy/dt in terms of p. So first we need to solve dp/dt and dq/dt. Now dp/dt is near in terms of y and dy/dt in terms of p. So you can use this and you can simply say that d2 y/dt square = dp/dt that is y here. So from this you can solve your d2y/dt square = y and this has a very simple relation and you can say that you can write your y as some linear combination of e to the power t and e to the power - t.

Similarly, you can write down that d2x/dt square = dq/dt that is x here. So again for the same thing, d2 x/dt square = x and you can write x as linear combination of e to the power t and e to the power - t. So we have written x = Ae to the power t + Be to the power - t and q is nothing by dx/dt. So dx/dt means q is written as differentiation of this with respect to t that is Ae to the power - Be to the power - t. So that is q here.

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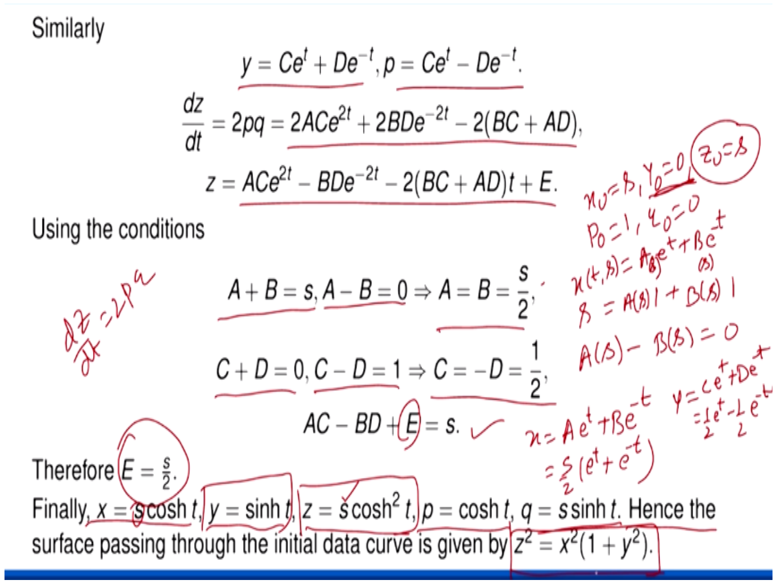

Similarly, you can find out y that is Ce to the power t + De to the power - t and p as dy/dt that is Ce to the power t - De to the power - t. So once we have p and q right now it is in terms of C and D, A, B. We can do one thing that we can use initial data right now to find out A, B, C, D or let us solve for dz/dt also. So dz/dt is 2pq and you can find out the value of p and q and you can write down this expression and when we simplify you have this expression.

Now our idea is to solve your A, B, C, D. Because to solve A, B, C, D we need to use initial data and initial data is what x0 = s, y0 = 0, z0 = s, p0 = that we have to p0 = 1 and q0 = 0 right. So since x(t, s) is given as so xt is given as what? Ae to the power t + Be to the power - t. So here when we write xt as this means that I am assuming that A and B are function of S. So As is basically what this is value s = A(s) e to the power t simply 1 + B(s) e to the power - t is again 1.

So A + B = s so that is this relation. Then using the relations for q that is A(s) - B(s) = q0 that is 0 here. So A - B = 0. So with this we can find out the value of A and B that is A = B = S/2. Similarly, using the relation that y0 = 0, we can write that C + D = 0 and p0 = 1 we can write C - D = 1. S0 C + D = 0 and C - D = 1 it means that C = - D = 1/2. And we also know that the z0 = s. So z0 is basically what you can get this from that here AC - BD - 2 * t0.

So this will be gone + E. So we have AC - BD + E = S. We already have A, B, C, D so we can find out the value of E and that is given by E = s/2. So what we have now is this x = Ae to the power t + Be to the power - t. A and B we already know so it is nothing but s/2(e to the power t + e to the power - t). So we can write this as s cos hyperbolic t, similarly you can write y as Ce to the power t + De to the power - t so it is what? 1/2 e to the power t - 1/2 e to the power - t and we can write this as y = sine hyperbolic t.

Now using this all values of A, B, C, D you can write down that z is coming out to be scos hyperbolic square t. Either using this or you first with the help of xy you got the value of p and q and with the help of p and q you can solve dz/dt = 2 pq and then you can get z = this value. So it is up to you whether you directly put the value or you use the value of p and q to find out z. It is up to you. So we can get your z as scos hyperbolic s square t.

Now we need to find out the integral surface formula which is not given in terms of 2 parameters. So you remove the 2 parameters s and t and once we remove the parameter s and t you can get our solution as x square = x square (1 + y square). So from this if you look at from x = s cos hyperbolic t and y = sin hyperbolic t you can get the value of s, right and once we have s and then you can put y as sin hyperbolic t.

So you can get the value of sin hyperbolic t and you can simplify this. So s is given as x/y and cot hyperbolic t. So you can simplify this z square = x square (1 + y square). So here we have solved this example also. So now we will consider 1 more example which I think by this we must have cleared our data and we know what we try to do in each and every example.

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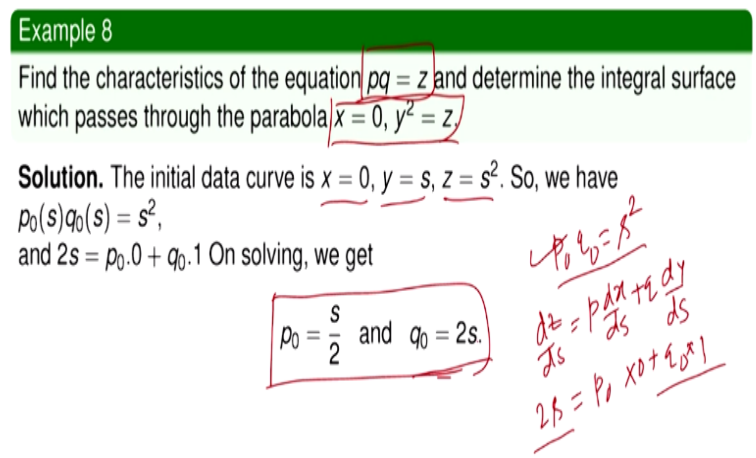

So let us consider 1 more example. So find the characteristics of the equation pq = z and determine the integral surface which passes through the parabola x = 0, y square = z. Now what is the characteristic r, the solution of differential equations. So those equations or those solutions are known as characteristics. Now let us do it is. So first thing is parameterized your initial curve that is x0 = 0, y0 = s, then z is automatically s square.

Once you have x0, y0, z0 use your pd that is pq = z so p0, q0 = z0 that is s square and dz/ds = pdx/ds + qdy/ds. So here dz/ds is what 2s = p0 dx/ds is 0 + q0 dy/ds is 1. So you can get q0 as 2s and p0 you can get it from this. When we have q0 you can find out the value of p0 that is s/2. So we have p0, q0. So now we have all the initial values x0 = 0, y0 = s, z0 = s square, p0 = s/2, q0 = 2s. Now look at the differential equation.

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The characteristic equations are

$$\frac{dx}{dt} = q$$
$$\frac{dy}{dt} = p$$
$$\frac{dz}{dt} = 2pq$$
$$\frac{dp}{dt} = p$$
$$\frac{dq}{dt} = q.$$

which implies that $p(t) = Ae^t$, $q(t) = Be^t$. At $t = 0$, we have $p_0 = \frac{s}{2}$ and $q_0 = 2s$ so $p(t)$ and $q(t)$ becomes $p(t) = \frac{s}{2}e^t$ and $q(t) = 2se^t$.

So differential equations are dx/dt = this thing. So for that you have equation pq = z. So pq - z = 0. So that is your f(x, y, z, p, q). Now here Fp is q, Fq = PFz = - 1. Fx = 0, Fy = 0. So dx/dt = Fp. We got it q. dy/dt = Fq that is p, dz/dt = PFp + QFq then again 2pq, dp/dt = p and - Fx + pFx so Fx = 0. So we will get Fz is - 1. So we will get d/dt = p and dq/dt = q. Now again look at here. dx/dt, dy/dt all these involve p and q. So first we have to solve.

We have to solve dp/dt and dq/dt. So dp/dt here it is quite easy. So here p is given as something constant C0 e to the power t and q is also some C1 e to the power t. Now how to fix these c0 so here at t = 0 we already know that p0 = what? So p0 is given as s/2 and q0 = 2s. So we can write it here. The value of c0 = s/2, and value of c1 = 2s. So we can write p(t) as s/2 e to the power t and q(t) as 2se to the power t. So p(t) and q(t) is now known to us.

Then we can write it dx/dt = q is 2s e to the power t and dy/dt= what? dy/dt = p that is s/2 e to the power t. So when you solve this, what you will get x(s, t) = 2 s e to the power t plus some constant let us say C3 and y of st will be what? s/2 e to the power t + some C4. Now how to get C3 so that initial values is what 0 and s.

So here it is 0 and here it is s. So t = 0 that is 0 = 2s + C3 and here t = 0 it is s so s = s/2 + C4. So C3 and C4 is C3 is - 2s and C4 = s/2. So using C3 and C4 x(s, t) will be what 2s e to the power t

- 2s. So you will get 2s e to the power t - 1. So we have x (s, t) = 2 s e to the power t - 1 similarly you can get your y of (s, t).

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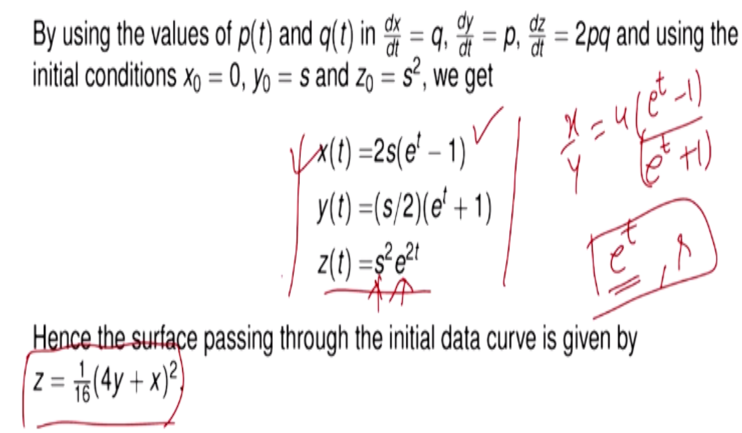

So let us look at here. So x(t) is 2s(e to the power t - 1), y(t) = (s/2)(e to the power t + 1). Similarly, once we have the value of p and q which we have already obtained here. You can get dz/dt = 2pq that is we can solve and you can get z(t) = s square e to the power t. So this is the integral surface given in 2 parameter family now to get to remove this parameter you can use, simplify and you can get the x/y = what? 4(e to the power t - 1/e + 1).

And from this you can get the value of et and you use the value of et and you can use any of the relation you can get your s here. So here this is I am leaving it to you to find out the value of e to the power of t and s and hence we can put it back to this equation and you can write down your relation of z in terms of x and y as z = 1/16(4y + x) whole square. So this is the integral surface which we want to obtain.

Now please look at here. If you look at if you solve only these equations, then we will get a solution in terms of constant. So it means that the solution of this without utilizing the initial data is the equation of characteristics, but if you utilize the initial data, then it will be an integral strip rather than characteristic strip. So, integral strips means that will contain the initial curve right.

So if you find out all these c1, c2, all the constant integration constant then it will be an integral strips.

And with the help of integral strips we have 2 parameter family of the solution that is what is written here. So this is an integral surface given in terms of 2 parameter family of representation. Now this is equally 5, but many at times people want that we should have an equation of surface in terms of z = g of some function of x and y. For that we have to remove the parameter s and t and that we can remove from this.

And we have an integral surface after removing the parameter s and t. So with this I end our lecture and I hope that you have better understanding of whatever theory we have already discussed in previous lecture. So I hope that you try to solve all the problems to get more insight of this. So with this I end this lecture. Thank you very much for this thing. Thank you.