Ordinary and Partial Differential Equations and Applications
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Lecture – 37
Cauchy Method of Characteristics - I

Hello friends! Welcome to this lecture. In today's lecture, we will discuss geometrical method, geometrical ideas based method to solve a nonlinear first order PDE which is known as Cauchy method of characteristics. So let us start with Cauchy method of characteristics to solve first order nonlinear PDE. So as I mentioned earlier here we are going to discuss the method based on the geometrical ideas.

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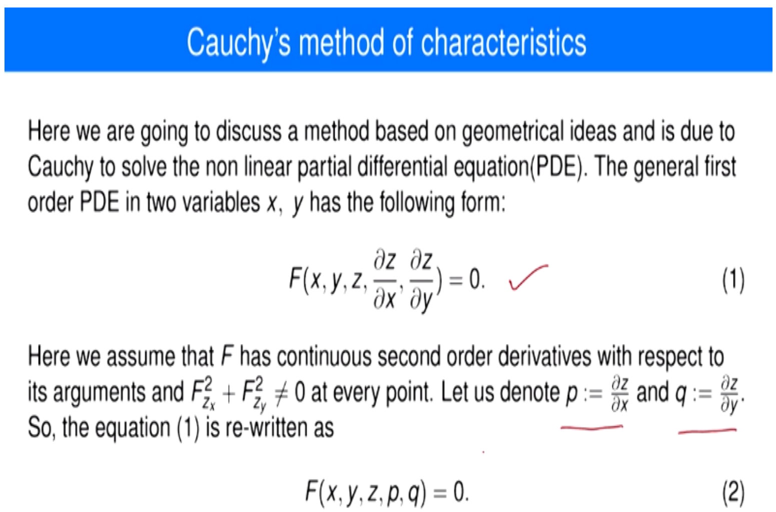

And it is due to Cauchy and to solve the nonlinear partial differential equation. The general first order PDE in 2 variable x, y has the following form. F(x, y, z, and the partial derivative of z with respect to the independent variable x, y that is dou z/dou x and dou z/dou y) and here we assume that F has continuous second order partial derivatives with respect to its arguments that is x and y and also we assume that Fzx square + Fzy whole square is != 0.

The idea behind this is that this equation number 1 is not independent of zx and zy. So in fact we can always solve this F(x, y, z, dou z/dou x and dou z/dou y) either in terms of zx is Fzx is nonzero and in terms of Fzy if Fzy is nonzero. Let us denote for simplicity let us denote that p =

dou z/dou x and q = dou z/dou y. So in terms of p and q we can rewrite our equation number 1 as follows that is F(x, y, z, p, q) = 0 and our aim is to solve this first order nonlinear PDE using the geometrical ideas.

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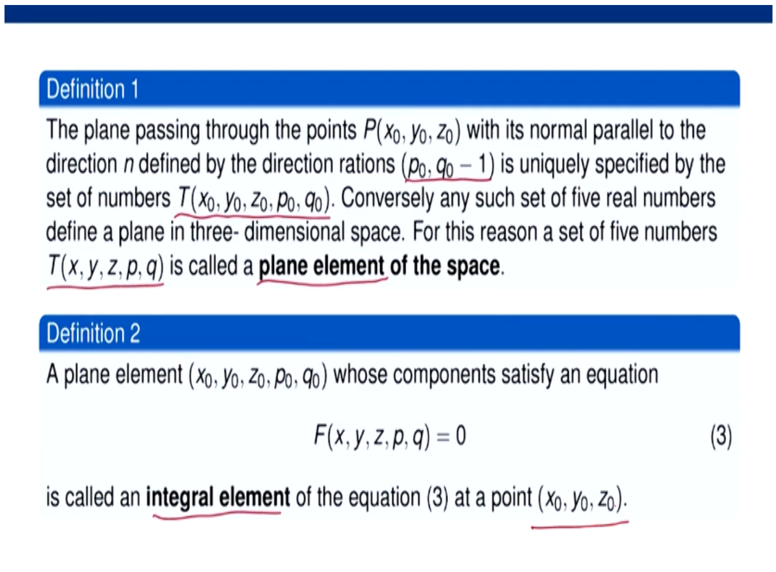

So to do this first let us do some basic things. So first you just recall that. The plane passing through a point P(x0, y0, z0) with its normal parallel to the direction n which is defined by the direction (p0, q0 - 1) is uniquely specified by the set of numbers T(x0, y0, z0, p0, q0).

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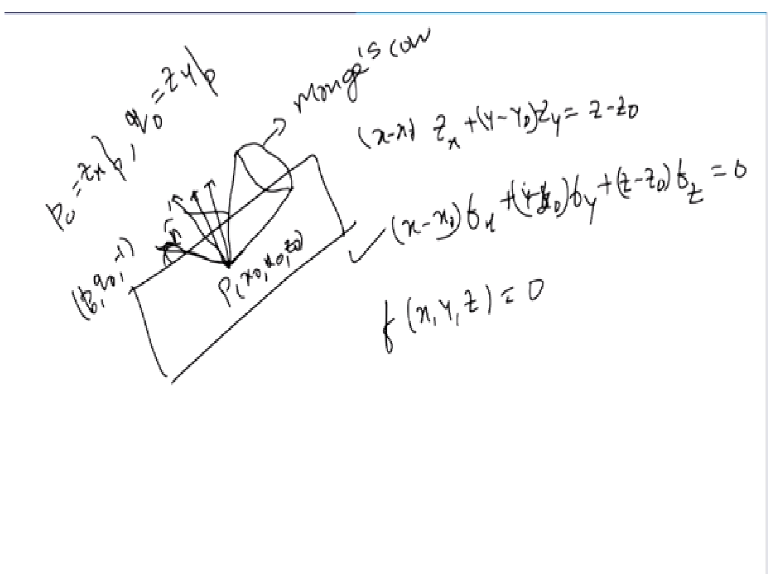

In fact, if you look at any plane and take any point which you call as P and let us say this is xo, yo, and zo and if it is this point is lying in a plane then if you know the say normal vector which is let us say that p0, q0, - 1 where p0 is your z of x given at the point p and q0 = zy given at point p. So if you know the direction factor let us say n, then you can write down the equation of plane and equation of plane you can write it like x - x0 say zx + (y - y0) zy = z - z0.

In fact, you can always write it like this equation (x - x0) fx + (y - y0) fy + (z - z0) fz = 0 if your equation is given as f (x, y, z) = 0. If your equation is given like f(x, y, z) = 0 then equation of the tangent plane you can always write it like this. So any plane you can say characterized by the point and the normal vector at that particular point and if it is a plane then you can characterize that plane by this 5 tuple that is x, y, z, and p, q. So conversely any such set of 5 real numbers define a plane in 3 dimensional space.

And for this reason a set of 5 numbers T(x, y, z, p, q) is called a plane element. So whenever we have these kind of 5 tuples we say that it is a plane element of this space. Now a plane element whose components satisfied an equation F(x, y, z, p, q) = 0 is called an integral element. So it means that if a plane element satisfy the partial differential equation F(x, y, z, p. q) = 0 then we call this plane element as an integral element. So these are some definitions which we want to write. So these are called integral element at a point x0, y0, z0.

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Since we have assumed that $F_p^2 + F_q^2 \neq 0$, so lets assume that we may solve an equation of the type (3) to obtain an expression

$$q = G(x, y, z, p) \tag{4}$$

which indicates that p and q are not independent to each other and we can calculate q when x, y, z and p are known. Keeping x_0, y_0 and z_0 fixed and varying p, we can obtain a set of plane elements $\{x_0, y_0, z_0, p, G(x_0, y_0, z_0, p)\}$, which depends on a single parameter p.

As p varies, we obtain a set of plane elements all of which pass through the point P and which therefore envelope a cone with vertex P, the cone so generated is called the **elementary cone** or **Monge's cone** of the equation (3) at the point P.

Here the term cone is used for any surface generated by a one parameter family of straight lines through the fixed point, that is, vortex.

Now it is already given that Fp square + Fq square is != 0. It means that we can always assume that either Fp is nonzero or Fq is nonzero at any given point. So let us assume that Fq is not 0.

Then we can always solve our equation F(x, y, z, p, q) = 0 in terms of q. That is q = G(x, y, z, p). Now here it indicates this equation indicates this equation that p and q are not independent at any given point.

Now we can calculate q when we know the value of x, y, and z and p at a given point. So keeping the point x0, y0 and z0 fixed. So given a point now you vary your value of p and when you vary your p we can obtain a set a plane element x0, y0, z0, p, G(x0, y0, z0, p) which depend on a single parameter p. For example, if you look at this at this particular point let us say it is x0, y0, z0 and then you take a P and you can get a value of q and you vary your p then you can have a different, different values of say q.

So we can have a different, different values of n here right. So in this way, you can say that your tangent line this can say trace a figure like this which is having vertex at the point p and here you can say that at every point you have this tangent vector. Here you have another normal vector. So we can say that these plane element trace a kind of curve which is we know as elementary cone, so that we are writing it like this as p varies.

We obtained a set of plane elements all of which pass through the point p and which therefore envelope a cone like figure and whose vertex is P and the cone so generated is called the elementary cone or Monge's cone right and this is the Monge's cone defined at the point p. So the cone is it is like this. This is what we call is Monge's cone.

Now here the term cone is used for any surface generated by a 1 parameter family of straight line passing through the fixed point that is vertex and here if you look at the p is your 1 parameter. Here you are vary your p and you are getting a line passing through this point p that is x0, y0, z0. So here we are getting a surface at a point p by varying your p and we call the generated surface as a cone and we call this cone as Monge's cone okay.

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Consider now a surface S whose equation is

$$z = g(x, y) \tag{5}$$

If the function $g(x, y)$ and its partial derivatives $g_x(x, y)$, $g_y(x, y)$ are continuous in a certain region R of the xy plane, then the tangent plane at each point of S determines a plane element of the type

$$\{x_0, y_0, g_x(x_0, y_0), g_y(x_0, y_0)\} \tag{6}$$

which we will call the **tangent element** of the surface S at a point $\{x_0, y_0, g_x(x_0, y_0)\}$.

How this is related to find out a solution surface of the partial differential equation. For that let us consider a surface s whose equation is z = g(x, y) and if the function g(x, y) and its partial derivatives gx and gy are continuous in a certain region say R of the xy plane, then the tangent plane at each point of the surface S determine a plane element of the type {x0, y0, gx, gy}.

So at any point of the surface S it determined a plane element like this. Now this plane element we call as tangent element of the surface S at the point. So it means that you look at the surface and look at the plane element here at any given point and we call this plane element as tangent element.

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Theorem 3

A necessary and sufficient condition that a surface be an integral surface of a PDE is that at each point its tangent element should touch the elementary cone of the equation.

A curve C with parametric equations

$$x = x(t), y = y(t), z = z(t) \tag{7}$$

lies on the surface (5) if $z(t) = g\{x(t), y(t)\}$ for all values of t in an interval I. If P_0 is a point on this curve at $t = t_0$, then the direction ratios of the tangent line P_0P_1 are $\{x'(t_0), y'(t_0), z'(t_0)\}$, where $x'(t_0)$ denote derivative of $x(t)$ at $t = t_0$, i.e. $\frac{dx}{dt}|_{t=t_0}$.

This direction will be perpendicular to the direction $(p_0, q_0, -1)$ if

$$z'(t_0) = p_0x_0'(t_0) + q_0y_0'(t_0).$$

Now once we have a tangent element then we can say that on the geometrical ground if this surface S is a solution of the given PDE that is the integral surface of the PDE, f(x, y, z, p, q) if it touches it at every point of the surface it touches the elementary cone or you can say it touches the Monge's cone so that is what we summarize in theorem 3. The necessary ends sufficient condition that a surface be an integral surface of PDE is that at each point, its tangent element touch the elementary cone of the equation.

So it means that a given surface is an integral surface provided that at every point of the surface it touches the Monge's cone generated at that particular point P. So now we want to find out the integral surface using this geometrical idea and we are trying to find out this integral surface as a totality of curve lying on the surface. So for that we just look back to the curve and here let us say a curve c with parametric equation x = xt, y = yt and z = zt lies on the surface z = g(x, y).

Now it will lie on the surface if it satisfy this equation for all t. So it means that z(t) = g(x(t)), y(t) for all values of t in some interval say let us say this is I. So for all t in this interval I your x(t), y(t), z(t) satisfies this relation. Now if P0 is a point on this curve so you have a curve and then so it means that it is like this we have a surface kind of thing and you have a curve like this which we call as c. Now take any point here let us say this is P0 and this is corresponding to the value of t at say t0. So let us say p0.

Then the direction ratio of the tangent line P0P1 let us say you take 1 more point say P1 on this curve and then you can always find out the direction ratio of the tangent line P0, P1 is just t0, y dash t0 and z dash t0 where these derivative represent the derivative of x(t), y(t), z(t) at the point t = t0. So you can find out the D.R.s of tangents and it is given by x(t), t0, y dash(t0) and z dash(t0). Now we say that if this tangent lying is perpendicular to the normal vector at this particular point say let us say you have a normal vector say n.

And it is given as say p0, q0, and - 1. So we say that if this direction will be perpendicular to the direction p0, q0, and - 1 then we have the following relation that is z dash(t0) = p0, x0 dash(t0) + q0y0 dash(t0). So here we have this x dash t0, y dash t0, z dash t0 dot this (p0, q0, and - 1) has to be 0. So it means that this direction and this direction has to be orthogonal to each other. So dot product has to be 0 and if you calculate the dot product we have this relation that is z dash(t0) = p0 x0 dash + q0, y0 dash.

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For this reason we say that any set

$$\{x(t), y(t), z(t), p(t), q(t)\} \quad (8)$$

of five real functions satisfying the condition

$$z'(t) = p(t)x'(t) + q(t)y'(t) \quad (9)$$

defines the strip at the point (x, y, z) of the curve C.

The set of functions (8) is an integral strip of equation (3) provided they satisfy condition (9) and the further condition

$$F\{x(t), y(t), z(t), p(t), q(t)\} = 0 \quad (10)$$

for all t in I.

For this reason we say that any set {x(t), y(t), z(t), p(t0, q(t)} of 5 real functions satisfy the condition this that is z dash(t) = p(t) x dash(t) + q(t) y dash(t). Then it defines a kind of strip. We say that it defines the strip at the point x, y, z, of the curve C. So it means that if this thing if p0, p1 satisfy this condition then we say that the x0, y0, z0, p0, q0 has a function of t. This will define a strip, right.

This is so any function any 5 function define like this who satisfy this orthogonality condition we call it a strip at p0 right. Now the set of function this (8) which define a strip is also called an integral strip provided they also satisfy the partial equation that is F(x, y, z, p, q) = 0. So satisfy the partial differential equation means that the value of f evaluated at x(t), y(t), z(t), p(t), and q(t) has to be = 0. So using these ideas now we want to find out the equation of the surface.

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If at each point the curve (7) touches a generator of the elementary cone, we call the corresponding strip, a **characteristic strip**.

We will first find the equations of a characteristic strip touching the Monge's cone. The point $(x + dx, y + dy, z + dz)$ lies in the tangent plane to the elementary cone at P if

$$dz = pdx + qdy \qquad (11)$$

$(dx, dy, dz) \cdot (p, q, -1) = 0$

where p and q satisfies the relation (3). Differentiating (11) with respect to p, we obtain

$$0 = dx + \frac{dq}{dp}dy. \qquad (12)$$

$F(x, y, z, p, q) = 0$

$q = G(x, y, z, p)$

So for that let us say that at each point the curve touches a generator of the elementary cone we call the corresponding strip a characteristic strip. So it means that you pick your curve right. You have this curve. Now if at each point if this strip you touches your Monge's cone that is this thing then we say that this strip is known as characteristic strip. Now what we try to do here, we try to generate this integral surface as totality of these characteristic strips.

So we want to find out first of all what is characteristic equations and then we look at the totality of characteristic equation and then with the help of this we try to find out the integral surface. So for that we need certain results which we are going to prove it. So first of all we try to understand what is characteristic strip? A strip which touches the Monge's cone at each point of the curve is called your characteristic strip. So we first find out the equation of a characteristic strip.

It means or we can say that the equation of the Monge's cone. Because you are characteristic strip you are touching the Monge's cone at each of its point. So first let us find out the equation of Monge's cone and generator of that. So the point x + dx, y + dy, z + dz which lines in the tangent plane of the elementary cone at p if it satisfy the following relation. So if this point lies in tangent plane it must satisfy the following relation that is dz = pdx + qdy.

So it is quite easy to find out say (dx, dy, dz) . (p, q, - 1) this has to be 0. So it means that here this point (x + dx, y + dy, z + dz) lies in the tangent plane to the elementary cone if the D.R.s of the tangent line that dx, dy, dz and normal vector to your elementary cone and normal vector to elementary cone is p, q, and - 1. That must be orthogonal to each other that is dz = pdx + qdy. Now here pq - 1 satisfy the equation 3 that is F(x, y, z, p, q) = 0.So it means that on elementary cone if you look at the normal vector it satisfy.

The partial differential equation which we have started with. Now what we try to do here we differentiate the equation number 11 with respect to p and we have the following relations that is dz with dp is simply 0. Now here we have p so it is dx + q is a function of p that we have already noted that q is great energy of x, y, z, p. so q is a function of p so we differentiate with respect to p we have dx + dq/dp * dy. So here from this compatibility relation that is dz = pdx + qdy we have the following relation that is known as dx + dq/dp, dy = 0.

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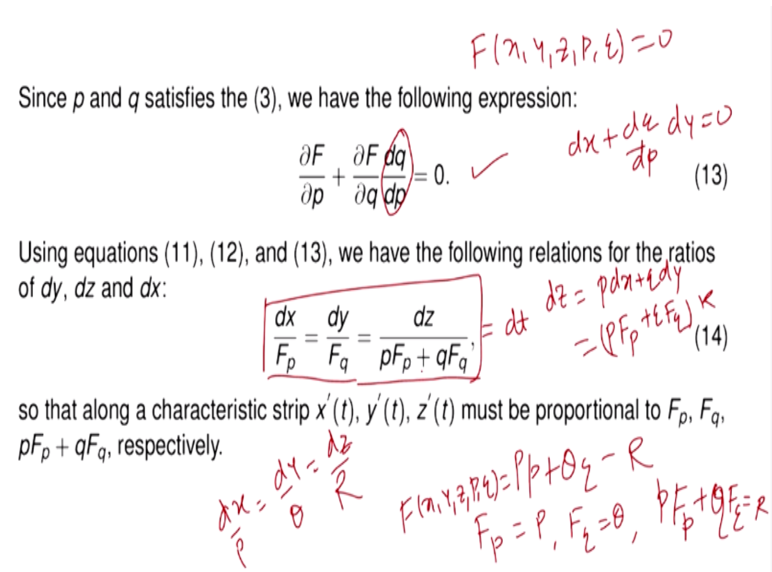

But also we know that this p and q satisfy the partial differential equation F(x, y, z, p, q) = 0. So differentiate this also with respect to p. So when you differentiate this with respect to p, then we have dou F/dou p+ dou F/dou q * dq/dp and = 0. So using the equation number 13 and the previous equation that is equation number 12, we can find out the relation between dx, dy, and dz and that we can write it here. So here if you recall is what you can say dx + dq/dp.

So here we have dx+ dq/dp * dy = 0 and here we have this relation. So we can equate the value dq/dp from these 2 equation and we can have the following relation that is dx/Fp = dy/Fq and you can get the ratio for dz that is pdx + qdy. So dx we can write here as ration between dz and this thing is PFp + qFq. So is a kind of a ratio let us say k is a ratio constant. So we have the following relation for the ratios of dx, dy, dz that is dx/Fp = dy/Fq = dz/pFp + qFq.

And when you solve this that will gives you the equation of Monge's cone that is equation of the elementary cone and here you please recall that in case of Cauchy linear equation rather than nonlinear equation in Cauchy linear equation your equation is Pp + Qq = R and if we denote this equation of F(x, y, z, p, q) = this let us write it - R here let us say it is - R. Then if you look at the value of Fp. Fp is your capital P and Fq = capital Q and PFp + QFq = R.

So this equation is now reduced to dx/p = dy/Q = dz/R and that is how you find out how you solve your Cauchy- linear problem by solving these set of differential equation and after solving this you can get the solution integral surface as a totality of integral curves. So now here we are doing the same thing. Now so that along the characteristic strip x dash(t), y dash(t), z dash(t) it must be proportional to Fp, Fq, and PFp + QFq respectively that is what this equation number 14 represent.

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Now we parameterize the curve by a parameter t in such a way that

$$x'(t) = F_p,\ y'(t) = F_q, \qquad (15)$$

and then

$$z'(t) = pF_p + qF_q. \qquad (16)$$

Also, note that along a characteristic strip, p and q are functions of t, so that

$$\begin{aligned} p'(t) &= \frac{\partial p}{\partial x}x'(t) + \frac{\partial p}{\partial y}y'(t) \\ &= \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial p}{\partial y}\frac{\partial F}{\partial q} \\ &= \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial q}{\partial x}\frac{\partial F}{\partial q} \end{aligned}$$

since $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$.

Now, what we do we can parameterize our curve as say let us say by taking this as dt here then we can write down our equations equation number 14 as the following thing x dash(t) = Fp that is here. dx/dt = Fp and dy/dt = Fq, dz/dt is PFp + QFq. So that is what I have written here as equation number 15 and equation number 16. Now along this curve which is parameterized in terms of t your p(t) and q(t) is also a function of t.

And we want to find out the p and t why because since a nonlinear problem corresponding to a given p you may have more than 1 values of q. So here you have to find out the equation for p and q as well. So we need to find out the relation for p dash t also. So to find out the p dash t value here you look at here. P dash t = dou p/dou x * x dash (t) + dou P/dou y * y dash (t). Here I am assuming that P(t) is a function of x(t), y(t) here.

So using this relation that p is a function of x and y and x and y are the functions of t so we can differentiate with respect to t and we have this relation using chain root. p dash(t) = doub p/dou x * x dash(t) + dou p/dou y * y dash (t). Now we already know the value of x dash(t) that is F(p) here. So dou p/dou x F(p) + dou P/dou y here y dash (t) is written as Fq that we are writing here. Now here we are using that dou P/dou Y is what if you use this, this is what p/dou y is basically dou/dou y. Now p is basically z of x that is dou/dou y and dou/dou x and z here

And you can write this as dou square z/dou y * dou x. Now if I assume that z is having second order continuous partial derivative then this is = dou square z/dou z dou y or you can say that here zxy = z of yx. So here I am assuming that your surface is smooth enough. So assuming this that zxy = zyx we can write down PY as q of x. So replacing py by qx we have p dash t = dou p/dou x dou F/dou P + dou q/dou x * dou F/dou q.

And we have the relation p dash t using that p is a function of xy and xy is a function of t, but we also know that p satisfy the equation f(x, y, z, p, q).

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Differentiating equation (3) with respect to x, we find that

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0$$

so that on characteristic strip

$$p'(t) = -(F_x + pF_z) \tag{17}$$

and similarly we can find the expression for $q'(t)$:

$$q'(t) = -(F_y + qF_z) \tag{18}$$

Collecting equations (15) to (18) together, we get

$$\begin{aligned} x'(t) = F_p, y'(t) = F_q, z'(t) = pF_p + qF_q \\ p'(t) = -F_x - pF_z, q'(t) = -F_y - qF_z \end{aligned} \tag{19}$$

which are known as the characteristic equations of the differential equation (3).

So using this we can differentiate the F{x(t), y(t), z(t), p(t), q(t)} = 0. If you differentiate this with respect to x we have the following relation dou F/dou x + dou F/dou Z *p + dou F/dou p * dou P/dou x + dou F/dou q * douq/dou x = 0. So right now I cannot use this because I really do not know whether this curve lies on this partial differential equation or curve lies on this surface this all this.

So right now we simply use that F(x, y, z, p, q) = 0. So differentiate with respect to x we have this condition and here we can get this value which is p dash (t) = this value. So here the characteristic strip p dash (t) = - of this value that is - (Fx + pFz) and similarly using the same thing it means that using that q is a function of x and y, we can get this kind of relation for q also and again we can differentiate the equation F(x, y, z, p, q) with respect to y.

And we can get this kind of relation as using those 2 relation we can get the value for q dash(t) and the expression for q dash(t) = - (Fy + qFz). So it means that if we collect all these differential equations then we have the 5 differential equation known as x dash(t) = Fp, y dash(t) = F of q, z dash(t) = pFp + qFq and p dash(t) = - Fx - pFz, q dash(t) = - Fy - qFz and we say that after solving these 5 set of differential equation we can get x(t), y(t), z(t), p(t) and q(t).

And we try to say that these functions define a strip and then see when this strip basically these strips are basically characteristic strip and then we say that totality of these strip will form a surface and we want to find out the condition under what this conditions this characteristic strips define a integral surface that we wanted to do. So here in this direction we have the following 2 results that

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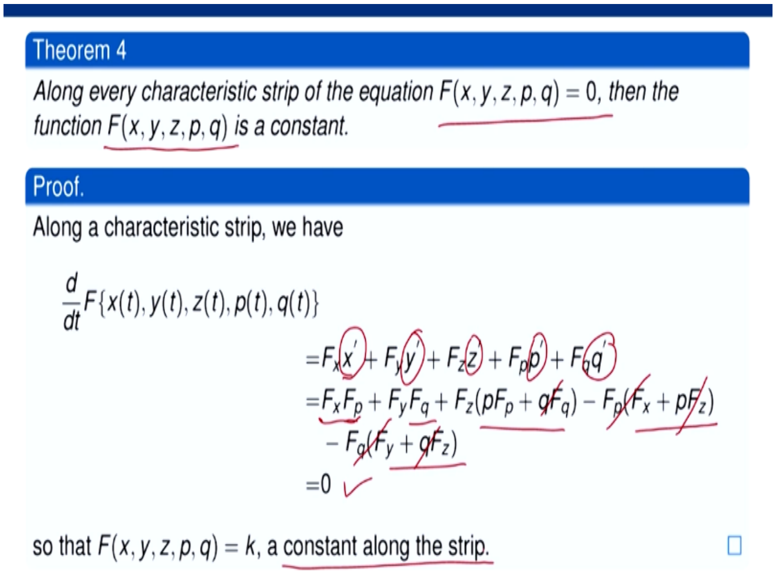

Now every characteristic strip of the equation F(x, y, z, p, q) = 0 then the value of F(x, y, z, p, q) remains constant. So to prove this you simply differentiate with respect to t of F{x(t), y(t), z(t), p(t), q(t)} when you differentiate look at F of x * x dash + F of y * y dash + F of z * z dash + F of p * p dash + F of q * q dash. Now here using the value of x as that is Fp, y dash as Fq, z dash as PFp + qFq, p dash = this thing and q dash = this thing.

When you simply this if you look at Fx and Fp and this can be cancelled out by this term. Fy * Fq this will be cancelled out here and Fz* PFp that will be cancelled out here. Similarly, this will be cancelled out here. So we can say that d/dt of F{x(t), y(t), z(t), p(t), q(t)} = 0. So it means that the value of F(x, y, z, p, q) = k will remain the constant along the strip right this so this simply says that along the characteristic the value of F(x, y, z, p, q) = constant, but we really do not know whether it is 0 or not. To make it 0 we look at 1 more small result that is.

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As a corollary, we immediately have,

Theorem 5

If a characteristic strip contains at-least one integral element of $F(x, y, z, p, q) = 0$, then it is an integral strip of the equation $F(x, y, z, z_x, z_y) = 0$

Now, suppose we wish to find the solution of PDE (2) which passes through a curve $C := \{C(s) : s \in [a, b]\}$ whose parametric equations are

$$x = f_1(s), y = f_2(s), z = f_3(s) \tag{20}$$

then in the solution

$$x = x(p_0, q_0, x_0, y_0, z_0, t_0, t) \tag{21}$$

of the characteristic equations (19), we can take

$$x = f_1(s), y = f_2(s), z = f_3(s)$$

as the initial values of x, y, z.

That if a characteristic strip contains at least 1 integral element, what is integral element? Integral element is the x, y the collection, tuple x, y, z, p, q says that the value of F(x, y, z, p, q) = 0. So if it contains 1 integral element it means that the characteristic curves contains the value x, y, z, p, q such that F(x, y, z, p, q) = 0 then of course the strip is an integral strip. It means value of F(x, y, z, p, q) = 0 along the characteristic for all values of t right what we have proven in this theorem 4 that along a characteristic your value of F(x, y, z, p, q) has to be 0.

Now if it contains a 1 single element it means that the value of F(x, y, z, p, q) = 0 for 1 particular integral element that is x, y, z, p, q then the characteristic strip will be a integral strip and once it is an integral strip then we simply look at all the integral strips and look at the totality of these integral strips and we say that this is an integral surface which we are looking for. So that is the idea of this Cauchy method of characteristic so we try to find out the characteristic strip which contain an integral element and this integral element is given in terms of initial data.

So let us look at here what is the strategy here. Now suppose you wish to find the solution of PDE(2) which PDE(2) here is F(x, y, z, p, q) = 0 and which passes through a curve C. Now this is the initial data means passes through means we want to find out a solution which contain this. It means that this curve is already on the surface which we are looking for. So it means that this is your integral, this will define an integral element.

So let us define this curve by a parameter say S. No S is lying in some interval let us say it is closed interval a, b and C can be characterized by these parametric representations that is x = f1(s), y = f2(s), z = f3(s). So here in a solution like this x has x(p0, q0, x0, y0, z0, t0, and t) of the characteristic equation we can take x = f1(s), y = f2(s), z = f3(s) as initial values of x, y, z.

So it means that if these are the initial values means at t = t0 the initial value of your characteristic strips are basically x = f1(s), f2(s) and f3(s) then it will contain an integral element and then the strips which we are looking for are integral strip and in this way we can find out the integral surface. So that is the idea of this. Now we have only x, y, z.

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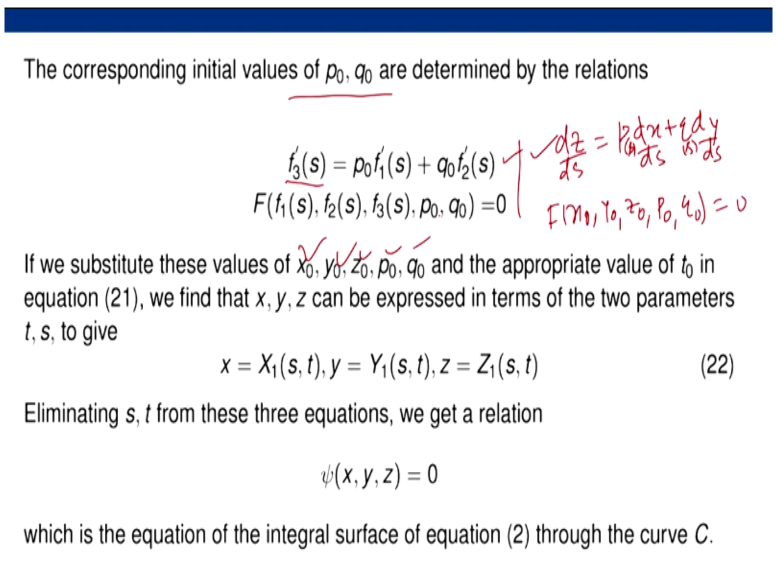

How do you find out p and q for that you can find out the initial value of p0, q0 as follows that is it satisfies this relation dz = p* dx + qdy. So we can write this as dz = f3(S) so dz/ds you can write it and this is p(s) and q(s). So initial curve is characterized by S so we can say along the solution curve your dz/ds is f3 dash(S) = p01 dash s + q0 * f2 dash (s). Also along with this because you want to find out p0 and q0 we need 2 relations.

So 1 relation is obtained by this relation that D.R.s of tangent is perpendicular to normal vector and another one is that these element will lie on the equation of the partial differential equation that is F(x, y, z, p, q) = 0. So F(f1(S), f2(S), f3(S), p0, q0) = 0. So when we solve these 2 equation we will get the initial value of x0, y0, z0, p0, q0 and we have 5 differential equations, 5 ordinary differential equations along with these initial values.

And if your Fp, Fq all these values like here which are coming here that these Fp, Fq these values are nice enough then we can get a unique solution. So here this will have a nice value because we have assumed that F has second order partial derivative in its arguments so it means that Fp, Fq all these are continuous function and we already know that in Euler differential equation if F(x, y) is continuous then also we have a solution.

And if we are assuming more than that that these function satisfy the, these functions are having 1 more partial derivatives. So we simply say that this will have a unique solution along with initial condition that we have already obtained in terms of x0, y0, z0, p0, and q0 and after solving this we have the following relation that is x = X1(s, t), y = Y1(s, t), z = Z1(s, t). So now we obtain the equation of the integral surface as 2 parameter family of surfaces.

Now if you want to find out the exact surface then you can remove your s and t from these 3 equation and we can write down these equation of the surface in implicit form like a psi of(X, Y, Z) = 0 and which is integral surface of the equation 2 through the curve C here. So that is the idea of this Cauchy method of characteristics so idea is what? That with the help of Cauchy method of characteristic we first find out the characteristic strip.

So it means that a strip which touches the Monge's cone at every point of its curve. So you look at the curve, and at every point if it touches the Monge's cone then we call this it will generate a strip, which is known as strip and now if it touches the Monge's cone we call this strip as characteristic strip and then we say that this characteristic strip is contained an integral element.

Then we call this as integral strips and we look at the totality of these integral strips then we say that we have an integral surface that is the idea of the Cauchy method of characteristics. We will see some example to see that how this method work? So that we are going to continue in next lecture, the example based on this method and so thanks for listening and we will continue in next lecture.