Ordinary and Partial Differential Equations and Applications

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Lecture - 30
Green's Function and its Application- I

Hello friends, welcome to this lecture. In this lecture, we will discuss the method of green functions to solve the non-homogenous boundary value problem with the help of Green's function. First of all, we should discuss the what is Green's function and what are the properties of green function and how to construct the Green's function and then with the help of Green's function how we can solve the-- how we can handle the non-homogenous boundary value problem.

In some case, we can get the solution in close form and in some cases we will get the solution in terms of say some kind of equation which is known as integral equation, so we will consider all these things in this lecture and the lecture afterward. So first let us start the first theorem.

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Green functions

Theorem Let the homogeneous problem

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0 \qquad (35)$$

with

$$l_1[y] = a_0 y(\alpha) + a_1 y'(\alpha) + b_0 y(\beta) + b_1 y'(\beta) = 0$$
$$l_2[y] = c_0 y(\alpha) + c_1 y'(\alpha) + d_0 y(\beta) + d_1 y'(\beta) = 0 \qquad (36)$$

have only the trivial solution. Then Green function exists for (35),(36) and there exists a unique solution $y(x)$ of the nonhomogeneous problem

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = r(x)$$
$$l_1[y] = a_0 y(\alpha) + a_1 y'(\alpha) + b_0 y(\beta) + b_1 y'(\beta) = 0$$
$$l_2[y] = c_0 y(\alpha) + c_1 y'(\alpha) + d_0 y(\beta) + d_1 y'(\beta) = 0$$

Let us that the homogenous problem that is p0 x y double dash+p1 x y dash+p2 x y=0 along with the two boundary condition l1 y that is a naught y alpha+a1 dash alpha+ b naught y beta+b1 y dash beta=0. So here we have a homogenous equation with two set of boundary conditions and here we are assuming that this l1 and l2 are not linearly dependent boundary condition. So it means that this a naught a1, b naught b1 is not a scalar multiple of c naught c1 d naught and d1.

And here we assume that this homogenous problem along with the boundary conditions has a trivial condition then Green function exists for this set of equation and they exists a unique solution for the non-homogenous problem p naught xy double dash+ p1 xy dash+ px xy= rx with this homogenous boundary conditions. So idea is that here we have homogenous equation along with homogenous boundary condition.

And if this has trivial solution then Green function exists and with the help of Green function we can find out a unique solution of the non-homogenous problem this along with the boundary condition and unique solution is given by the following formula that is yp=alpha to beta g of x t rt dt. Here alpha beta is the interval in which we are finding the solution.

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can be represented by

$$y_p = \int_\alpha^\beta G(x,t)r(t)dt,$$

where $G(x,t)$ is a Green function for the problem (35), (36).

So this means that your homogenous problem is given in terms of this alpha n beta here. So here I am denoting the unknown function is y and it is function of x and x is lying between alpha and beta. So here we will try to prove this later on but first we discuss the property of this equation. So here yp=alpha to beta G x,t r t dt. So if we know this function G x,t which is known as Green functions for the problem 35 and 36. So if we know the Green function then we can find out the solution of the non-homogenous problem that is going to be unique.

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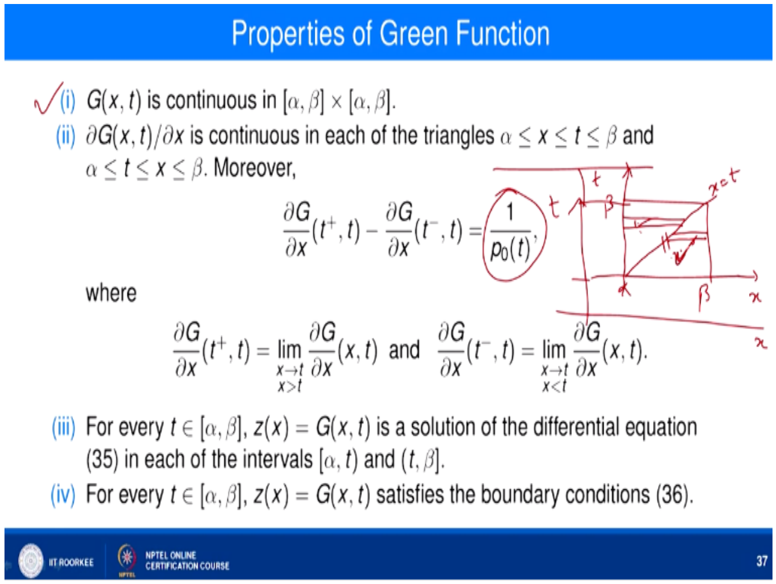

So let us first discuss the properties of Green functions. So here some properties of Green function, some property is that G x, t is continuous in alpha, beta x alpha, beta. So here x is in alpha, beta and t is in alpha, beta. So G x, t is continuous in this rectangle. Now, dou g/dou x is continuous in each of the triangles that is x < or = t and t > or = to x. And alpha <=t <=beta.

So it means that here it is basically continuous in each of the triangles. And it also at the diagonal term, let me writ it here. This is say alpha, beta and this is alpha beta here, so here we have this and then we have the triangles here. So we have two triangles. So in each of the triangles you can say that your dou g/dou x is continuous and at the say diagonal term we have jump this continuity that is dou g/dou x t+, t- dou g/dou x, t/s, t =1 upon p0 A.

So here we discuss the properties of Green function. So first property is that G x,t is continuous in this rectangle alpha, beta to cross alpha beta. So here let us say that here we have this access represent the x and this represent the t here, so x is from alpha, beta and t is also from alpha, beta, so we are writing it here, so it is somewhere here so x-axis and t. So this represent alpha and this represent your beta.

So now what is given here that in this whole activity your G x, t is continuous. And also second property is that dou g/dou x is continuous in each of the triangles, the first triangle is alpha < or =

x < or = t that is this here. So here dou g/dou x is continuous in this triangle and also in this triangle that is alpha < or = t < or = x. So here your this in this region your dou g/dou x is continuous. And when we look at when we look at the diagonal term of this triangle then rectangle then we have this jump discontinuity.

Means in each of the separate region dou g/dou x is continuous but at the diagonal it has a jump discontinuity; jump discontinuity is given by quantity one upon p0 t. So it is given by dou g/dou x from t+, t-dou g/dou x t-t=1 upon p0 t. So here, so it means that if you calculate the dou g/dou x here in this reason – dou g/dou x in this reason then it has the jump discontinuity of measure 1 upon p0 t.

So here dou g/dou x t+t is defined as that limit extending to t x is coming from the positive side of t that is x is coming from this region. So here your x is bigger than t. So dou g/dou x t+, t is the derivative of dou g/dou x, when x is reaching to this diagonal term. And dou g/dou x t-t is when you reaching towards the diagonal term from this region then this represent dou g/dou x t-, t is limit extending to t when x is < t dou g/dou x,t.

So here one is dou g, G x,t is continuous dou g/dou x continuous in this triangle and on the diagonal term it has a jump discontinuity and it is given by 1 upon p0 t and next property is that for every element in alpha, beta your z x G x,t is a solution of the differential equation in each of the interval alpha, t and t, beta. So it means that G x,t is a solution of the differential equation and the region alpha t and t, beta.

So at t it may not be a solution here. For every t and alpha, beta z x= G x,t satisfies the boundary condition. So first of all it is the solution of the differential equation when t is not equal to x and it satisfy the boundary condition which we have provided along with the equation.

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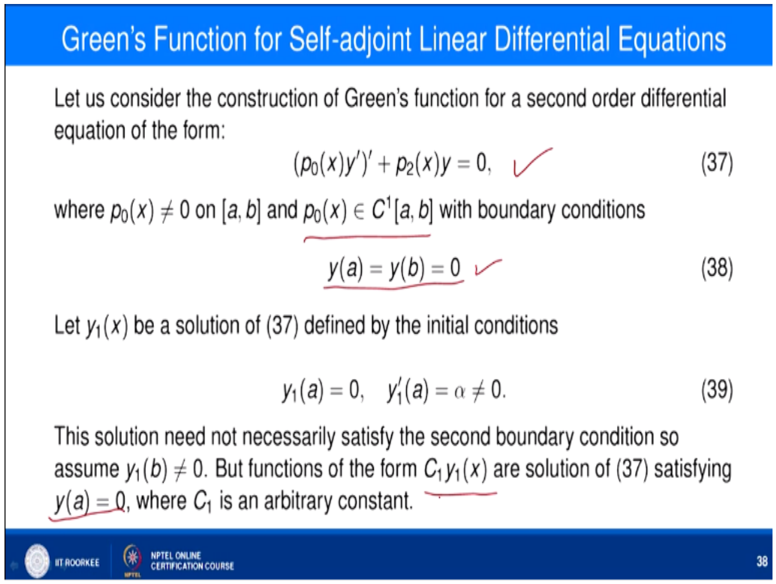

So now let us now construct the Green function for self-adjoint Linear Differential Equation. And so let us consider the construction of Green's function for a second order differential equation of the form p0 x y dash whole dash+p2 x y=0, where p0 x is non-zero on this interval A, B and p0 x is basically a c1 function means continuous differential function with the boundary condition that y of A=y of B=0.

So basically it is a Strum-Liouville problem which we are considering. So it is a self-adjoint problem along with boundary condition. We want to find out the Green function for this particular boundary value problem. So let y1 x be a solution of this equation 37 defined by the initial condition. We already know that if we have some initial condition then this is a initial value problem and it must have a same mission because your coefficients are continuous in this close interval A, B.

So by existence uniqueness of initial value problem this must have a solution, let us denote that solution as y1x and it satisfy the initial condition y1 A=0 and y1 dash A= some alpha where alpha is some non-zero value. So this solution need not necessarily satisfy the second boundary condition that is here y1 B may not be 0 but if y1 B=0 then we have already solved our homogenous problem and we need not to consider for anything more.

So but here it may not always possible to have solution for which y1 B is not = 0. But any multiple of this y1 will have this property that c1 y1 x be a solution of this equation that is Sturm-Liouville equation and it also satisfy the initial condition that is y of A=0. So any multiple of y1 will have this particular property.

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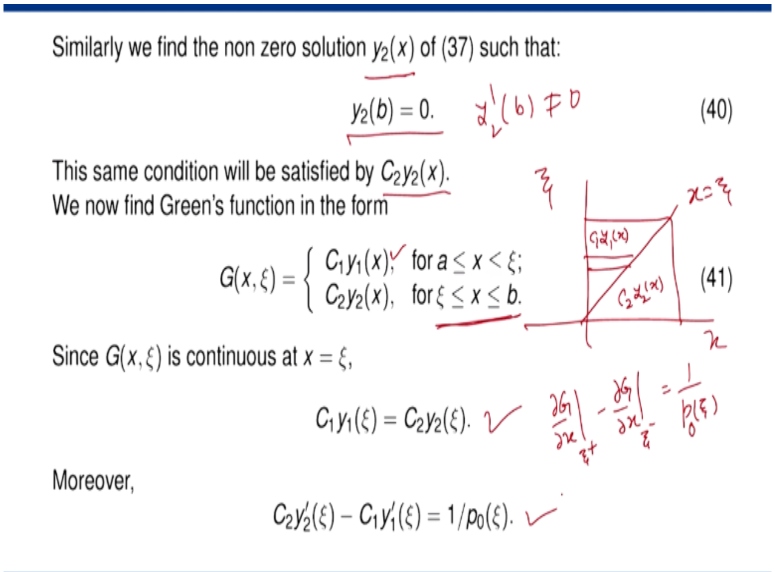

Now look at another solution y2 x which satisfy the equation and the other boundary condition that is y2 B=0. Here we can assume that y2 dash B is some non-zero value so that it is not a trivial solution, because if you take y2 B=0 and y2 dash B=0 then we have only a trivial solution on the homogenous problem. So here we assume that y2 dash B is not = 0. This same condition will be satisfied by any multiple of y2.

So now we have, we propose our Green function in the following form that G x, psi = C1 y1 x, for x defined from a to psi and it is C2y2 x when psi is from x—x is from psi to b. So here we if you look at this you have x and you have psi here and let us write it a to here this is x=psi here. So x from A to psi so from this here you have C1y1 x and here you have C2y2 x and you need to find out this C1 and C2 based on the properties of Green function which we have assumed here.

So first property we have assumed that G x, psi is continuous at x=psi so on this diagonal it is continuous so it means that C1y1 psi = C2y2 psi. So if we come from the left or the right the value must be equal. So if C1y1 psi = C2y2 psi because here we are approaching from this side and approaching from this side at x=psi it must be equal. So C1y1 psi = C2y2 psi. That is from the continuity of G x,t.

Now we also assume that dou g/dou x is continuous in triangle but has a jump discontinuity in the diagonal term that is x=psi here. So here we simply say that dou g/dou x at say psi+ - dou g/dou x at psi - = 1 upon the coefficient term that is p0 here, so this is. So dou g/dou x when psi + so psi + is the region x is bigger than psi so it means that this represent this psi + region. So C2y2 dash at point psi so C2y2 psi – C1y1 dash psi = 1 upon p0 psi here.

So we have these two conditions.

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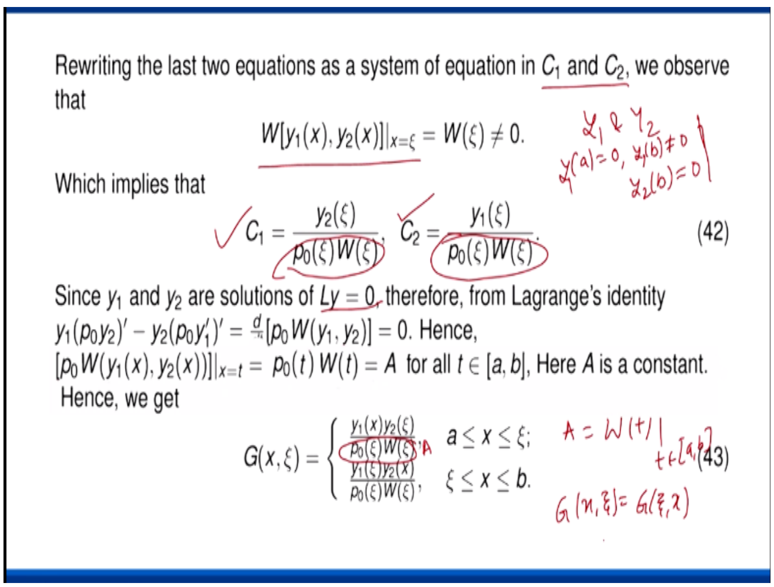

Now when you simplify these two thing to find out C1 and C2 because if you know C1 and C2 then our Green function is known to us. So now we already have rewriting the last two equations as a system of equation in C1 and C2, we observe that this Wronskian of y1 and y2 that is y1 y2 dash – y2 y1 dash is basically Wronskian of y1 and y2 at the point x=psi is non-zero y because here your y1 and y2 are linearly independent solution.

How we say that if you look at we have already assumed that y A=0 but y B is not equal to 0. And we have this is y1 and we have assumed our y2 B is something which is satisfying 0 value at point B. So y1 B is non-zero and but y2 B is 0, so from this we can say that y1 and y2 are linearly independent to each other. So it means that Wronskian is going to be non-zero and let us call this Wronskian is w psi here and with the help of this Wronskian we can find c1 as y2 psi upon p0 psi w psi and c2 as y1 psi upon p0 psi would have psi.

In fact, it is basically what, you can write it here as this equation you can write it here that c1, c2= here 0 and 1 upon p0 psi here and here c1 c2 means you can write it herey1 psi and here it is –y psi, here it is what –y1 dash psi here y2 dash psi. If we write down these two equation in terms of metrics equation e x = b kind of thing. So here from first equation we can write y1 psi – y2 psi c1 c2 = 0. And last equation can be written in this term –y1 dash psi c1+c2 y dash y2 dash psi = 1 upon p0 psi. So we can write down these things in terms of this.

Now, if you look at the determinant of this is basically the Wronskian up 1 and y2 at the point psi and we have assumed that since y1 and y2 are linearly dependent so this Wronskian is going to be non-zero throughout the interval. So with the help of Wroskian we can write down this solution c1 and c2 and here we have to note down one thing that this the denominator this p0 psi w psi is going to be a constant value.

For that we utilize the Lagrange's Identity and we note that y1 and y2 are solution of ly=0. And therefore from Lagrange's Identity y1 p0 y2 dash-y2 p0 y1 dash whole dash = d/dt p0 w y1 y2. Now we already know that y1 y2 is basically solution of this so this is going to be 0 here. So we can write that p0 w y1 y2 is basically a constant value. So p0 w y1 y2 at any point x=t which I denote as wt is going to be a constant value.

So that is what, so if using this constant value, I can write p0 w=A, so for all values of x. So here whether you write p0 psi w psi or you simply write any constant A. So here any constant means the value given at w at t. So w at t is given a some constant. So here either you write this or you simply replace this quantity by constant A where A is basically Wronskian of t at any point t in this interval A to B. Is that okay? So G x, psi is not known to us.

So it satisfies all the properties of Green function. And moreover this also satisfy one very important property that is G x psi = g psi x. So here when you interchange the rule of x and psi then we have symmetric. So G x psi = g psi x, so here it is a symmetric. And this symmetricity is coming from the fact that we are considering the self-adjoint second order linear differential equation. So because of self-adjointness the Green function coming out to be a symmetrical function. If we consider non-self-adjoint or any type of equation, then this symmetricity may not be preserved.

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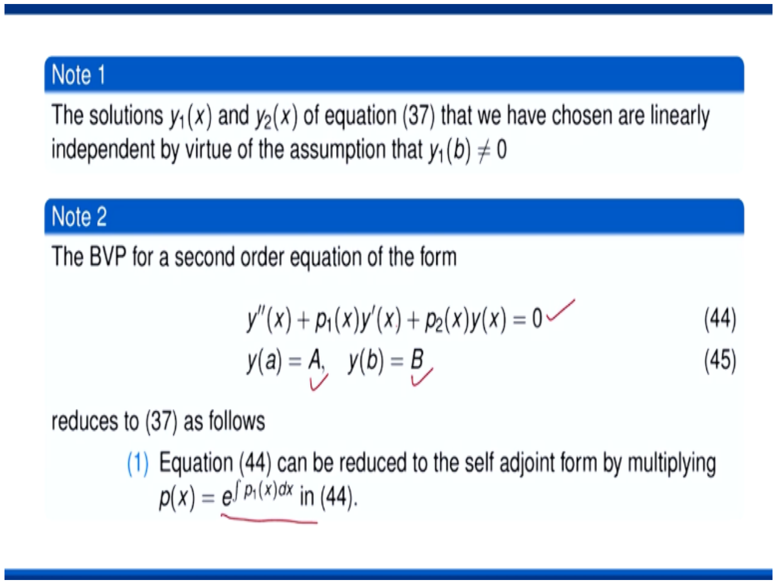

So now here this is one some remark, first remark is that solution y1 and y2 of equation 37 which we have chosen are linearly independent by because we have assumed that y1 b is not equal to 0 and y2 b is =0. So that we have already noted down. Now second remark which we wanted to note it here, that we have given the method of constructing the Green function for self-adjoint equation.

But if it is not self-adjoint then we have already discussed that how a given linear ordinary differential equation can be converted into a self-adjoint linear ordinary differential equation. And again I am repeating that if we have equation y double x+p1x y dash x+p2 xyx=0 here purposefully I am denoting the coefficient of y double dash x1. And along with the boundary condition y A=A and y B=B.

Here also I am assuming that we have a non-homogenous boundary condition. So first thing is that how to retain the self-adjointness of this equation and that we can do by multiplying function e to the integration of p1 x dx. If you multiply this, then this 44 can be converted into self-adjoint form. And now look at the non-homogenous boundary condition.

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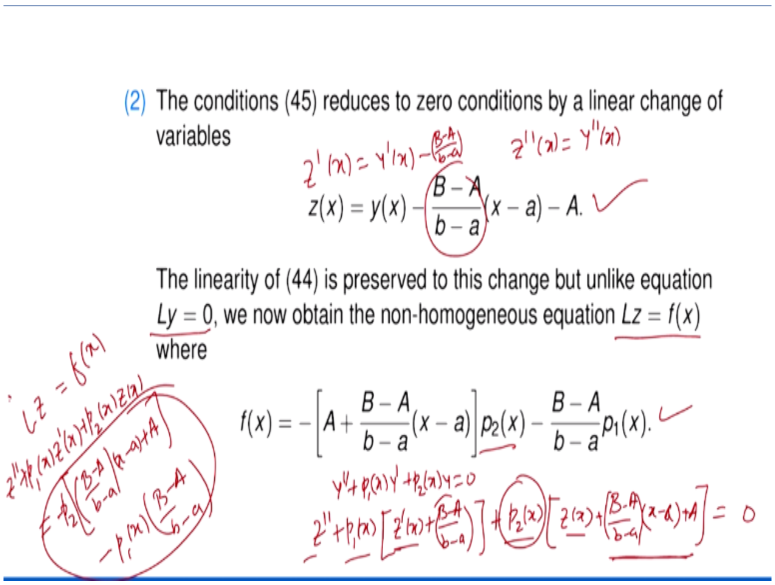

That can be rectified by a linear change of variable that is if we assume zx=yx-b-c upon b-A*x-A-A, then this zx has a, infact what we have done here, we have considered z of x = y of x+ax +b, sorry I have to use some other variable because a and b are already given here. So let us assume this as say alpha and beta. Alpha, beta I am not taking. So here let us say that alpha x+beta. We need to find out alpha, beta such that z of a = 0 and z of b = 0.

So using this you can find out alpha and beta. So if you look at this zero= y of a =a+alpha a+beta b=0, so beta = b + alpha b + beta. So by using this you can find out the value of alpha and beta here. That is what I have written here. That you can find out alpha, beta and it is given by this. So z of x = y of x –b – a upon B –a divided by b-a x-a, so if you use this, then we can change our boundary condition to homogeneous boundary condition.

But if you do this then our problem will also change. So problem initially it is given as ly = 0, is now changed to something like lz=f of x. so how we can look at here. z of x is given by this. So you can write down z of x as y dash x - some quantity let us say this * this is simply 1, so you will get this quantity that is b-A/b-a. And if you look at z double dash x, then we have y double dash x. Is it okay?

Then using these values, if you put it back then what you will get. Let me write it here. So your equation is basically what equation is this, y double dash plus p 1 x y dash+p * x y = 0. Im just using the values of y double dash. So y double dash is z double dash + p1 x. Now y dash is basically this so it is z dash plus p – a upon b-a+p2x+y a is basically this z of x – b- a, x-a=0. I think this will be + here. So here we have this.

And if you simplify what you will get. You will get here z double dash+p 1 x z dash x. So this term we are taking + p 2 x z of x, and what left we are receptive they are, so here we have this quantity, first thing is this that p2x when you shift it there, then we have a negative sign so here p2 within bracket we have B-a/b-a and x-a- so this is –a, right. So we are taking this term along with these two terms. So –p2 * within bracket what you have.

And –p1x we are considering now this term, p1x B-a upon b-a. So that is what we have written. So it means that now our problem is. This is what, this is Lz and we call this quantity as some function of x. So now by changing the non-homogenous boundary condition to homogenous boundary condition, we have used this transformation but this transformation changes the boundary condition to homogenous boundary condition but it also changes the homogenous equation to a non-homogenous equation.

So here what we have done is that we have considered the homogeneous equation with non-homogenous boundary condition. When we do this change, then we have non-homogeneous equation with homogeneous boundary condition. That is what we have seen here. So now we should know how to solve this non-homogenous equation with homogeneous boundary condition.

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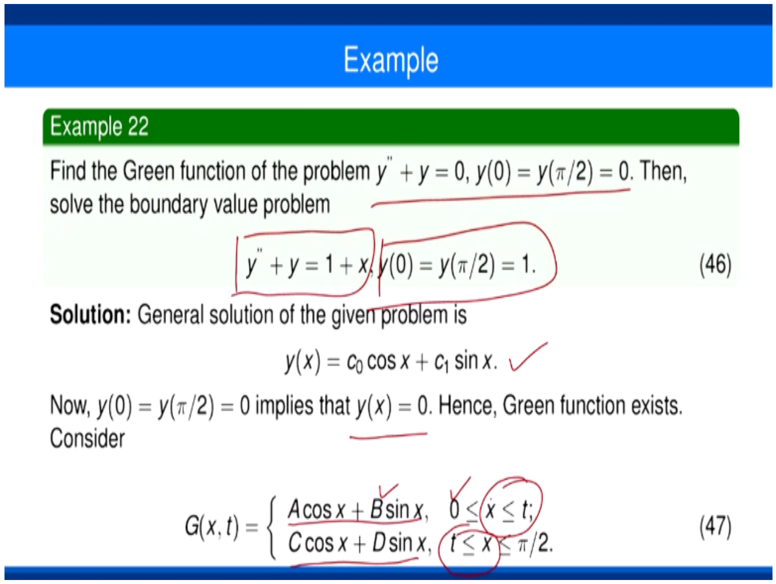

So now let us consider one simple example based on whatever we have just presented. So find the Green function of the problem y double dash+y = 0 at the boundary condition y = y pi by 2 =0 and then we want to solve the non-homogenous problem that is y double dash+y = 1+x and non-homogenous boundary condition as well that is y 0=1 and y pi by 2=1. So we start with homogenous problem and with the help of Green function we try to solve the non-homogenous equation along with non-homogenous boundary condition here.

So first let us first find out the Green function. So to find out the Green function we have to look at the homogenous equation along with homogenous boundary condition and if you solve that, we have yx=c naught cos of x + c1 sin of x. and if you look at these boundary condition, then y0=0=ypi by 2=0, we have only a trivial solution.

So it means that homogenous problem with homogenous boundary condition has only trivial solution so by the first theorem which we discussed in the beginning of the lecture that in this case we have Green function exist and with the help of Green function we can solve the non-homogenous problem, okay. So, first we have Green function exists and now we want to find out the Green function.

Now please note down here that the homogenous equation is given in self-adjoint form, so we can easily find out the G x, t with the procedure we have already discussed so G x, t =a cos x+b sin x where x is < or = to t, look at this term only and c cos x+d sin t when t is < or = x. So we have two reason, x < or = t and t > or = x. So here we need to fix your a, b, c, d depending on the properties of this.

So here first you have that this a cos x+b sin x must satisfy the boundary condition given at the point x = 0. And c cos x + d sin x satisfy the boundary condition given at the point x = pi/2. So here we already know that for each t from 0 to pi by 2, yx= G x, t is a solution of the differential equation and satisfies the boundary condition so y zero = 0 implies that a = 0 and y pi by 2 =0 implies that y = 0.

So here if you look at this, in this domain x = 0 is coming so the boundary so G x,t will satisfy the boundary condition at x=0 by this function and G x, t will satisfy the other boundary condition defined at pi/2 by this function.

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Since for each $t \in [0, \pi/2]$, $y(x) = G(x, t)$ is a solution of the differential equation and satisfies the boundary conditions. So, $y(0) = 0$ implies that $A = 0$ and $y(\pi/2) = 0$ implies that $D = 0$. Thus (47) reduces to

$$G(x,t) = \begin{cases} B\sin x, & 0 \leq x \leq t; \\ C\cos x, & t \leq x \leq \pi/2. \end{cases} \quad (48)$$

Since Green function is continuous on $[0, \pi/2]$ so, we have

$$B\sin t = C\cos t \quad (49)$$

for each $t \in [0, \pi/2]$ and by using $\frac{\partial}{\partial x} G(x,t)\Big|_{x=t^-}^{x=t^+} = \frac{1}{p(t)}$, we get

$$C\sin t + B\cos t = -1 \quad (50)$$

Solving (49) and (50), we get $B = -\cos t$ and $C = -\sin t$. So, Green function becomes

$$G(x,t) = \begin{cases} -\cos t \sin x, & 0 \leq x \leq t; \\ -\sin t \cos x, & t \leq x \leq \pi/2. \end{cases}$$

So it means that you can find out that y of 0 =0 implies that a = 0 because this is the solution which contains the reason x = 0, so here we will get a = 0. And similarly y pi/2 = 0 means your d= 0. So our G x, t is now reduced to this that is G x, t = b sin x when x is lying between 0 to t and c cos x when lying between t to pi/2. So now we have to find out this b and c. So what we have utilized so far. We have utilized so far the G x, t is a solution of the differential equation.

So with the help of this we have constructed our G x, t and that G x, t satisfies the boundary condition. So with the help of boundary condition we have fixed our a and one of the constant. Now let us fix the other remaining constants b and c. and for that we have two conditions that G x, t is continuous and G x, t has a jump discontinuity. So let us say that since Green function is continuous on 0 to pi/2.

So let us check the continuity at x = t, so it means that if we come from this reason or this reason, we will get the same value. So b sin t has to be equal to c cos t. And then the other condition that it has the jump discontinuity so it means that dou g/dou x from t+ to t- =1 upon p t. So if you look at dou g/dou x from t+, so t+ is this. So here you can see that -c sin of t, we are looking at x =t, -b cos of t = 1 upon p t, p t is basically 1 here, so it is 1.

So we can simplify this and write c sin t+b cos t = -1. This minus we are shifting here. So we are with equation 49 and equation 50. And with the help of this we can find out the constant b and c. So when we find out the constant b and c it is coming out to minus cos t and minus sin t. And with the help of b, c now we are able to write down the Green function that is G x, t is -cos t sin x from x from 0 to t and -sin t cos x when x is from t to pi/ 2.

And you can see that your symmetricity is preserved here. So here it means that G x, t can be written as replaced by G x, t. Anyway. So now let us fix the boundary condition as well. So we have not fixed the boundary condition, so for fixing the boundary condition—

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Now, consider $z(x) = y(x) - 1$; then (46) becomes

$$z'' + z = x, \quad z(0) = z(\pi/2) = 0.$$

Now, solution of the above differential equation is

$$z(x) = \int_0^{\pi/2} G(x,t) f(t) dt = \int_0^x G(x,t)\, t\, dt + \int_x^{\pi/2} G(x,t)\, t\, dt$$

$$= -\cos x \int_0^x t \sin t dt - \sin x \int_x^{\pi/2} t \cos t dt.$$

On solving this, we get $z(x) = x - \frac{\pi}{2} \sin x$. Since, $z(x) = y(x) - 1$, so we have $y(x) = 1 + x - \frac{\pi}{2} \sin x$.

$$y'' + y = 1 + x, \quad y(0) = 1, \quad y(\pi/2) = 1$$

So let us assume that z x = y x -1. In fact, you can assume z x as yx+alpha x+beta and you can simplify your alpha and beta and it is coming out that alpha is coming out to be zero and beta is = minus 1. So zx = yx -1. Now with the help of this reduce your non-homogeneous problem that is y double dash + y = 1+x and these boundary conditions. So these boundry conditions are already fixed.

Now z of zero = 0 and z of pi by 2 = 0 and if you fix this then you have z double dash + z =x and with this now if you look at the homogeneous boundary condition and a non-homogeneous equation. And we already know that the solution is given by z of x is from 0 to pi by 2. G x, t * ft dt, where ft represents the right hand side of this equation and here your ft is basically t. so z of x = 0 to pi by 2. G x, t *t dt.

Now here, depending on the domain you have to use different values of G x, t. So you can simply write it, 0 to t G x, t t dt + t 2 pi by 2 G x, t * t dt. So we have to look at in this reason when here, sorry, I have to write 0 to x. so let me write it here, 0 to x. So here o to x and x to pi/2. So here you look at that t is < or = x. So in this way t can go up to x. t< or =x means x is > t. So if you look at x > t means – sin t cos x. so here we use minus sin t cos x.

So here we are using this value. So G x, t is minus sin t cos x, here we are using it. And here in this way, your t is bigger than x means -cos t sin x. so this will be used here. So here we use this

part, minus cos t sin x. so - cos t will go outside and sin x is taken outside because integral is with respect to t. when you simplify you will get your solution zx = x – pi by 2 sin x.
I am not doing this integration because it is simple integration. You can do it.

So once we have z of x then since we already know zx = yx-1, so we can have our solution of the problem that is yx = 1 + x – pi by 2 sin x and this is the solution of the problem by y double dash + y = 1+x with condition y zero = 1 and y pi by 2 = 1. So what we have seen in this example is how to construct a Green function and with the help of green function how to solve a non-homogeneous equation with non-homogeneous boundary value problem.

So, here I will conclude our lecture and in next lecture we will discuss some more result on green function and discuss some more example based on this. So thank you for listening us. Thank you.