Ordinary and Partial Differential Equations and Applications
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Lecture - 29
Sturm-Liouville Problem and its Applications

Hello friends, welcome to this lecture. In this lecture we continue our study of regular Sturm-Liouville boundary value problem. And in our previous lecture we have discussed the existence of eigenvalue and Eigen functions and we have discussed some of the properties of Eigen function which is one very important property of Eigen function that is orthogonality with respect to weight function.

So there we have assumed two type of boundary conditions. So let me write it here.

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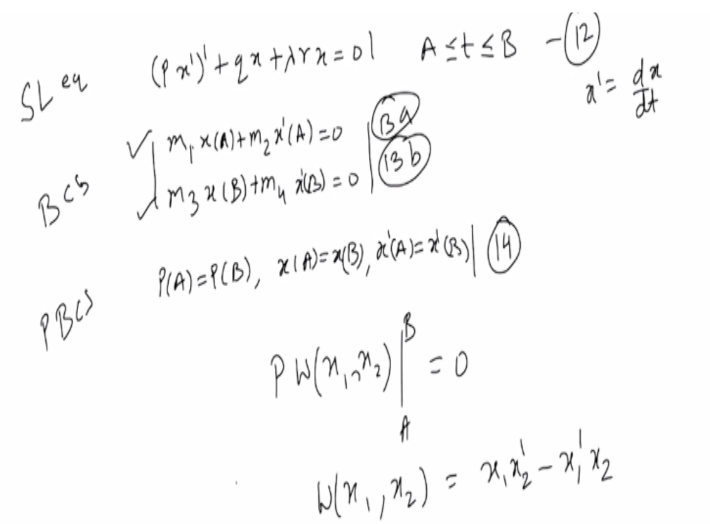

We have considered this Sturm-Liouville that is p x dash dash+q x+ lambda r x= 0 here. So this equation is known as Sturm-Liouville equation and here this x dash means dx/dt. So t is ranging between A to B here and we have discussed two type of boundary condition that is one is separated boundary condition that is m1 x A+m2 x dash A=0 and this is defined at the point x=A -- t=A and then other boundary condition that is m3 x B+m4 x dash B =0.

So this is defined at A and this defined at B. So this is one type of boundary condition. And another type of boundary condition is periodic boundary conditions that is p A=p B, p is coefficient of x double dash so p A = p B; x A = x B and x dash A=x dash B. So this is periodic boundary condition, and with the help of this we have also shown that p w say x1, x2 given at A and B is coming out to be 0, that is the content of the last two results.

So here w x1, x2 is Wronskian of x1 and x2 where x1 and x2 are two independent solutions of SL equations. So here it is x1 x2 dash – x1 dash x2. So here we have shown that it satisfies this particular property. Now we are going we are discussing one more important property of Eigen functions which Sturm-Liouville problem.

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Theorem 20

Let the hypothesis of Theorem 15 holds. Suppose that $r(t)$ is positive on [A, B] or $r(t)$ is negative on [A, B]. Then all the eigenvalues of BVP (12), (13a), (13b) or (12), (14) are real.

Proof: Let $\lambda = a + ib$ be an eigenvalue and let $x(t) = m(t) + in(t)$ be a corresponding eigenfunction of the given BVP. It is clear that $a, b, m(t)$ and $n(t)$ are real. So, using the values of λ and corresponding eigenfunction, we have

$$(pm' + ipn')' + q(m + in) + (a + ib)r(m + in) = 0.$$

Equating the real and imaginary parts, we have the following:

$$(pm')' + (q + ar)m - brn = 0$$
$$(pn')' + (q + ar)n + brm = 0.$$

So that property is that let the hypothesis of theorem 15 holds, this theorem 15 is the existence of eigenvalues and Eigen functions. And also suppose that r t is positive on the entire interval A B or r t is negative or on A, B so it means that r t is non-zero in this particular interval whether it is positive or negative it will keep one sin on it. Then in this case then all the eigenvalues of boundary value problem 12 along with 13a 13b boundary condition or 12 with 14 r real.

Here 12 is the SL equation and 13a 13b are separated boundary condition and forth 14a is the periodic boundary condition. So it says that if r is positive or r is negative then-- and it satisfies all the condition of the theorem 15 that is that A and B are finite and coefficient p is positive, it means that it is also non-zero. Then the eigenvalues of this are r all real. So eigenvalues of 12 or 13a or 13a 13b or 12 with 14 r are all real, that is what we wanted to show in this particular theorem.

So let us assume that that lambda is a complex we want to proof that it is real. So let us first assume that it is a complex and then we try to show that it is not the case. So let lambda = A+iB so where A and B are both real constants, be an eigenvalue and corresponding Eigen functions are given at x t which is again complex we are assuming mt+int; be a corresponding Eigen function of the given boundary value problem.

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Eliminating $(q + ar)$ in the above two equalities, we get

$$-b(m^2 + n^2)r = m(pn')' - n(pm')' = \frac{d}{dt}[(pn')m - (pm')n].$$

On integrating both the sides with respect to t, we get

$$-b\int_A^B (m^2(s) + n^2(s))r(s)ds = [(pn')m - (pm')n]_A^B. \qquad (21)$$

Since m and n satisfy either of the boundary conditions (13a), (13b) or (14), we have, as shown earlier,

$$\Big[p(n'm - m'n)\Big]_A^B = \Big[pW(m,n)\Big]_A^B = 0. \qquad (22)$$

Also, $\int_A^B [m^2(s) + n^2(s)]r(s)ds \neq 0$ by the assumptions. Hence, from (21) and (22), it is clear that $b = 0$, which means that λ is real. This completes the proof.

So whether it is 12a 13a, so it is either 12a 13a 13b or 12 of 14 so here boundary value problem is either this one, so any of this. So it is already clear that by assumption A and B are constant and mt and nt are real value function. So using the values of lambda and corresponding Eigen function we now put it the values of lambda and the corresponding Eigen function here in p x dash dash+q x+ lambda r x=0.

So here we are putting the value of x and lambda. So it is p so this is x dash is lambda+i n. So here x dash is basically m dash+i n dash. So we are writing here +q x+lambda r n x = 0. So once we have put it here then we simplify this in terms of real and imaginary part. So when you equate the real and imaginary part here I can write 0+ this 0 is basically 0+i 0. So equating the real and imaginary part we have the following 2 equations that is pm dash dash+q+ar m – brn=0 and p n dash whole dash+q+ar n+brm =0. That is not very difficult to look at here.

So if you at this is what here it is pm dash and then okay. So here you have to do little bit of calculation. So once if you do the calculation then you can get that it satisfies these two equations here. Okay. So now once we have these two equation then we want to ultimately our aim is to prove that this B is basically 0. If B is 0 then lambda is real. So we want to prove that this B is 0. So what we try to do here, let us simplify this.

And for simplifying this you just multiply the first question by n and second equation by m and try to simple try to make it simple. So when you eliminate q+ar in the above two equalities means you just multiply n here, you multiply m here then you can eliminate and you subtract these two then you can eliminate these q+ar. So when you do this thing then what you will have. Let me write it here. That is pm dash whole dash n + q + ar mn – brn square = 0.

Similarly, second equation is pm dash whole dash m+q+ar multiplied by mn+brm square=0. When you subtract these two - - - so this will cancel out, so what do you will get it is a following thing that –b*m square+ n square r that is the last term if you look at here, if you look at this thing then we have brn square+m square that is what it is written her -b m square+n square r and the opposite side we have this thing that m p n dash whole dash –n pm dash whole dash.

So that is what I have written here. So now we have already seen that this can be simplified further and this can be written as d/dt of pn dash m–pm dash n. So this we have because if you simplify this you will see that you will get this. So here we have this equation –br m square+n square = d/dt pn dash m – pm dash n.

Now what we do we just integrate with respect to t from A to B, so what do you will get –b b0 is just a constant so we can take it out so –b integration from A to B m square s + n square s and r s ds. And here since it is derivative respect to t and if you integrate you will get pn dash m – pm dash n from B to A, right. Now we already know that this m and n satisfy either the boundary condition 13a 13b or 14.

So we already know this m and n both will satisfy the boundary condition and we have already seen that if they satisfy the boundary condition then this quantity which is p w mn from A to B = 0, that is the content of the previous two theorems. So it means that if it is 0 then this part is simply 0. So it means that –b A to B m square s+ n square s rs ds = 0. Now if you look at the integrant, integrant is what, this is all positive and r is either positive or negative but it cannot be 0.

So it means that A to B m square s + n square s into rs ds is non-zero because we have assumed so that r is never 0 either it is all the time positive or negative. So it means that this quantity has to be either positive or negative depending on the sin of rs. So it means that –b into something which is non-zero = 0 implies that we have only option that is B=0. When B=0 it means that lambda is real.

Why because lambda we have assumed that it is A+iB. Now if b is 0 then we have only option that is lambda=A. So what we have proved here that your eigenvalue is purely real. And that completely-- it means that what we have proved here that if the hypothesis of theorem 15 holds means we are considering the regular Sturm-Liouville boundary value problem with the boundary condition either separated boundary condition or periodic boundary condition.

And if r t is either positive or negative then all the eigenvalues of this Sturm-Liouville boundary value problem are all real that is the content of this theorem. And let us move ahead.

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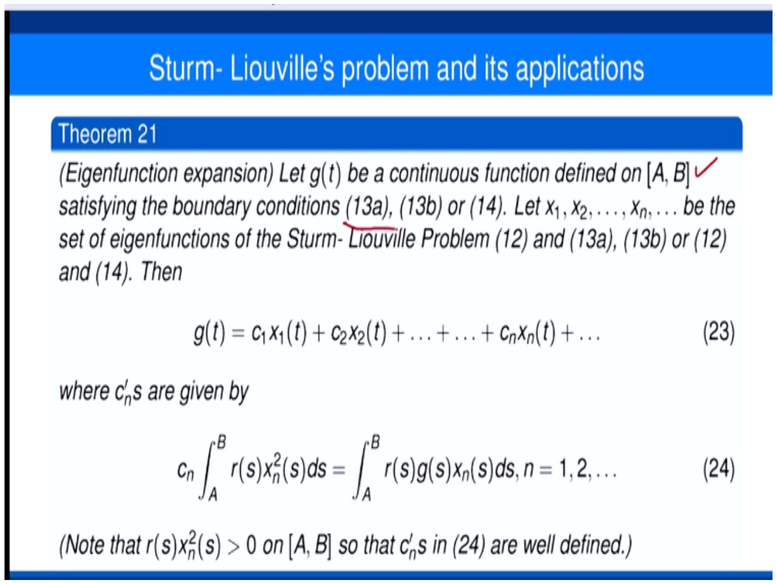

Let us consider one very important application of the Sturm-Liouville problem and that application is the expansion of a class of function in terms of Eigen functions. So that is the content of this theorem 21. So let gt be a continuous function defined on close interval A B and satisfying the boundary condition either 13a 13b or 14 and let us assume that we have certain Eigen function let us say x1, x2, xn be the set of Eigen function of which Sturm-Liouville problem 12 and 13a 13b or 12 along with 14.

Then you can expand your g t in terms of these Eigen functions that is c1 x1 t+c2x2 t and so on cn xn t and so on. So here all these exercise are Eigen functions of the Strum-Liouveille problem. And here you can calculate the coefficient say ci by the following formula that is cn from A to B rs xn square s ds=A to B rs gs xn s ds. Now if you look we already know rs we know gs we know xn s so we can say that this quantity is also non-zero because rs we are assuming that it is positive and this xn square is again positive.

So it means that cn you can easily find out by dividing this strum. So it means that you can get the values of cn and once you have all these cn you can write g t in terms of these xi, so that is what is the content of this theorem and this is proof is quite easy, let me write it here.

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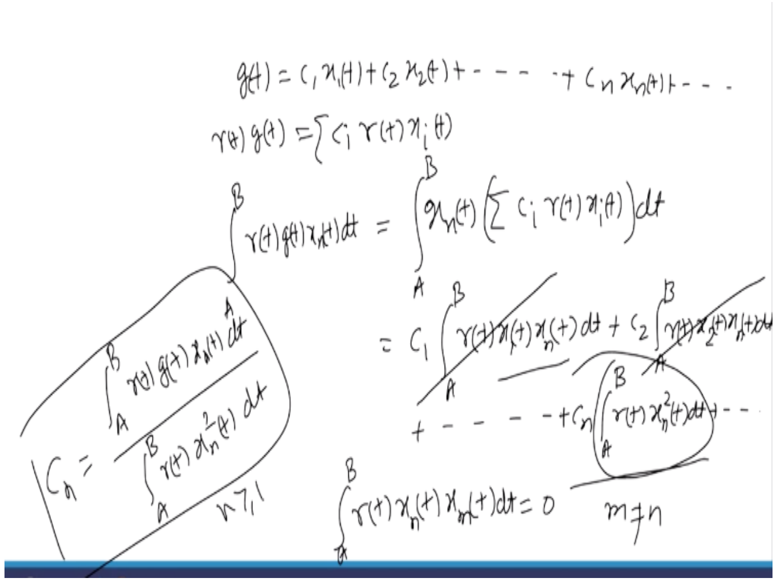

So a proof is this that you have g of t = c1x1 t+c2x2 t and so on so cn xn t and so on. So what we try to do here we just multiply both side by rt so rt gt=c1 let me write it here in a summation form ci rt xi t, right. Then what we try to do here, we just multiply by say xn say. So let us multiply both side by xn and integrate with respect to t so we will get A to B rt gt xn t dt and here what you will have.

Here you will have let me expand it, it is xn t * summation it is A to B summation ci rt xi A dt, right. Or if you want to write it in a simpler form this is what c1 your integration of A to B rt x1 t xn t dt+c2 A to B rt x2 t xn t dt and so on. Here it is cn A to B rt xn square t dt and so on. Now we have already discussed that these Eigen functions are orthogonal to each other with respect to weight function rt.

So it means that if n is not equal to 1 then x1 and xn r two distinct Eigen function. So this has to be say 0. So by orthogonal property this has to be 0 similarly this has to be 0 similarly this has to be 0 the only term which is left non-zero is this term where we have m=n, so it means that here using this property that A to B rt xn t xm t dt = 0 when m is not equal to n. So using this orthogonal property we can see that A to B rt gt xn t dt = cn A to B rt xn squared d dt.

And we already know that this term is going to be non-zero. So by dividing this term we have cn=A to B rt gt xn t dt/A to B rt xn squared t dt. So once we have cn and this you can write for all n >= 1 so it means that you can expand your gt in terms of xn t. I am using the following formula to find out the coefficient here. So that is very, very important use of Sturm-Liouville boundary value problem and we have seen some of the applications in expanding, say given function in terms of Legendre polynomial or Bessel polynomial and so on.

In Legendre polynomial we do not have any weight function and we do not require any say boundary conditions, so we simply say that we have already seen this kind of application in terms of Legendre polynomial and Bessel functions. So now let us move ahead and discuss one another important boundary value problem. So here this boundary value problem is that you consider that--

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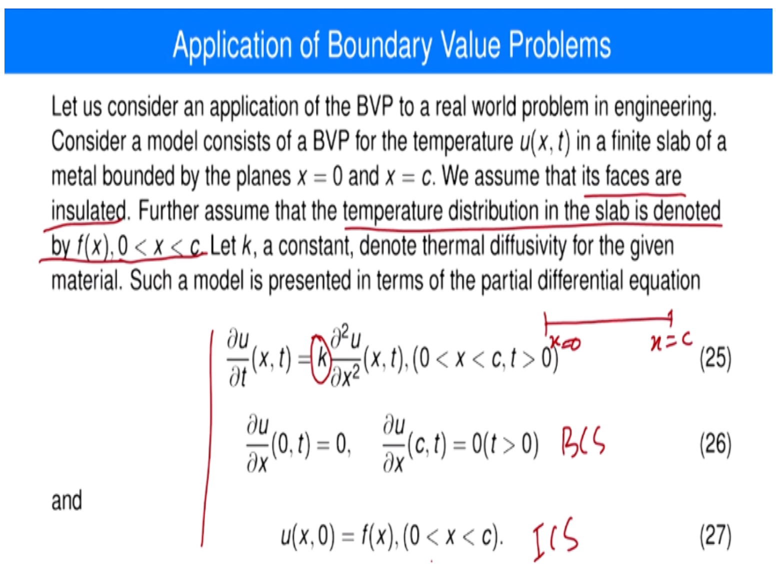

You consider a finite rod, right. So here finite rod is that it maybe consider a metal rod so one end is x=0 another end is x=c and we want to look at the temperature distribution on this finite length rod. So we assume that its faces are insulated. And further assume that the temperature distribution in this slab is denoted by f of x from x from 0 to c. So this is temperature distribution at the initial is given by f of x between x to c.

Let k is the constant denote thermal diffusivity for the given material. So this k is a constant depending on the material of the metal basically metal of the-- that rod. So now if we use model to write down the mathematical formula for this temperature distribution then we have the following partial differential equation that is dou u/dou t xt= k dou u/ dou x square x, t and x is lying between 0 to c and t is > 0. And on the boundary since it is insulated and so dou u/dou x at x=0 is 0 and double u/double x at c, t =0 for all t >0.

So these are the boundary conditions and this is your initial conditions, initial condition is that the initial temperature at time t=0 your initial temperature is ux0=f of x that is what we have given here. So now we wanted to solve this using this Sturm-Liouville boundary value problem. So far we have not discussed this how to obtain this problem and we have not discussed any property of this equation this partial differential equation which is commonly known as heat conduction problem or heat equation.

So let us assume that we know how to define partial derivative. So let us if you not gone through the partial differential equation, so let us assume that we have a partial differential equation some equation related with partial derivative and let us try to solve this.

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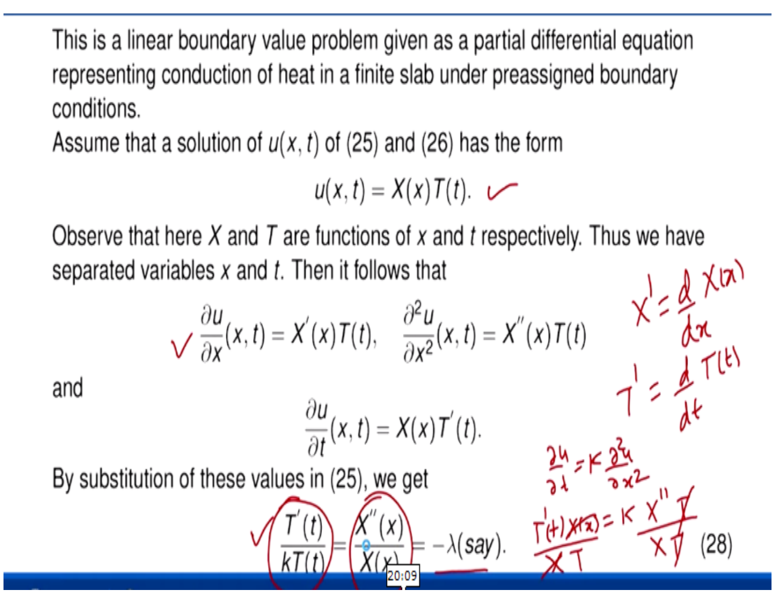

So to solve this we simply say that observe that this is a linear boundary value problem given as a partial differential equation representing conduction of heat in a finite slab under preassigned boundary conditions. So now let us assume that the solution of this problem which is solution means ux t we are searching for u x t represent the temperature distribution at the point x at the time t, right.

And we want to show that let us assume that the solution of ux t has the following form that is x of x * t of t. So it means that here solution can be separated function of x and t. So let us assume that ux t = x of x and t of t. We will discuss later on that why we assume this form of the solution. This is known as the separation variable form of the solution. So here we assume that here x is the function of small x and t is function of temperature here.

So let us calculate dou u/ by dou x, so when you calculate this then it is total derivative of x and T t and dou 2 u/ dou x c square is x double dash T t. Here this x dash is basically two t derivative d/dx of x and t dash is basically d/dt of T t. So here this represents the total derivative and partial derivative is given by this dou u/dou x and dou 2u/dou x . And similarly we can calculate the dou u/dou t and it is x is T dash t.

So if you look at our equation is what equation is basically dou u/dou t = k dou 2u/ dou x square. So when you use this, this is what T dash t x x= k times this x double dash and td. So divide by your x*t. So let us divide by xt here also you will write xt. So when you divide by this what you will left this will cancel out and here this x will cancel out. So we have T dash t/k t you take k here so T dash t/kT t = x double dash upon x.

Now here if you look at this is the function of t alone and this is the function of x alone and this can be equal only when the ratio is a constant value. So let us say that the constant value is some – lambda. Now see why it is – lambda you can use +lambda no problem, it is just a constant. So using that since this ratio this is a function of t this is a function of x this can be equal only when each ratio has to be a constant. So let us denote that constant as –lambda.

So we have this equation L we have also boundary condition, boundary condition means what that dou u dou x = t = 0, so if you simplify this is, this is what x dash 0 T t = 0 and here we have x dash c T t = 0. Now since we are assuming that we are looking for non-trivial solution so it means that none of this x of x is T t are trivial solution means these are not 0 solution.

So it means that if T t is non-zero so this condition satisfies means that we have x dash 0 =0 and similarly, since T t is not eigenvalue 0 solution so we can say that x dash c=0. So it means that not only we have two set of conditions that is T dash t upon k T t equal x double dash upon x=-lambda also we have boundary condition, that is--

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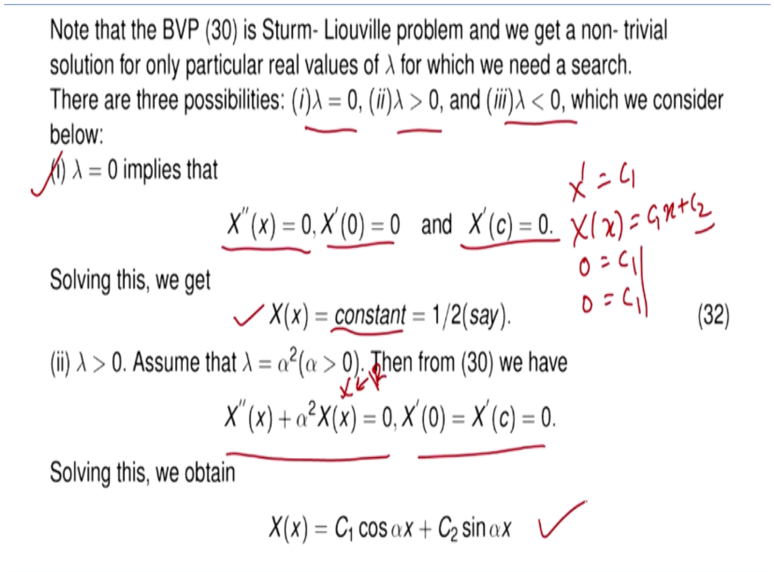

X dash 0=x dash c=0. And here we have assumed that t of t is not equal to 0. So it means that now our partial differential equation is now reduced to two set of boundary value problem in terms of ordinary differential equation. So this is one set of boundary value problem that is x double dash x+lambda x of x=0 and x dash 0=x dash c=0, so this is a boundary value problem in terms of x of x.

And we have another differential equation T dash t+lambda k T t=0 here. So here we do not have any boundary condition in terms of T t, but we have a one condition left that is the condition initial condition of u of x 0 = f of x that we can utilize little bit later. So first we try to solve this equation number 13, this is a boundary value problem and if you look at this is a Sturm-Liouville boundary value problem.

So here we can apply our discussion of Sturm-Liouville boundary value problem to solve this boundary value problem. So let us solve this and here since lambda is a parameter and lambda can take distinct value so lambda=0; lambda > 0 or lambda < 0. So we look at these conditions separately. So let us consider condition that lambda=0. So when lambda 0 our equation reduce to that x double dash = 0 and x dash = 0 and x dash c are 0. These are the boundary conditions. So when you solve this x double dash x=00 your condition is x of x = you write it c1, c1 x+ c2.

You have to find out this c1 and c2 using this condition. So when you see that you will say that x dash 0 is 0 means this is 0=c1 and means because x dash is basically c1 basically. So x dash 0 means c1=0 and x dash c=0 means you have what 0=c1 so that is what we get. So it means that we have only c1=0 so means that x of x has to be c2 so it means that solving this we get x of x= just a constant value.

So let us for simplification let us assume this as 1/ 2 and see why this we are assuming 1/2, in fact you can use any constant alpha, beta, gamma anything you want to use it. But for the simplification we are assuming it is 1/2. Now let us-- so it means that this lambda=0 is an Eigen function-- is an eigenvalue and the corresponding Eigen function is a constant value. Let us say that it is 1/2. So the lambda=0 is an eigenvalue.

So now let us consider lambda > 0. So assume that lambda=alpha square because lambda is positive so let us assume that alpha square and alpha is belonging to real number. So when you put this value lambda alpha square then we have equation as x double dash + alpha square x and boundary condition x dash 0 = x dash c=0. So when you solve this equation x double dash+alpha square x we have relation c1 cos alpha x+c2 sin alpha x.

And now let us find out this c1 and c2 using the boundary condition.

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and

$$X'(x) = -C_1\alpha \sin \alpha x + \alpha C_2 \cos \alpha x$$

Now $X'(0) = \alpha C_2 = 0$ implies that $C_2 = 0$ (since $\alpha \neq 0$) and $X'(c) = -C_1\alpha \sin \alpha c = 0$.

Here $C_1 \neq 0$ since we are searching $u(x,t) \not\equiv 0$.

Therefore the only choice left is $\alpha = \frac{n\pi}{c}$, $n = 0, 1, 2, \ldots$ Hence, when $C_1 = 1$, we write

$$X(x) = \cos \frac{n\pi x}{c}, n = 0, 1, 2, \ldots \tag{33}$$

$\sin \alpha c = \sin n\pi$

$\lambda = \alpha^2 = \frac{n^2\pi^2}{c^2}, n = 0, 1, 2 \ldots$

So let us find out see derivatives of this x dash x =-c1 alpha sin alpha x+alpha c2 cos of alpha x. And we have x dash 0=0 so this implies the your alpha c2=0. Now alpha cannot be 0 so c2 has to be 0 because alpha is positive real number. Alpha is any real number and if we take alpha=0 then lambda has to be 0 and which in this case we have already considered. So here we assume that alpha is some value which is not 0 basically.

So here c2 has to be 0 and x dash c=-c1 alpha sin alpha c=0. Now since c2 is already 0 so if we take c1=0 then we have only trivial solution which we really do not want. So what we try to do here we try to show that this alpha sin alpha c has to be 0. So we want see non-zero since we are searching for non-trivial solution of the problem, so therefore only choice left is that alpha has to be n pi/c.

So here we assume that sin of alpha c=sin of n pi. So it means that alpha c=n pi/n is running from 0, 1, 2 in fact n is coming from integer values. So here let us assume that c1 can be any constant so let us assume that it is one for the timing and Eigen functions we have x of x = cos of n pi x /c where n is from 0,1,2 and your lambda is basically n is alpha square that is n square pi square/c square. So it means that here we have another set of and n is from 0, 1, 2 any number here any natural number, in fact any integer number.

So here eigenvalues are n square pi square/c square and the corresponding Eigen function x of x = cos of n pi x/c.

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(iii) $\lambda < 0$. Assume that $\lambda = -\alpha^2$, $\alpha > 0$. Then the BVP (30) takes the form

$$X''(x) - \alpha^2 X(x) = 0, X'(0) = X'(c) = 0.$$

Then we have

$$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

and

$$X'(x) = c_1 \alpha e^{\alpha x} - c_2 \alpha e^{-\alpha x}.$$

Here $X'(0) = 0$ implies $c_1 = c_2$ and hence $X(x) = c_1(e^{\alpha x} + e^{-\alpha x}) = 2c_1 \cosh \alpha x$.
Further $X'(c) = c_1 \alpha \sinh \alpha c = 0$ implies that $c_1 = 0$.
These two facts together mean that $X(x) = 0$ or $u(x,t) = 0$. The above conclusions provide us a non- trivial solution $u(x,t)$ only when $\lambda = \lambda_0 = 0$ and $\lambda = \lambda_n = \alpha^2 = \left(\frac{n\pi}{c}\right)^2$, $n = 1, 2, \ldots$ These are the eigenvalues and the respective solutions of the BVP (30) namely $X(x) = X_0(x) = 1/2$ and $X(x) = X_n(x) = \cos\frac{n\pi x}{c}$, $n = 1, 2, \ldots$ are the eigenfunctions.

So now let us consider one more term that one more possibility that is lambda < 0 and here assume that lambda = - alpha square alpha is positive. Then the boundary value takes the following form x double dash – alpha square x=0 and x dash 0=x dash c=0. So when you simplify we have this relation x of x=c1 alpha x+ c e to the power of alpha x.

I leave it to you that when you put this condition x dash 0 = x dash c=0 then we have only this thing that x of x has to be 0, so when you use these condition then we have only a trivial solution. So we do not want trivial solution so it means that this lambda < 0 is not eigenvalues. So it means that we get only two set of eigenvalues one is corresponding to 0, so lambda not equal to 0 is an eigenvalue and corresponding Eigen function is basically some constant.

Let us assume that it is 1/2. And another set of eigenvalues we have obtained for the case when lambda is positive that in that case we assume that lambda=lambda n=alpha square which is given as n pi/c. So here we write lambda n as n pi/c whole square where n is 1, 2, 3 because if we put n=0 then this reduce to this lambdA0. So here we assume that n is non-zero integer value. So these are the eigenvalue and the respective solutions of boundary value 30 is corresponding to lambda not we have this solution so it is Eigen function corresponding to lambda not equal to 0.

And corresponding to lambda n we have Eigen functions given by this. So when we basically Eigen function means non-trivial solution of the boundary value problem. So when you summarize this we can get all the possibilities of x of x.

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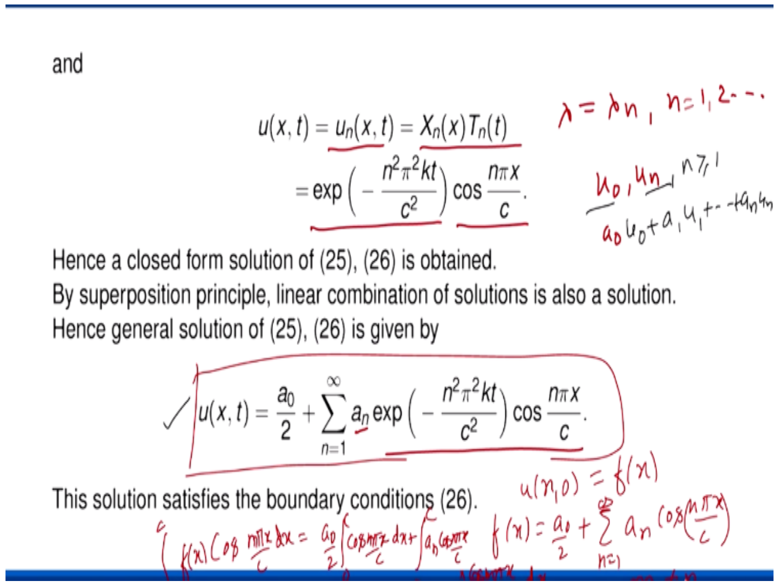

Now solve the other part that is in terms of T of t so that is the problem we have here that is T dash t lambda T t=0. Here already, you already know lambda n u k is already given because material is given so we can simply solve this simple ordinary differential first order ordinary differential equation and the solution of this problem is given as T t= c3 exponential - lambda kt and see there is a arbitrary constant so let us assume that c3=1, if you do not assume then also you can write it but c3=1.

Then substituting the eigenvalues so eigenvalues is that lambda not equal to 0 so when you put lambda not equal to 0 then your T t which we denote as t naught t it is coming out to be 1. And when you take lambda as lambda n then your T and t which is the solution corresponding to lambda n it is exponential of -n square pi square kt on C. So here we have solution obtained for lambda not equal to 0 solution is given by x naught x * T naught t.

And since x naught x is 1/2 and t naught t is coming out to be 1 here. So this is for lambdA0 equal to 0. So u x t= u naught x t for lambdA0 equal to 0. Similarly, u x t is un xt when we are assuming lambda s lambda n and n=1, 2 and so on. So in this case the solution xn x * Tn t. So xn x is cos n pi x/c and Tn t is exponential of e to the power – n square pi square kt upon c square. So we obtain this solution in different cases that is lambdA0 which is 0 and lambda n when is n is from 1 to 2

And we here we utilize one more thing if we have some solution say y1 and y2 as a solution of linear problem then there linear combination is also a solution of the-- that linear homogenous problem. So here we are utilizing the property of homogeneity that your boundary value problem that your problem is homogenous and problem is linear. So here we can apply that if we have solutions then linear combination of solution will also be a solution.

So here linear combination of so here you say that A0 and u naught so A0 u naught is A0/2 + An and this is your xn un xt. So we already have solution u naught and un. So we say that, that A0 u naught+a1 u1+1 An un will also be a solution. So if these are solution then linear combination of this will also be a solution. So utilizing this we have solution ux t=A0/2+ summation n from 1 to infinite An exponential of –n square pi square kt upon c square * cos n pi x /c. So here we have this solution.

Now the only thing is that this is a complete solution provided we know these coefficient A0 and An's and because so far we have utilized only the boundary condition we have not utilized the initial condition so we have the possibilities of Ai's . So to obtain this Ai's we can use the initial condition that is u of x 0 = f of x. So using this we can simplify this so we can write it f of x = A0 by 2+summation n is from 1 to infinite, your An and cos of n pi x/c, right.

And then we can obtain our coefficient n naught An. So here what we do here we just multiply by cos n pi so here what we do here we utilize the orthogonality of cos of n pi x/c. So you multiply by cos of m pi x/c on both the side and integrate. So it means that you just multiply here f of x and between interval I think it is 0 to c, we have written 0 to c; fx*cos n pi x/c tx= this A0/2 integer 0 to c cos of n pi x/c m pi x/c dx + here it is what, here we have this An cos of n pi x/c*cos of m pi x/c p of x and 0 to c.

So here using orthogonality you can find out the value of A0/2, so you consider the case when m is not equal to n then you can find out the corresponding values of coefficient. So this part I am leaving it to you and you can easily find out the coefficient A0 An using orthogonality of cos n pi x/c, as we have done the expansion of gx in terms of Eigen functions. So here I will end our lecture.

And in this lecture what we have seen we have seen the that how this Sturm-Liouville boundary value problem is useful in solving many important problem. So first is that it is use in expansion of a given class of function in terms of Eigen functions another is solving real world problem such as partial differential equation in terms of separation variable form. So we have seen one example based on this.

So here we end. I will continue our discussion in next lecture. Thank you very much for listening. Thank you.