Ordinary and Partial Differential Equations and Applications
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Lecture - 28
Sturm-Liouville Problem and its Properties

Hello friends, welcome to this lecture. In this lecture we continue our study of boundary value problems. And in this lecture we will discuss the very special kind of boundary value problem that is Sturm-Liouville boundary value problem. So let us first define what is a Sturm-Liouville boundary value problem.

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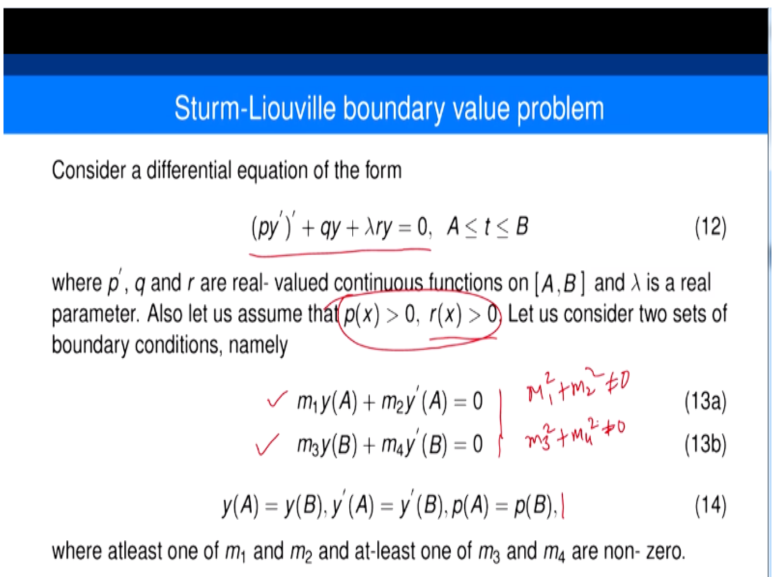

So first we discuss the differential equation of the form py dash dash+qy+lambda ry = 0, here. So here this is a by looking we can say that this is the self adjoint differential equation, here p dash q and r are real- valued continuous functions on close interval A and B and lambda is a real parameter. And also let us assume that this p x is never 0 in this interval, so without loss (()) (01:17) let us assume that P(x) is > 0 and also we assume that the coefficient of y function that r(x) is also > 0.

So this is known as Sturm-Liouville equation. Sturm-Liouville equation without putting this condition that p x and r x are non-negative. So here we simply consider this equation given is equation number 12, we say that it a self adjoint equation and given in this form and we call this as Sturm-Liouville equation. And with the condition that p dash q and r are the real-valued continuous functions.

And here we consider the boundary conditions, so Sturm-Liouville equation along with boundary condition then it is known as Sturm-Liouville boundary value problem. So boundary conditions are given either in this form or this form. So here the first set of boundary condition is this that m1y of A + m2y dash A=0 and m3y of A + m4y dash A= 0 here. So here boundary condition is given in 13a and 13b.

So here if you look at the one condition is given at the point x=A; t=A and the other boundary condition given in the at the point of B, so here we have m1y of A + m2y dash A=0 and m3y of B+ m4y dash B=0, so this kind of boundary conditions are known as separated boundary conditions because they are separately defined at point A and B. And there is one more kind of boundary condition that is periodic boundary condition that is y of A= y of B and y dash A=y dash B.

And if you are considering one more condition that is p of A= p of B. So we call these kind of conditions has periodic boundary conditions. So initial whatever be the value at the initial point that is value given at the end point, so it means that y of A = y of B; y dash A=y dash B and so on. So and here we are assuming that in 13 the variable m1, m2 must non-zero similarly one of the coefficient that is m3, m4 are non-zero constant.

So here we assume that m1 square+m2 square is non-zero, similarly m3 square+m4 square is non-zero. So at least we are saying that these are not trivial boundary conditions.

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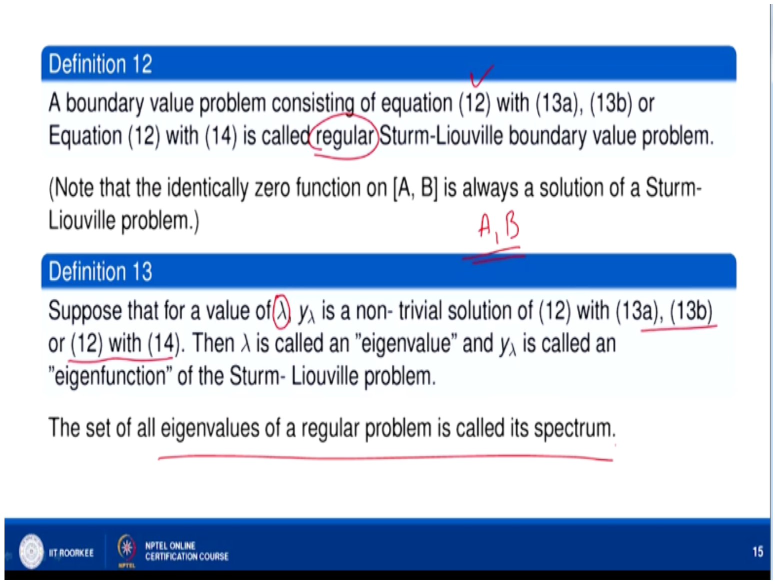

So a boundary value problem consisting of equation number 12 that is Sturm-Liouville equation, along with boundary condition whether it is 13a or this 14, this, we say that strum is called regular Sturm-Liouville boundary value problem. Here regular means what? First of all, we define what is Sturm-Liouville boundary value problem for defining Sturm-Liouville boundary value problem we consider Sturm-Liouville equation that is SL equation.

And one of the boundary conditions either it is 13a or 14 then we call this as Sturm-Liouville boundary value problem. And then we define a regular Sturm-Liouville boundary value problem, it means that our problem is regular. In the sense that, your point A, B these are finite end point that none of A and B are infinite, and the coefficient of highest order derivative that is y double dash that is p here is non-zero in this interval.

If it is happening then we call this SL boundary value problem as regular Sturm-Liouville boundary value problem, okay. So here we say that we already know that, since it is a homogenous equation so trivial solution is always exists. So it means that ideally 0 function on A, B is always a solution of Sturm-Liouville problem and we see that this solution is a trivial solution of this boundary value problem.

But, we are more interested in finding the non-trivial solution. So we say that for a value of this parameter lambda let us say that y lambda is a non-trivial solution of 12 with 13a, 13b or 12 with

14. Then this parameter the value of this parameter for which we have a non-trivial solution the value is known as eigenvalue and the corresponding solution is known as eigen function.

So our idea of considering the Sturm-Liouville boundary value problem is to find out the parameter lambda for which this has a non-trivial solutions and we see that this has a very, very useful application and in fact we say that the application is one very important application is that, that this system has a complete orthonormal system of eigen functions and we use these eigen functions in expanding a certain class of function.

That we see as an application of a Sturm-Liouville boundary value problem. And you can see that some very important boundary value Sturm-Liouville boundary value problems are example of Legendre polynomial and Bessel polynomials and we say we have already seen that how a given function can be written as expansion of Legendre polynomial or Bessel polynomials. So here we see the general theory behind this.

So here idea is that in throughout this lecture we always try to find out the value of the parameter for which we have a non-trivial solution. And this set of all eigenvalues of a regular problem is called its spectrums. So set of eigenvalues is known as spectrum.

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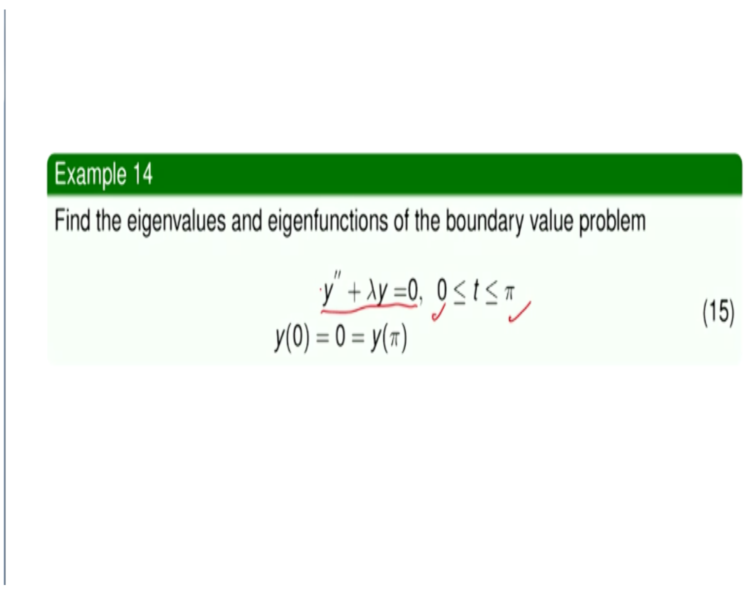

So now let us consider one example of finding all the eigenvalues and eigen functions of the boundary value problem, so if you look at y double dash + lambda y=0. So first let us see that this equation is given in terms of self adjoint form. So here coefficient of y double dash is 1 and coefficient of y dash is simply 0, so we can say that this equation is given in self adjoint form.

Now, we have a boundary condition, boundary condition is given as y0=0 and y pi=0, so these boundary conditions are separated boundary conditions, so it means that this system of boundary value problem is a regular boundary, regular Strum-Liouville boundary value problem because here endpoints are also finite and coefficient of y double dash is non-zero. So we say that this system is a regular Strum-Liouville boundary value problem.

And we want to find out the parameter lambda for which we have a non-trivial solution.

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Solution:

1. When $\lambda = 0$, then $y(t) = 0$ is the solution, so we doesn't have any non- trivial solution in this case. Hence, $\lambda = 0$ is not an eigenvalue of (15).
2. When $\lambda < 0$, then we can write $\lambda = -\mu^2$ for $\mu \in \mathbb{R}$. Solving (15), we get $y(t) = 0$ is the solution, so we doesn't have any non- trivial solution in this case. Hence, $\lambda < 0$ is not an eigenvalue of (15).
3. When $\lambda > 0$, then we can write $\lambda = \mu^2$ for some $\mu \in \mathbb{R}$. Then, general solution of (15) can be written as

$$y(t) = c_0 \cos \mu t + c_1 \sin \mu t.$$

Now, $y(0) = 0$ implies that $c_0 = 0$ and $y(\pi) = 0$ implies that $c_1 \sin \mu\pi = 0$. For non-trivial solution of (15), we must have $\mu\pi = n\pi$, where $n \in \mathbb{Z}$ i.e. $\mu_n = n$ for each $n \in \mathbb{Z}$ and in this case, $\lambda_n = n^2$ is the eigenvalue and $\phi_n(t) = \sin nt$ is the corresponding eigenfunction for each $n \in \mathbb{Z}$.

For which, let us consider several cases. So since lambda is a real numbers so it can take either 0 value, positive value or negative value, so we will consider one by one. So first let us consider the simplest case that is lambda=0 and in this case equation reduced by y double dash=0. And we already know that the solution is simply a linear solution so it means y of x=A of t+B t A t+B and we can find out the coefficient A and B, sorry it is y of t not y.

So you can find out the coefficient A and B by this y of 0=0 and y of pi=0 and you can simply say that since y0=0 this implies at b=0. Now y pi=0 means you can say that we have that A=0. So in case when lambda=0 then we have only 0 solution that y t=0 is the only solution. And we say that we do not have a non-trivial solution, so this is not very useful to us and we simply say that lambda=0 is not an eigenvalue because eigenvalue we say when we have a non-trivial solution corresponding to the value of the parameter.

So we say that lambda=0 is not an eigen value of equation number 15. So now let us consider the next case that is lambda < 0. So when lambda is < 0 we can write it lambda as –Mu square where Mu is some real number. And we say that solving this we have this equation y double dash –Mu square y=0 and we already know the solution of this, solution of this is basically what, here we write it—

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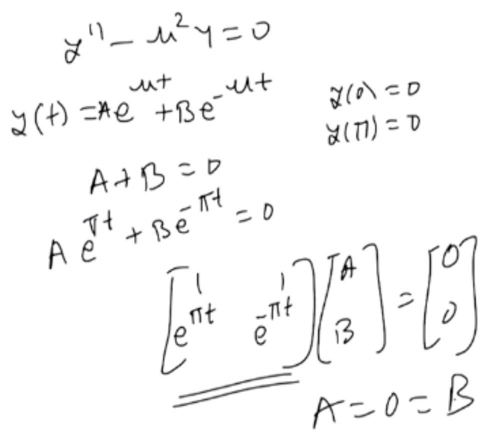

y double dash – Mu square y=0, so we can write it solution of y t = e to the power Mu t, let me write it A+B e to the power –Mu of t. And we need to find out the coefficient A and B for that we let utilize this condition y 0=0 and y of pi=0 when you apply y 0=0 then it is A+B=0 and when you apply Y pi=0 that is e to the power pi t + B e to the power –pi of t=0. And when you simplify this that is 1 1 e to the power pi t and e to the power –pi t and here we have A B=0 0.

And you can simply say that the determined of this coefficient metrics is non-zero so it means that this is invertible and you can say that your solution is only that A=0=B. So it means that here we have only A trivial solution because here A and B are 0 so we have only A trivial solution. So we say that we do not have any non-trivial solution so lambda < 0 is also not the eigenvalue. So it means that any lambda for which, any non-negative value of lambda is also not an eigenvalue of 15.

So now let us move to next condition that is lambda > 0. So for that let us assume that lambda=Mu square where Mu is any real number and hence we can write it our equation as

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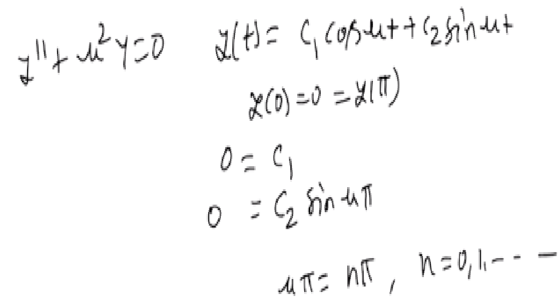

y double dash + Mu square y=0 and we already know that solution is y t = C1 cos of Mu t+C2 sin of Mu t, and we need to find out this C1 and C2 for that y0=0=y of pi. When you put y of 0=0 so the 0=, here we have C1 because, so C1 is coming out to be 0, now y pi=0. When we put y pi=0 we have 0=C2 sin of Mu pi, so it means that we have C2 sin Mu pi. Now we already know that if C2 is 0 then we have only a trivial solution which is not useful to us.

So let us find out the values of Mu say that this C2 sin Mu pi has to be so we say that if Mu of pi is n pi where n is 0, 1, 2, 3 and so on then we simply say that we have a non-trivial, in that case we can choose any value of C2. So here we can say that when lambda is > 0 then we can write lambda=Mu square as we have pointed out, solution is given as y t=C naught cos of Mu t+C1 sin of Mu t and when you use y0=0 you know that C naught = 0 and y pi=0 and pi z C1 sin Mu pi=0. And we have a non-trivial solution only when Mu pi= n pi where n is any integer value.

So it means that calling that-- we say that for each n= each n belonging to z we have solution that is sin of n t, right. And in this case we say that our lambda n which is Mu square that is any square is the eigenvalue so any square is the eigenvalue so and the corresponding solution is sin of n t. So you're your eigenvalues are what eigenvalues are every integer value is an eigenvalue and this corresponding eigen function is given as sin of n t.

So here we are able to find out eigenvalues and eigen function of a given Sturm-Liouville boundary value problem that is this here. So now proceed further.

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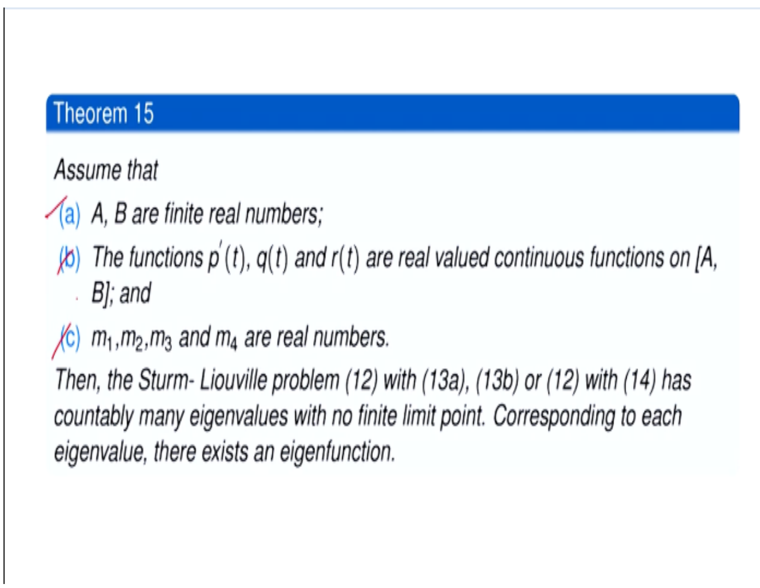

And we have a-- this very, very important result which says that assume that A and B are finite real number, so you have a finite endpoint. And the functions qt and rt are real value continuous function on this close interval A, B and that m1, m2, m3 and m4 that is the coefficient present in a boundary conditions are real numbers then the Sturm-Liouville problem 12 with 13a, 13b or 12 with 14.

Now 12 is what, Sturm-Liouville equation, and 13a, 13b are separated boundary condition and 14 is the periodic boundary condition. So we say that if A, B and C satisfy then we have countably many eigenvalues with no finite limit point, corresponding to each eigenvalue we have eigen functions. So this is the existence theorem of eigen function. Here I am not going to prove this, we assume the statement and the result of this theorem as a result which gives us license to work with eigenvalues and eigen functions.

So without proving without providing the proof of this we are using this theorem as an existing theorem for eigenvalues and eigen functions. And we say that eigenvalues are countably many and corresponding to each eigen values we have eigen functions. Here we have a eigenvalues are like discrete eigenvalues and we say that discrete eigenvalues means these are not continuous spectrum it is only discrete one, okay.

So all these we have utilized so it means that in regular Strum-Liouville boundary value problem we have this theorem to provide the existence of eigenvalues and eigen functions. Now once we have existence of eigenvalues and eigen function then we start working with these eigenvalues and eigen functions.

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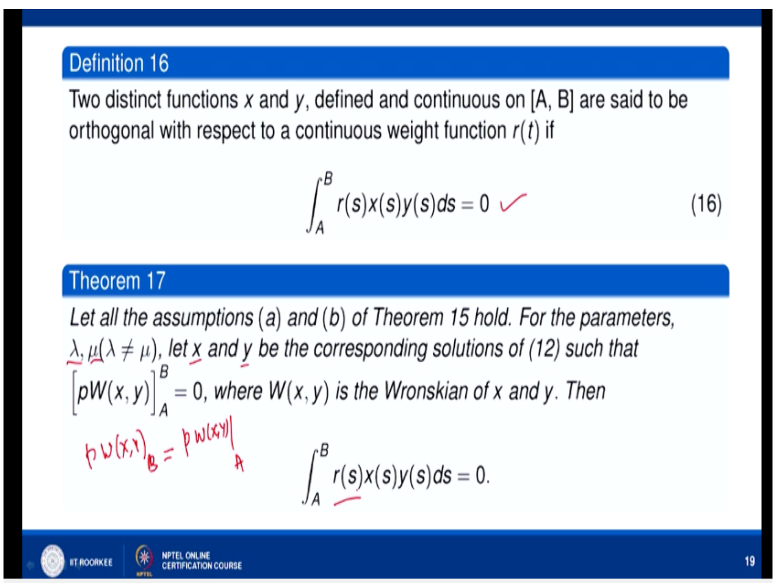

So next very important result is in the direction of orthogonal eigen functions. So before that what do you mean by orthogonal eigen functions let us define it. So two distinct functions x and y, defined and continuous on this close interval A, B are said to be orthogonal with respect to a continuous weight function r t if, A to B r s, x s, y s ds=0. So here when this condition holds then we call this as that x and y are orthogonal to each other with respect to the weight function r of s.

And this is very, very important property which is preserved by if you look at Legendre polynomials, Bessel polynomial and many more special function which is nothing but a bi-product of the Sturm-Liouville boundary value problem. So this is very, very important property, and it says that this Regular Strum-Liouville boundary value problem consist a complete orthogonal system of eigen functions. So let us that we are going to discuss in next class. Now let us discuss a next theorem.

So let all the assumptions a and b of Theorem 15, Theorem 15 is existence of eigen functions hold and for the parameter lambda and Mu where lambda is not equal to Mu, let x and y be the corresponding solution of 12 such that this value that is p Wronskian of x, y given at B and A=0, basically it is what it is p w, x y given at B – or you can = p w, x y given at A are same. So this quantity is same. Then we say that A to B r of x, x of y, y of x ds = 0.

So we say that if this condition hold what are conditions, condition is that A and B are finite real number and the function p dash t, q t and r t are real valued continuous functions then the Strum-Liouville boundary value problem has orthogonal solution corresponding to non-distinct eigenvalues. So it means that even lambda and Mu are not equal then corresponding to lambda let us say x is the solution of the Strum-Liouville boundary value problem and corresponding to Mu we have y has solution of the boundary value problem.

Then x and y are orthogonal to each other with respect to the function r of s. So let us prove this and this is true when we have the following condition that is this.

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Proof: From the hypothesis of the theorem, it is seen that

$$(px')' + qx + \lambda rx = 0$$
$$(py')' + qy + \mu ry = 0.$$

Multiplying first equation by y and second by x and subtracting, we have

$$(\lambda - \mu)rxy = (py')'x - (px')'y$$

i.e.(adding and subtracting $py'x'$)

$$(\lambda - \mu)rxy = \frac{d}{dt}[(py')x - (px')y]. \qquad (17)$$

Integrating equation (17), we have

$$(\lambda - \mu)\int_A^B r(s)x(s)y(s)ds = [(py')x - (px')y]_A^B.$$

Since $\lambda \neq \mu$, it follows from the assumptions that $\int_A^B r(s)x(s)y(s)ds = 0$.

So let us proof this. So from the hypothesis of the theorem we know that corresponding to lambda x is the solution of this Sturm-Liouville equation. So px dash dash + qx + lambda rx=0 and py dash whole dash+qy+Mu ry=0, so that is what is given here that x is the solution of this and y is the solution of this, so it satisfies this. Now simplify this by multiplying the first equation by y and multiply the second equation by x and just subtract it.

So here when you multiply what you will get, this Sturm will be canceled out qx y-qy x it is gone. So here in, let me write it here, so it is what p x dash dash y+q x y+lambda r x y=0 and here we have p y dash x+q y+Mu rxy=0, when you subtract it this sample gone and we have lambda–Mu rxy, so we have lambda–Mu rxy=0 p y dash whole dash x-px dash dash y, so we have this thing.

Now, we can simplify and we can write it lambda – Mu rxy=dy/dt of py of x-px dash y. Here we simply add this term py dash x dash and we can write it like this, in fact you can easily check that if you expand it you will get this only. So here we have done this p y dash dash x+py dash x dash - py dash x dash – px dash whole dash y, and if you simplify you have this thing, that d/dt of py dash x–px dash y.

So then we integrate this to with respect to t from A to B. So when you integrate this last equation 17 we have lambda – Mu A to B r of s, x of s - t s=, now here this derivative will go out and we have p y dash x–p x dash y evaluated at B-A. So now look at this equation is already given. What is given here that p Wronskain of x y at A to B = 0, means what p-- now Wronskain x y is x y dash – x dash y given at the = 0, so this already given which is written here.

p x dash y B at A. So here we can say that quantity is going to be 0 because that is given as assumption here. This is the condition we have given. So it means that this right hand side is simply gone so we have lambda – Mu*A to B r s, x s, d s = 0. Now lambda is not equal to Mu that we have already assumed, so it means that from the assumption we have that this quantity has to be 0.

That is what we wanted to prove that the solution corresponding to different distinct eigenvalues are orthogonal to each other that is what we wanted to prove.

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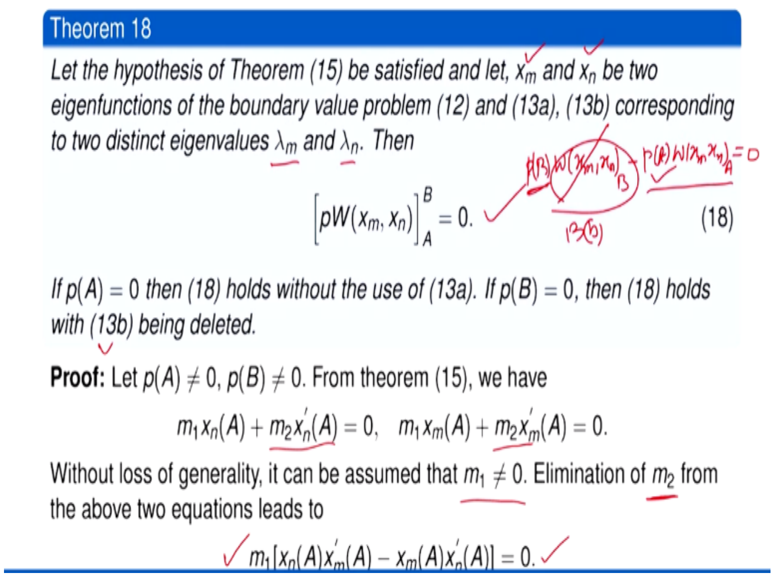

Now next we want to show that, that we have already proved that if certain condition hold that is p of Wronskain of x y add B=p w xy at A if these two are equal then distinct eigen function corresponding to distinct eigenvalues are orthogonal to each other. Now see that how this boundary condition which we have assumed that is separated boundary condition and periodic boundary condition are helpful to find out that this condition is trivially hold.

So for that let us consider this following theorem. Let the hypothesis of Theorem 15 that is the existence of eigen function theorem be satisfied and let xm and xn be two eigen functions of the boundary value problem corresponding to two distinct values lambda m and lambda n. Then this boundary condition holds true. Because if you look at, we have assumed that if this boundary condition holds then the corresponding eigen function that is xm and xn are orthogonal to each other with weight function r of x.

Now you want to show that if we have boundary condition that is 13a 13b is true then this condition is automatically true. Now here, in addition to this we say that if p A=0 then this condition holds without use of 13b, right 13a. And if p B=0 then this equation holds with the 13b being deleted. So it means that if p A=0 then we have to use only one condition, if p B=0 then we can use only one condition.

So without loss of generality let us assume that, that none of this p A and P B is 0. So let us assume that p A is not equal to 0 and p B not equal to 0. In fact, let me write it here this is what this is p B Wronskian of x of xn given at B–p of A Wronskian of xn at A=0. Now if p B is 0 then I need not to use this only this is sufficient that is what it is written here. Then if p B=0 then we have to use only one that is 13a.

And if p A=0 then we have to use only one condition that is 13b and the boundary condition given and the point t = B. So let us assume that, that none of this p A and p B is 0. So from this theorem we have, that since xn and xm both are solution of this so let us write it m1 that they must satisfy the boundary condition that is m1 xn A+m2 xn dash A=0, similarly xn will also satisfy the boundary condition that is m1 xn A+m2 xm dash A=0.

So here let us assume that this m1 is not equal to 0 because if m1is =0 then this is quite easy, we see what is going to be easy because if m1 is not equal to 0 then we can simplify these two equation by eliminating this m2, so here you can eliminate m2 from these two equations and we can write our equation as m1 xn A xm dash A–xm A xn dash A=0. So here we assuming that m1 is not equal to 0.

If m1 is equal to 0 then this is very easily we can obtain like this but coefficient is going to be m2. Now we have this, so since m1 is != 0 so we can cancel it out and we have this thing.

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Since $m_1 \neq 0$, it is seen that

$$x_n(A)x_m'(A) - x_m(A)x_n'(A) = 0. \quad (19)$$

Similarly,if $m_3 \neq 0$ in (13b), it is seen that

$$x_n(B)x_m'(B) - x_m(B)x_n'(B) = 0. \quad (20)$$

From the relations (19) and (20), it is obvious that (18) is satisfied. If $p(A) = 0$, then the relation (18) holds since

$$\left[pW(x_m, x_n)\right]_A^B = p(B)[x_n(B)x_m'(B) - x_m(B)x_n'(B)] = 0,$$

in view of the conclusion (20). Similar is the case when $p(B) = 0$. This completes the proof.

That xn xm dash A - xm A xn dash A=0, so this we have obtained from 13a that is the boundary condition given at the point t=A. Similarly, if we assume that use 13b is let us assume that m3 is not equal to 0 then in a similar manner we can prove that, that xn B xm dash B - xn B xm dash B=0. And we can easily see that if p A=0 then this 19 is not useful. And if p B=0 then this is not useful.

And we can say that from equation number 8 let us simplify this p w xn xm given at B at A=p B xn B xm dash B –xn B xm dash B. Now since p A=0 we are not looking at the condition given at xt=A, so we have this. Now if you look at the in a bracket whatever is here it is trivially 0 because we have already proved in equation number 12. So it means that boundary condition 13b implies that 20th is true.

Now 20 is true means Wronskian this pw xn, xm given at the B and A=0 if p A=0. Similarly, we can handle the case and p B=0. So it means that if p A and p B both are non-zero then we can simply say that this is what this is written as p B and this is xn B, xn B xm dash B – xn B xm dash B – p of A xn A xm dash A – xn A xm dash A, so we have from 19th this is 0, this is 20th this is 0, so it means that quantity is going to be true.

So we have seen that the condition this condition that is p w xn xm at A and B=0 is true when we assume the condition that 13a and 13b that is the separated boundary condition if you use then this is automatically true.

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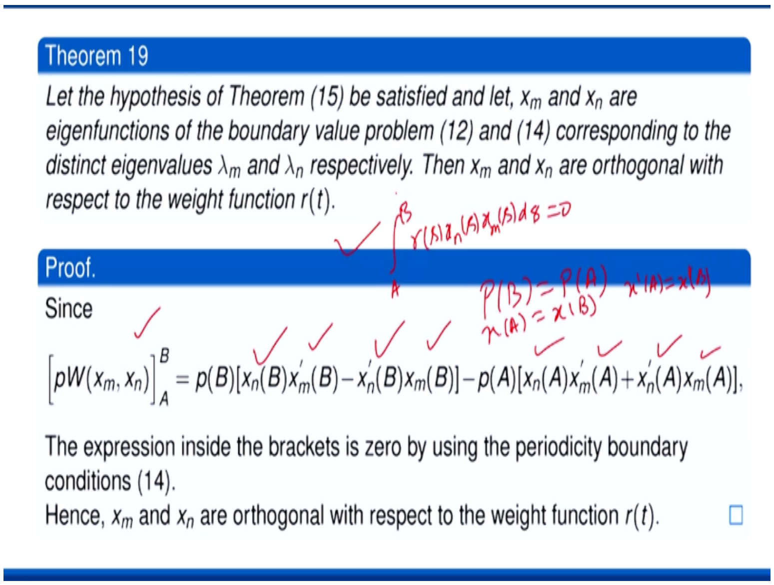

So now we look at the other case and consider that hypothesis of Theorem 15 we satisfied and let xm and xn are eigen functions of the boundary value problem 12 and 14 that is periodic boundary conditions are true corresponding to distinct lambda m and lambda n then xm and xn are orthogonal with respect to the weight function r t.

In fact, only thing we had to show is this quantity this is going to be 0 here. So let us look at here we since p w xm, xn B-- A is can be expanded in this form p B * B Wronskian of xm, xn that is xn B xm B-xm dash B xm B-p of A xm A – xm dash A+xn dash A+xm A. Now here we simply say that if we use periodic boundary condition then we can make this quantity 0. So here we use xm B=xn A. xm dash B = xn dash A; xn dash B = xm dash A; xn B = xn m.

And we have also assume that p of B= p of A. So we can say that here this periodic periodicity if you apply then this quantity is =0, so it means that if this quantity is 0 we have already proved that if this condition is true then A to B r s, xn s, xm s d of s = 0 here. So in case of periodic boundary condition that is p B=p A; x A=x B and x dash A=x dash B, this quantity is 0 and hence your xn and xm are orthogonal to each other with respect to the weight function r of s.

And previous theorem says that if we have the separated boundary condition then this is automatically true, here we need not to assume that p A = p B. But when we consider the case of periodic boundary condition then we have to consider the function p x also. So here we have discussed some important properties of eigen functions, what we have and discussed is that we are considering the regular Strum-Liouville boundary value problem.

And we have seen that, that eigen functions existence of eigen functions that is theorem 15 we have discussed and not only this, we have discussed that when the case of separated boundary condition or periodic boundary condition the regular Strum-Liouville boundary value problem has orthogonal set of eigen function corresponding to distinct eigenvalues.

So and in next class we will some more properties of eigen function and eigenvalues and the use of eigen functions in many other field. So with this I am just summarizing my lecture and we continue in next lecture. Thank you very much for listening, thank you.