Ordinary and Partial Differential Equations and Applications

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Lecture - 27
Self Adjoint Form

Hello friends, welcome to this lecture. In this lecture we will continue our study of boundary value problems. In this lecture and in coming lecture we will discuss a special kind of boundary value problem known as the Sturm - Liouville boundary value problems. But before that, let us consider some preliminary for Sturm - Liouville boundary value problem. So first let us start discussing the concept known as Adjoint Equation.

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$$y' = f(x, y) \Rightarrow \frac{d}{dx}\left(g(x, y)\right) = 0$$

So Exact Equation is basically what, we have already seen the exactness conditions for first order differential equation. And in this case, when you talk about first order differential equation y dash =f(x, y) or you can say that, when this kind of equation is written as d/dx of some g(x, y), then we say that this equation which is now written in this form is known to be exact. Now let us extend this concept for second order and higher order linear differential equation.

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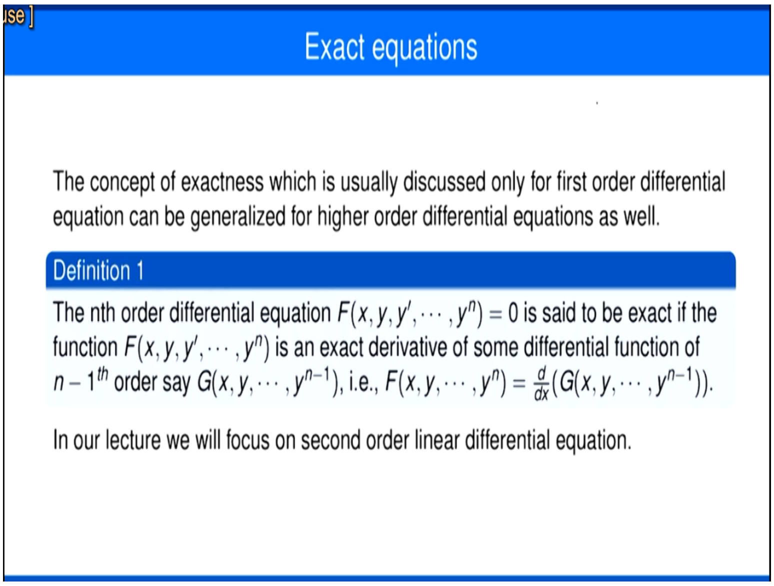

So the concept of exactness which is hugely discussed only for first order differential equation can be generalized for higher order differential equations as well. So first of all we need to understand what exactness means. So the nth order differential equation given as like F (x, y, y...) = 0 is said to be exact if the function F(x, y, y dash ,..., yn) is an exact derivative of some differential function of n-1th order means just one less order, say G(x,y,.....,yn-1). Or we can say that here F(x, y, ...yn) is written as d/dx of G(x, y, ..., yn-1).

And so in this case when this equation, this left hand side is now written as exact derivative of some differential function of one less order, we say that our differential equation is an exact differential equation. And in our lectures we will discuss only on second order. We can say that this is a general definition which is applicable for any order >= 1, but in particular we will discuss only for second order in this lecture.

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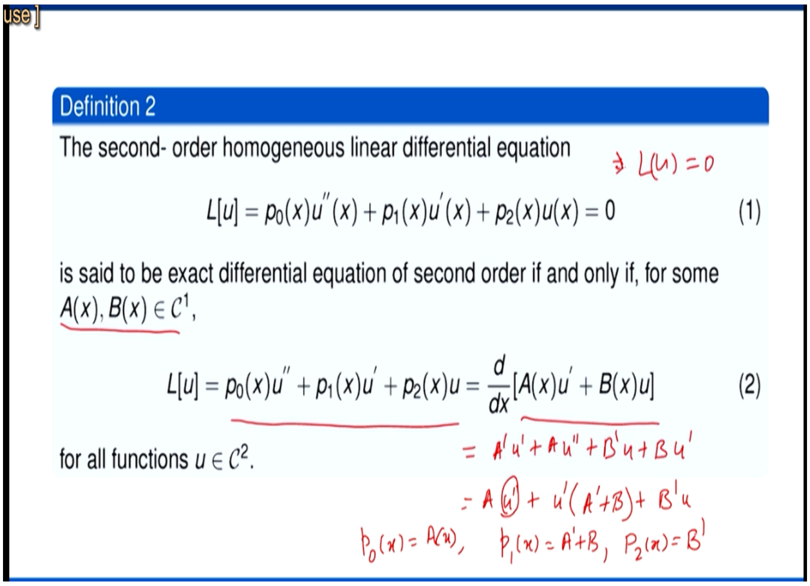

So if we reformulate our definition for second order, we say that second order homogenous linear differential equation L (u) = p0(x)*u double dash(x)+p1(x)*u dash(x)+p2(x)*u(x) = 0 which is written as the operator form as L (u) = 0. So in operator form it is written as L (u) = 0 is said to be exact differential equation of second order if and only if, for some A(x) and B(x) which is a differentiable function we can write this differential equation p0(x)*u double dash+p1(x)*u dash+p2(x)*u as d/dx of A(x)*u dash+B(x)*u.

So if we can write this term (x)*u double dash+p1(x)*u dash+ p2(x)*u as exact derivative of some differentiable form of one less order that is first order, 2-1 it is first order. And then we say that this equation L (u) = 0 is an exact differential equation. So we can say that this L (u) = 0 is exact if and only if we have this relation.

How we get this relation, if you look at this, if you simplify this, this is what you can write it here as A dash *u dash+A*u double dash+B dash *u+B*u dash. So if you collect the coefficient, it is A*u double dash+u dash*(A dash +B)+B dash *u. So it means that, if you compare the coefficient here, coefficient will be what, coefficient of u double dash is A here and here it is p0(x). So we can write p0(x) = A(x). If you look at the coefficient of p1(x), p1(x) = A dash +B and p2(x) = B dash.

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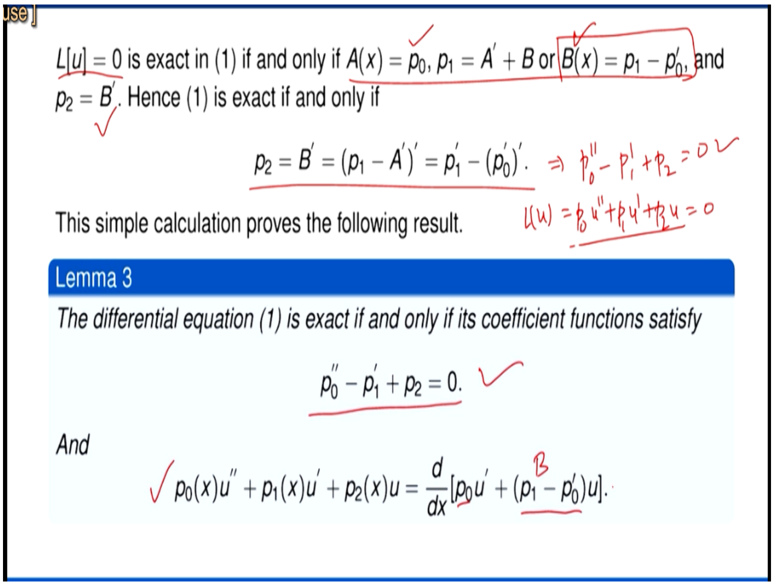

So, here by comparing we can get the value of A and B which we have written here, A(x) = p0 and p1 = A dash +B or we can simplify this to like B(x) = p1- p0 dash and p2 = B dash . So using the value of B(x) and B dash we can say that the previous equation L (u) = 0 is exact provided we have this p2 = B dash or B is p1- p0 dash . Now p1 is what, p1 is A, so we can say that equation is exact if this condition holds true.

So it means that this condition can be written as, if we simplify this, it is what p0 double dash – p1 dash+p2 = 0. So it means that this L (u) is now written as p0*u double dash+p1*u dash+p2*u = 0 is exact if the coefficients for this differential equation satisfy this condition. And we call this condition as exactness condition. So the differential equation (1) is exact if and only if its coefficient functions satisfy the following condition: p0 double dash – p1 dash+p2 = 0.

And not only this we can also write the equation p0(x)*u double dash+p1(x)*u dash+p2(x)*u = d/dx of, now A is p0 here, so A*u dash+ (p1- p0)*u, so this is the value of B which we have calculated here. So it means that I can write this as d/dx of p0*u dash+(p1- p0 dash)*u. So it means that literally we are able to find out the exactness condition which is given by this but we can also write this differentiable form as derivative of this equation.

So it means that if this equation is not given in the exact form it means that the coefficient functions does not satisfy this relation. Then we can find out any other function which makes this equation as an exact equation. Such a function is known as integrating factor. This is the same concept which we have discussed in first order differential equation.

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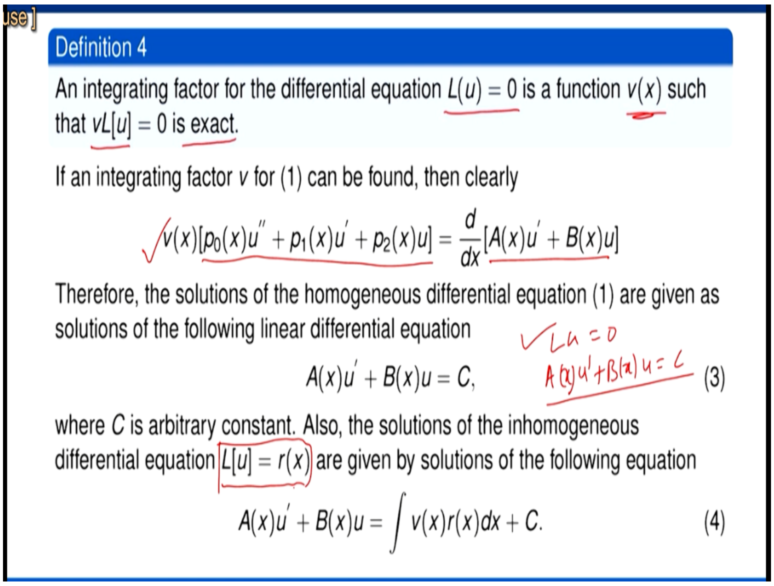

So here we write an integrating factor for the differential equation L (u) = 0 is a function v(x) such that v*L[u] = 0 is an exact differential equation. So how we can find out this function v(x), so if you look at by multiplying by v, this much by exact differential equation. It means that v(x), though this part must be some exact derivative of some differentiable form of one order less i.e., d/dx of A(x)*u dash+B(x)*u and solving this we can find out the function v(x) such that this L(u) is an exact differential equation.

And what is the use of this, we can use this, because now your L (u) = 0 is now reduced to this equation A(x)*u dash +B(x)*u = c. So solution of this L(u) = 0 is now reduced to solution of differential equation of one order less which is known as A(x)*u dash +B(x)*u = constant; C is some constant which we can determine later on.

And in particular if you look at the inhomogeneous differential equation L (u) = r(x), the solution of this inhomogeneous equation is given by the solution of A(x)*u dash +B(x)*u = integral of v(x)*r(x)*dx+ some constant. So this integral has a limit depending on the condition given along with this inhomogeneous differential equation.

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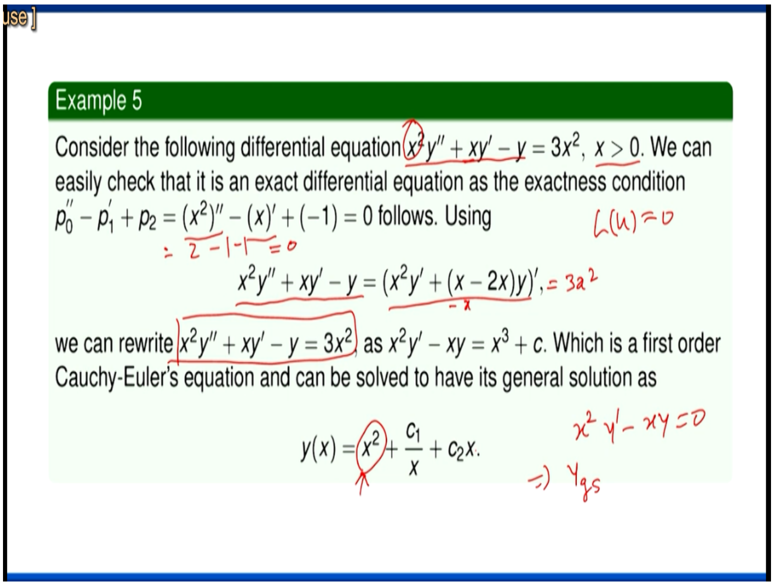

So how to find out the integrating factor which we will discuss little bit later but before that let us consider one example. So consider the following differential equation x square 2*y double dash+x*y dash – y = 3 x square 2. Basically it is a inhomogeneous Cauchy-Euler kind of equation, here without loss of --- let us assume that x is not 0, either x is positive or x is less than 0. Now we can easily check that it is an exact differential equation because the exactness condition is what, the exactness condition is p0 double dash – p1 dash+p2

Let us calculate this quantity, now p0 is the coefficient of x", find out the double --- coming out to be (x square 2) double dash - p1 dash is p1 is the coefficient of y dash, we write it like that (x) dash +p2 is the coefficient of y that is -1 here. If you simplify this is what, this is simply 2-1-1, it is coming out to be 0. It means that this differential equation L (u) = 0 satisfy the exactness condition. So we say that it is an exact differential equation.

So it means that now we can write it x square 2*y double dash – x*y dash – y as a derivative of this following quantity that is p0*y dash +(p1- p0 dash)*y so we can write x square 2*y double dash – x*y dash – y as (x square 2 *y dash +(x – 2*x) *y) dash , now x – 2*x is nothing but –x. So we can say that this inhomogeneous differential equation x square 2*y double dash – x*y dash – y = 3*x square 2 is now written as x square 2*y dash – x*y = x cube 3 - c.

Here we are using this thing because this is written as 3*x square 2, now integrating both sides we will get that x square 2*y dash – x*y = x cube 3+c. Now we can say that this is a first order Cauchy-Euler's equation and we can solve it quite easily. And we can solve it by seeing that first solve the homogenous problem that is x square 2*y dash – x*y = 0 and find out the general solution here. So once we have ygs general solution and the particular solution which we can check that it is x square 2.

So in this case we can write the general solution as y(x) = particular solution that is x square 2 plus solution of the homogenous part, that is c1/x+c2*x, so now the general solution is given by this.

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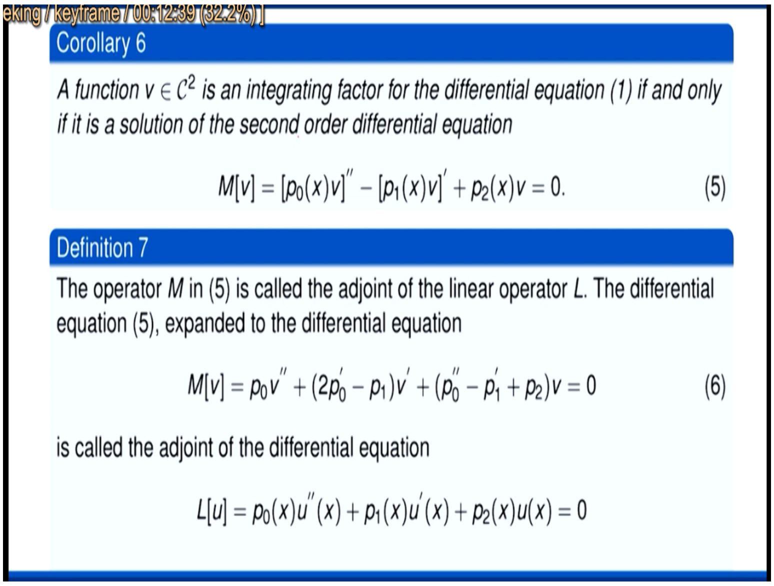

So now as we have already discussed that a function v belongs to c2 is an integrating factor for the differential equation 1 if and only if it is a solution of the second order differential equation [p0(x)*v] double dash – [p1(x)*v] dash +p2(x)*v = 0.

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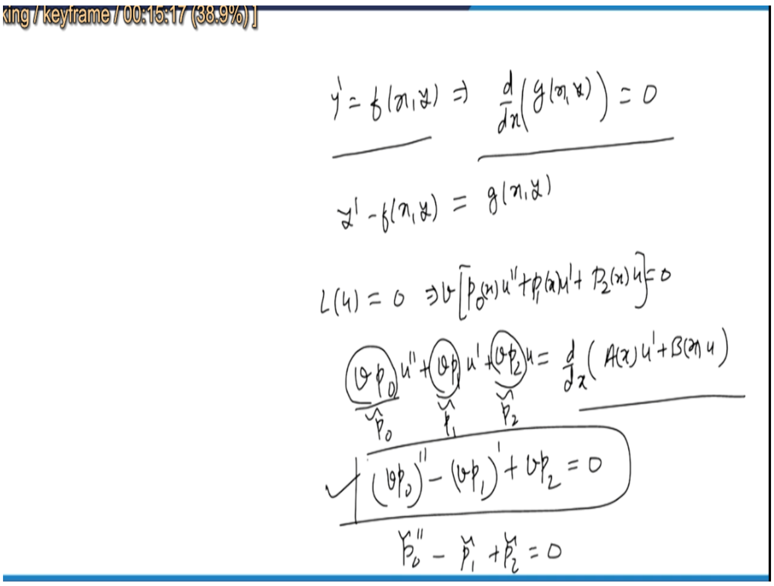

How it comes let us see like this, here we already know that L(u) = 0 is not exact without (()) (12:56) let us assume that L(u) = 0 is not exact, because if it is exact then our integrating factor is nothing but any function is an integrating factor but let us assume that it is not exact differential equation. So here it means that p0(x)*u double dash+p1(x)*u dash+p2(x)*u = 0 is not exact.

So it means that now we are finding function v, if you multiply this then it is an exact differential equation. It means that now v*p0*u double dash+v*p1*u dash +v*p2*u = d/dx of A(x)*u dash+B(x)*u. Now we already know that this is an exact if we have this relation, we say that coefficient of u double dash that is (v*p0) double dash – coefficient of u dash that is (v*p1) dash +coefficient of u that is v*p2 = 0.

So we, by the exactness condition we know that this happens only if we have this relation that is (v*p0) double dash - (v*p1) dash +v*p2 = 0. In fact, we have just taken the coefficient of u double dash. This is now p0 dash, I should not write p0 dash, let us use some other notation. We call it p0~ and this is as p1~ and this as p2~. So we already know that this is an exact provided that p0~ - p1~+p2~ =0. Now I am just writing the value of p0~, p1~ and p2~ and we have this relation.

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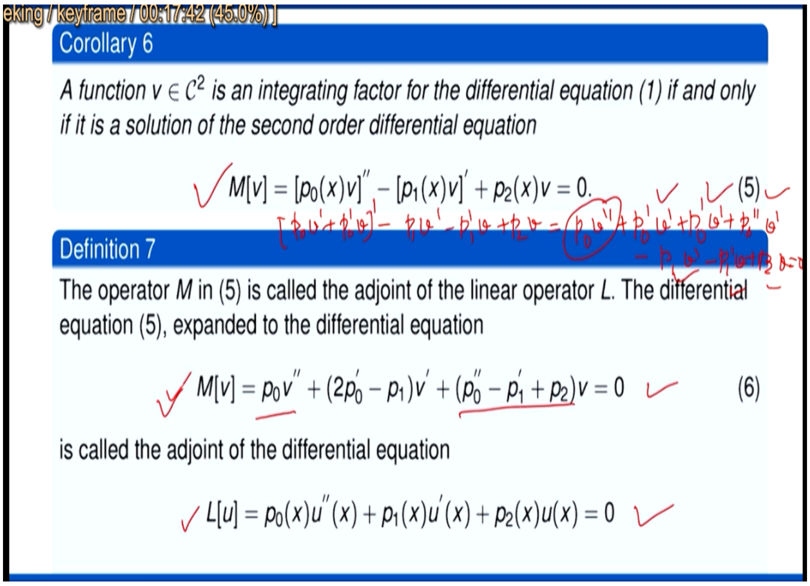

So it means that a function v is belonging to c2 category, c2 function is an integrating factor for the differential equation (1) if and only if it is a solution of the second order differential equation [p0(x)*v] double dash – [p1(x)*v] dash +p2(x)*v = 0. Now if you simplify this we can get a relation like this, we can get this as [p0*v dash +p0 dash*v] dash – p1*v dash – p1 dash*v+p2*v. Now further simplifying we have p0*v double dash+p0 dash*v dash + p0 dash*v dash +p0 double dash*v dash – p1*v dash - p1 dash*v+p2*v = 0.

When we collect this then we have this relation p0*v double dash that is here+(2*p0 dash- p1)*v dash , so we collect the coefficient of v dash . Coefficient of v dash is here, here, here and here. That is (2*p0 dash- p1)*v dash +the remaining part that is we have here, here and here. So we can write it here. So by collecting this we can get this term (p0 double dash- p1 dash+p2)*v = 0. So we say that the operator M in (5) is called the adjoint of the linear operator L. The differential equation (5) expanded to the differential equation.

So we simply expand this differential equation given in, this M[v] = 0. If we expand we have this, p0*v double dash+(2*p0 dash- p1)*v dash +(p0 double dash –p1 dash +p2)*v = 0 and we call this differential equation as the adjoint of the differential equation L(u) = 0, where L(u) is defined as p0(x)*u double dash +p1(x)*u dash(x)+p2(x)*u(x) = 0. So adjoint of L is given as M. And we can easily check that if you find out the adjoint of M then we can have some other equation.

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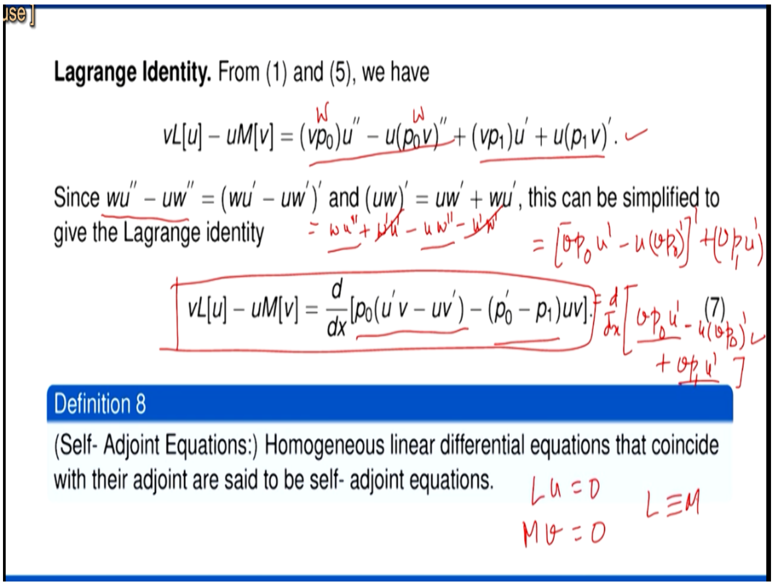

So now here is where the important identity known as the Lagrange Identity which we can calculate from equation number (1) and (5) which is known as Lagrange Identity. So here from (1) and (5) we can say that v*L[u] – u*M[v], if you want to calculate this quantity, then we will get is what is known as Lagrange Identity. So let us first calculate this quantity, so v*L[u] – u*M[v].

So when you write v, L[u] is basically what, previous slide it is already given L[u] now you just multiply v here – u*M[v], multiply u here and just simplify and you will get the following inequality that is (v*p0)*u double dash – u*(p0*v) double dash +(v*p1)*u dash+u*(p1*v) dash .

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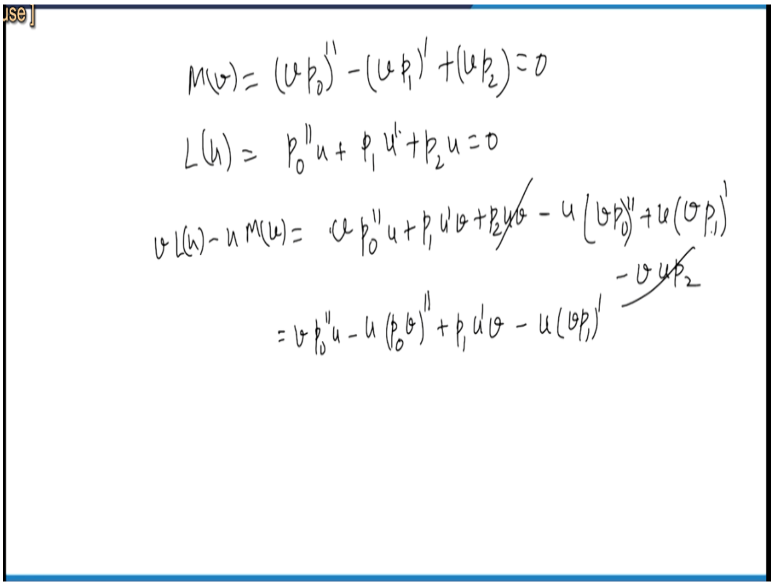

So let us do it in here. So here your M[u] is given as (v*p0) double dash – (v*p1) dash + (v*p2). So that is your M[v]. If you calculate L[u], L[u] is basically p0 double dash*u +p1 dash*u dash +p2*u = 0. So here this is not dash it is simply p0. So now let us calculate, now we want to calculate this quantity v*L[u] – u*M[v], so v*L[u] – u*M[v] that we want to calculate. So v*L[u] is what v*p0 double dash*u+p1*u dash*v+p2*u*v – u*M[v]. So u*M[v] is u*(v*p0) double dash – u *(v*p1) dash – u*v*p2.

So this will cancel out here and what is left here is v*p0 double dash*u – let us write it here u*(p0*v) dash +p1*u dash*v – u*(v*p1) dash. So now we want to simplify this, that is what it is written here u*(p0*v) dash +p1*u dash*v – u*(v*p1) dash, that is what it is written here. Now let us simplify this, I think there is, it is double dash here. So when you simplify, u*M[v] I think there is a plus sign here, u*(v*p1) dash here it is+sign.

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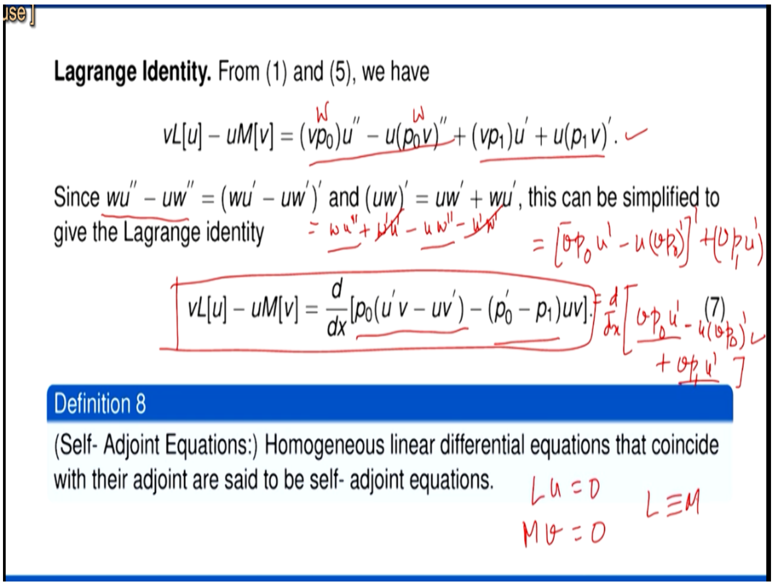

So here, now let us look at this term here (v*p0)*u double dash – u*(p0*v) double dash, so here we use this inequality w*u double dash – u*w", this can be written as (w*u dash- u*w dash) dash . How we can write this, if you simplify this it is what w*u double dash+ w dash *u dash- u*w" – u dash*w dash. So now this will cancel out and we have w*u double dash – u*w", that is what is written here. Using this, here this is simply (u*w) dash = u*w dash +w*u dash, this is simply product rule for functions.

So using this w is your v*p0, so using this I can write this term as, we can simply write this as [v*p0*u dash – u*(v*p0) dash] dash +this we write (v*p1*u) dash. So using this and if we simplify this we can write it this as v*L[u] – u*M[v] as d/dx of p0*(u dash*v – u*v dash) – (p0 dash – p1)*u*v]. So when you simplify you will get this. So it means that this is written as what, you simply write d/dx of and here you simplify v*p0*u dash – u*(v*p0) dash +v*p1*u dash.

So using this you can write it that p0 dash-p1 is u*v. So here you have to simplify this. When you simplify this you will get this v*L[u] – u*M[v] = d/dx of p0*(u dash*v – u*v dash) – (p0 dash – p1)*u*v and this identity is known as Lagrange Identity. So when you simplify this you will get this. Now let us consider this Lagrange Identity as very useful and we will see how this is useful in further lecture.

So now we define one more important type of equation known as self adjoint equation. Homogenous differential equations that coincide with their adjoints are said to be self adjoint equations. It means that, we have already discussed that corresponding to L (u) = 0 we have adjoint equation M(v)=0 and if your operator L is ideally equal to M then we say that our equations are given in self adjoint equation form. So it means that, let us find out the conditions under which equations are known as self adjoint equations.

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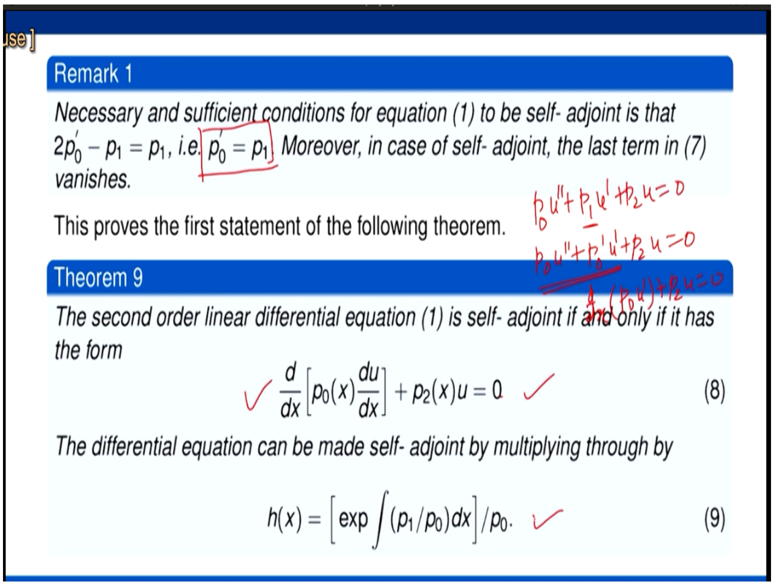

So necessary and sufficient conditions for equation (1) to be self-adjoint is that 2*p0 dash- p1 = p1.

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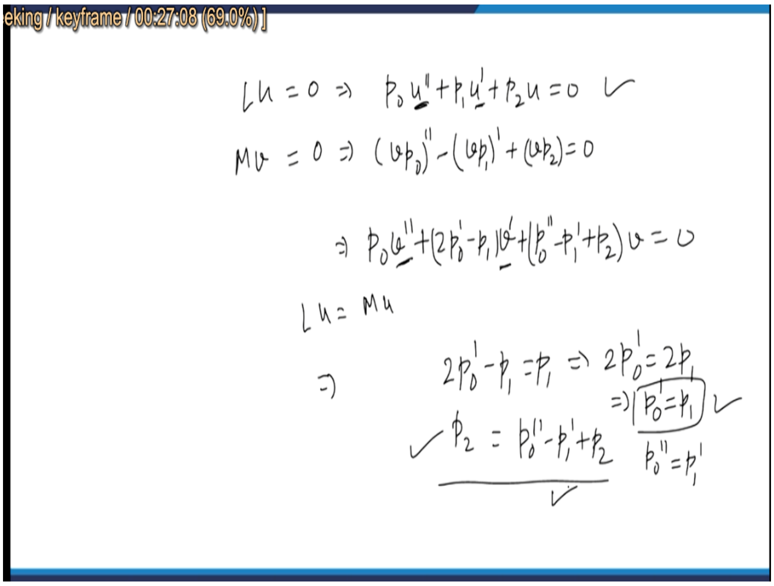

How we can identify this, so if you look at here we have this L (u) = 0. So this is what p0*u double dash+p1*u dash+p2*u = 0. And we have obtained our M (v)=0 provided that here we have (v*p0) double dash+(v*p1) dash +(v*p2) = 0. There is a small mistake here, the mistake is that this is – in place of +, so it is – here. So if you simplify, we can write this as, I think this is the following format, this we have already calculated that p0*v double dash, so we can write it here p0*v double dash+(2*p0 dash-p1)*v dash +(p0 double dash – p1 dash+p2)*v = 0.

So we say that L (u) = M (v) provided that the corresponding coefficients are same. So look at the corresponding coefficients. Here the corresponding coefficient of u double dash is p0, here it is p0, so no problem. If you look at the coefficient of u dash it is p1 here but here it is 2*p0 dash – p1. So it means that 2*p0 dash- p1 has to be equal to p1. So this is one set of condition which you need to. Look at the coefficient of u, p2 and here it is p0 double dash – p1 dash +p2 here.

Now if you simplify this, this you can get it 2*p0 dash = 2*p1. So we can write it here, this as p0 dash = p1. So if you use this then p0 double dash = p1 dash. So we can say that this condition is also truly satisfied by the next equation that if you take p0 dash = p1 then this condition is automatically satisfied. So we say that L (u) is same as M (u) provided that we have p0 dash = p1. That is what is written here that the necessary and sufficient conditions for equation (1) to be self-adjoint is that condition that 2*p0 dash- p1 = p1 or we can write it p0 dash-p1.

So it means that the coefficient of u double dash = coefficient of u dash. That is what the condition is given here. And this proves the first statement of the following theorem. The following theorem says that the second order linear differential equation (1) is self adjoint if and only if it has the following form. So here if you look at, our equation is reduced to what p0*u double dash+p1*u dash+p2*u = 0. So now in place of p1 I can write it as p0 dash. So we can write p0*u double dash+p0 dash*u dash+p2*u = 0.

Now if you look at these two term, then I can write this as d/dx (p0*u dash) +p2*u = 0. So it means that now I can write our differential equation in the following form that d/dx of p0(x)*du/dx+p2(x)*u = 0. So if a particular differential equation is written in this particular form then it automatically satisfies the necessary and sufficient conditions for self adjoint. So it means that this kind of differential equation is known as self-adjoint differential equation.

If it is not given in self-adjoint equation, then we can always find out the integral factor by which we can make differential equation into a self-adjoint form. We claim that the differential equation can be made self-adjoint by multiplying thoroughly by this integral factor h(x) = [exp integral (p1/p0)*dx]/p0. This is our claim; let us prove this claim here.

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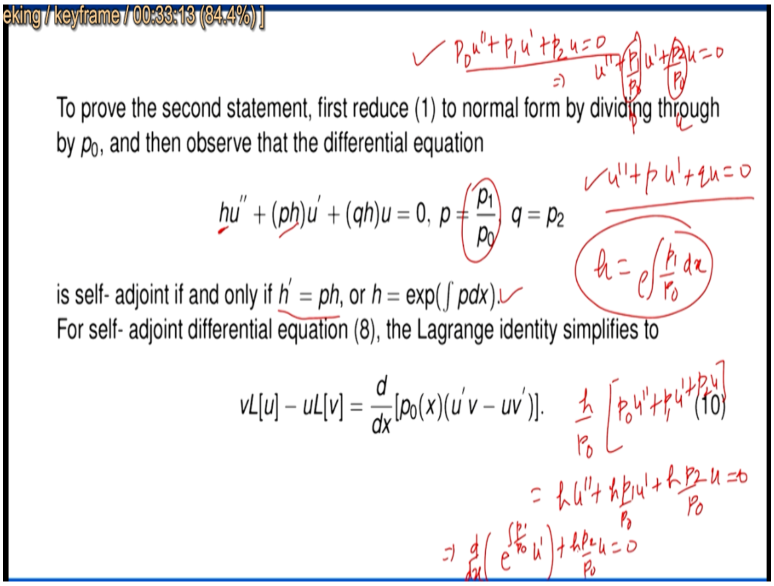

So to prove the second statement, we assume that our equation p0*u double dash+p1*u dash+p2*u = 0 is not given in self-adjoint form. So it means that your p0 dash is not equal to p1. So what we try to do, here we multiply by some function by which we say that it is your integrating factor. So before that multiplying let us assume that p0 is never 0 so that we can divide it by p0. So when you divide by p0, we write it as u double dash+(p1/p0)*u dash+(p2/p0)*u=0.

Now multiply by h, so when you multiply by h, you have h*u double dash+(p*h)*u dash+(q*h)*u. Here p*h is, p is p1/p0, so let us call this as p here and call this as q. So it means that we are writing u double dash+p*u dash+q*u = 0. Then multiply by h. So we have h*u double dash+(p*h)*u dash+(q*h) u = 0 where p is p1/p0 and q is p2 here. Now our claim is that here this is an exact differential equation, so it means that this is given in self adjoint form.

So it means that if it is self-adjoint form then (()) (30:39) of this u double dash must be equal to the coefficient of u dash. So it means that h dash must be equal to p*h. This equation is in self-adjoint form if we have h dash= p*h and this is first order homogenous differential equation, so we can easily solve this equation to find out the value of h. If you solve, then equation of h is given in exp (p*dx) where p is this p1/p0. So it means that the equation (8) can be made self-adjoint, so it means that if you multiply here h by p0 then this equation is in the form of self-adjoint form because this h we are multiplying in this kind of equation.

If we have p0, first we have to divide by p0 and then multiply by h. So it means that the function h which we need to multiply to make this equation as self-adjoint form is the following function h(x) =[exp(p1/p0)*dx]/p0. So if you multiply this equation, then our equation is reduced in a self-adjoint form. Let me, look at here, so h is coming out to what? H is coming out to be the exponential of p1/p0*dx.

So h we are multiplying here in this equation. So it means that if you multiply h/p0 into this equation that is p0*u double dash +p1*u dash+p2*u then this is what, this is h*u double dash+h*p1/p0*u dash+h*p2/p0*u = 0. So if h is this, then this can be written as d/dx of (exp(p1/p0)*u dash)+h*(p2/p0)*u] = 0. So if you looking at making this as self adjoint the only rule is of the coefficient of u dash and coefficient of u double dash. So you need not to worry about coefficient of u at all.

So now we say that self adjoint equation is very common and it is written in this particular form d/dx of p0(x)*du/dx]+p2(x)*u = 0 and if it is not given in self adjoint form then we can always multiply by function which we have denoted as the equation number 9. We can make linear differential equation into a self-adjoint form. So it means that now onward without (()) (33:48) we can always consider that a given second order linear differential equation is given in self-adjoint form.

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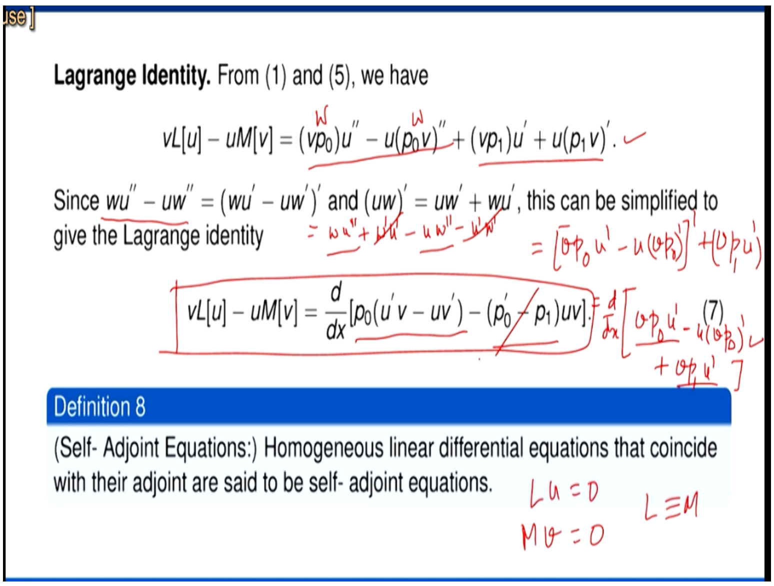

And here note one thing that in this case self-adjoint differential equation the Lagrange Identity is now simplified and it is now given as v*L[u] – u*L[v] = d/dx of p0(x)*(u dash*v-u*v dash)]. In fact it is d/dx of p0(x)*w(u,v). So that is what, why because in self-adjoint form p0 dash=p1 and if you look at, go to Lagrange Identity. Lagrange Identity the coefficient of u*v is (p0 dash-p1). So in self adjoint form p0 dash= p1.

So this will cancel out in the case of self-adjoint equation forms. So what remains is d/dx of p0*(u dash*v – u*v dash). That is what it is written here d/dx of p0(x)*w(u,v). So here Lagrange Identity simplifies to v*L[u] –u* L[v] = d/dx of p0(x)*(u dash*v – u*v dash) and which I can write it as –d/dx of p0(x)*w(u,v). So now let us consider one example.

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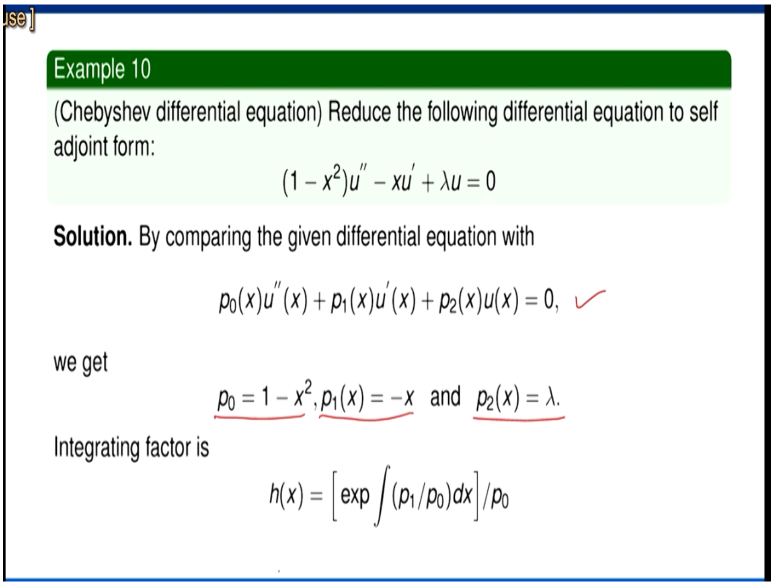

So consider this Chebyshev differential equation. So here we want to reduce the following differential equation to self adjoint form. Right now it is not given in self-adjoint form, because if you differentiate this (1 - x square 2) it is given as -2*x which is not the coefficient of u dash. Coefficient of u dash is only x. So right now this differential equation is not given in self-adjoint form.

So let us make it into self adjoint form. So here we just compare from the differential equation and we say that p0 = 1- x square 2, p1(x) = -x and p2(x) = lambda. So we already know that the factor by which we can make this equation as self adjoint equation. We can calculate h(x) as [exp (p1/p0)*dx]/p0. So p1 is –x and p0 is 1- x square 2.

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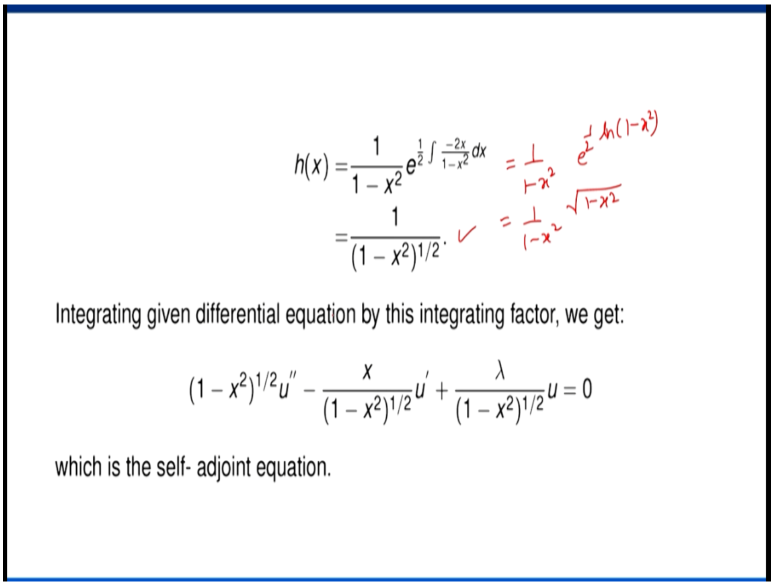

So we can calculate this h(x) as 1 upon (1-x square 2) this is p0*e to the power ½*(-2*x/(1-x square 2)). This we can simplify, this is what, this I can write as 1 upon (1-x square 2)*[e1/2*(ln*(1-x square 2))]. So you can write this as 1 upon 1-x square 2(square root (1-x square 2)). We can simplify and we can have h(x) as 1 upon (1-x square 2) power ½. So it means that by multiplying this function h(x) we can make our differential equation as self-adjoint differential equation.

So multiply throughout by 1 upon 1-x square 2, we have this following differential equation ((1-x square 2) to the power 1/2)*u double dash – (x/(1-x square 2) to the power 1/2)*u dash +(lambda/(1-x square 2) to the power 1/2)*u = 0. You can easily check that here the derivative of this is given by this. So this is given in self-adjoint form.

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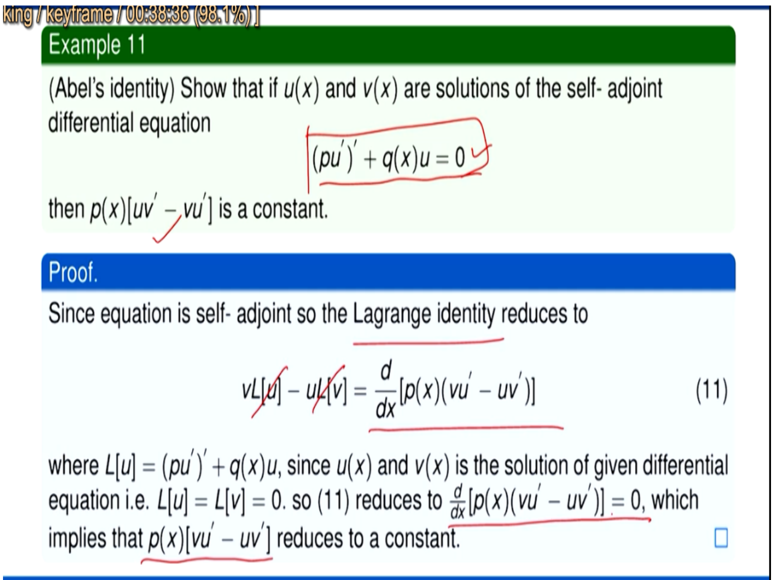

Now let us look at one more example, so that if u and v are solution of self-adjoint differential equation then this quantity (p*u dash) dash +q(x)*u = 0, this is the self-adjoint equation given. Then p(x)*[u *v dash - v*u dash] is a constant. So this is the self adjoint equation given in this way and we say that u and v are solutions of this then the following quantity is a constant quantity.

This is straight forward example because the given equation is self-adjoint form then we have v*L[u] – u*L[v] = d/dx of p(x)*(v*u dash- u*v dash). So we already know that u, v are solution of this, so L[u] = 0 and L[v] = 0. So this part is going to be 0, so it means that p(x)*[u*v dash -v*u dash] is going to be 0. This says that the given p(x)*[v*u dash- u*v dash] is a constant value. So this example is straight forward example based on Lagrange Identity.

By using Lagrange Identity for self adjoint form we can easily say that this quantity d/dx*[p(x)*(v*u dash- u*v dash)] = 0 and by integrating we can say that p(x)*[v*u dash- u*v dash] is equal to a constant value. So here we end our discussion and in the next session we will start with Sturm - Liouville boundary value problem. That is all for this lecture, thank you very much for listening. Thank you.